Numer. Math. 42, 259–269 (1983)

Numerische Mathematik

The QD-Algorithm and Multivariate Padé-Approximants

Dedicated to Prof. Dr. P. Henrici on the occasion of his 60th birthday

Annie A.M. Cuyt*

Department of Mathematics U.I.A., University of Antwerp,
Universiteitsplein 1, B-2610 Wilrijk, Belgium

Summary. The quotient-difference (QD) algorithm can be used to construct univariate Padé-approximants [1]. In this paper we see that it can also be used to construct the multivariate Padé-approximants introduced in [3], just by reformulating the quotient-difference algorithm as in Sect. 1. The multivariate Padé-approximants and the multivariate QD-scheme are treated in the Sects. 2 and 3 respectively. Thus for this type of multivariate Padé-approximants a link with the theory of multivariate continued fractions is established.

Subject Classifications: AMS(MOS): 65D15, CR: 5.13.

0. Introduction

It is well-known that the quotient-difference algorithm can be used to compute Padé-approximants for a univariate function [1]. They are obtained then as convergents of a continued fraction.

Several authors already have tried to generalize the concept of Padé-approximants to the multivariate case. We refer to [2, 6–9]. For all those generalizations there is no link with continued fractions. Other authors have introduced multivariate continued fractions without really obtaining Padé-approximants [11]. But if the multivariate Padé-approximants are defined, using a shift of the degrees of numerator and denominator [3], then one can construct multivariate continued fractions that provide those Padé-approximants and thus one can use the QD-algorithm to compute them. The shift of the degrees is also necessary to obtain a nontrivial denominator, as will be illustrated. For an extensive study of the properties of this type of Padé-approximants the interested reader is referred to [4].

In the last section the newly introduced multivariate QD-scheme will be used to calculate approximations for the poles and zeros of the two-variable beta-function.

* Aspirant M.F.W.O. (Belgium)

1. The Quotient-Difference Scheme

Our presentation of the QD-algorithm differs slightly from the usual one. In the one-dimensional case the two approaches are equivalent; however our approach can be generalized to the multivariate case.

Let us consider a univariate function

$$f(x)=\sum_{k=0}^{\infty} c_k x^k$$

analytic in the origin.

We call the series f normal if

$$\begin{vmatrix} c_n x^n & c_{n+1} x^{n+1} & \dots & c_{n+k-1} x^{n+k-1} \\ c_{n+1} x^{n+1} & & & \\ \vdots & \vdots & & \vdots \\ c_{n+k-1} x^{n+k-1} & & & c_{n+2k-2} x^{n+2k-2} \end{vmatrix} \not\equiv 0$$

for all $n \geqq 0$ and $k \geqq 1$. This determinant is a monomial of degree $k(n+k-1)$ in x. The condition of nontriviality is equivalent with the condition that the same determinant with $x=1$ is nonzero.

For a normal series we can construct a table with double entry of numbers $q_k^{(n)}$ and $e_k^{(n)}$ defined as follows:

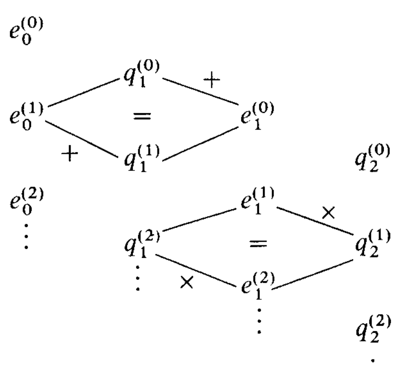
$$e_0^{(n)}=0 \qquad n=0, 1, \dots,$$

$$q_1^{(n)}=\frac{c_{n+1} x^{n+1}}{c_n x^n} \qquad n=0, 1, \dots,$$

$$e_k^{(n)}=q_k^{(n+1)}+e_{k-1}^{(n+1)}-q_k^{(n)} \qquad n=0, 1, 2, \dots \quad k=1, 2, \dots,$$

$$q_{k+1}^{(n)}=q_k^{(n+1)}\, e_k^{(n+1)}/e_k^{(n)} \qquad n=0, 1, 2, \dots \quad k=1, 2, \dots.$$

These rules can easily be remembered by means of the following scheme (the superscript (n) indicates a diagonal while the subscript k indicates a column):



The QD-algorithm can now be used to construct Padé-approximants to the function f. For a normal series the Padé-table is likewise normal [1].

The (ℓ, m) Padé-approximant (numerator of degree ℓ and denominator of degree m) for $\ell \geqq m$ is equal to the $(2m)^{\text{th}}$ convergent K_{2m} of the continued fraction

$$c_0 + c_1 x + \ldots + c_{\ell-m} x^{\ell-m} + \frac{c_{\ell-m+1} x^{\ell-m+1}|}{|1} - \frac{q_1^{(\ell-m+1)}|}{|1} - \frac{e_1^{(\ell-m+1)}|}{|1} - \frac{q_2^{(\ell-m+1)}|}{|1} - \frac{e_2^{(\ell-m+1)}|}{|1} - \ldots$$

if $K_0 = \sum_{k=0}^{\ell-m} c_k x^k$, and the $(2m+1)^{\text{th}}$ convergent K_{2m+1} of the continued fraction

$$c_0 + c_1 x + \ldots + c_{\ell-m-1} x^{\ell-m-1} + \frac{c_{\ell-m} x^{\ell-m}|}{|1} - \frac{q_1^{(\ell-m)}|}{|1} - \frac{e_1^{(\ell-m)}|}{|1} - \frac{q_2^{(\ell-m)}|}{|1} - \frac{e_2^{(\ell-m)}|}{|1} - \ldots$$

if $K_0 = \sum_{k=0}^{\ell-m-1} c_k x^k$ [1].

The terms $q_k^{(n)}$ and $e_k^{(n)}$ each contain a factor x now because of the definition of $q_1^{(n)}$. We shall find this property also in the multivariate QD-scheme.

2. Multivariate Padé-Approximants

During the last ten years several ways have been tried to generalize the concept of Padé-approximant to multivariate functions. For most of the generalizations there is no link with continued fractions. The definition of the multivariate Padé-approximant which follows, enables one to construct multivariate continued fractions, the convergents of which provide the Padé-approximants.

Let

$$f(x_1, \ldots, x_p) = \sum_{k_1, \ldots, k_p = 0}^{\infty} c_{k_1 \ldots k_p} x_1^{k_1} \ldots x_p^{k_p}.$$

If we introduce the notation

$$c_k(x) = \sum_{k_1 + \ldots + k_p = k} c_{k_1 \ldots k_p} x_1^{k_1} \ldots x_p^{k_p}$$

we can also write for $f(x_1, \ldots, x_p)$:

$$f(x) = \sum_{k=0}^{\infty} c_k(x)$$

where now $x = (x_1, \ldots, x_p)$ is a vector.

Now find

$$p_{[\ell,m]}(x) = \sum_{i=\ell m}^{\ell m+\ell} \sum_{i_1 + \ldots + i_p = i} a_{i_1 \ldots i_p} x_1^{i_1} \ldots x_p^{i_p} = \sum_{i=\ell m}^{\ell m+\ell} a_i(x)$$

and

$$q_{[\ell,m]}(x)=\sum_{j=\ell m}^{\ell m+m}\sum_{j_1+\ldots+j_p=j} b_{j_1\ldots j_p}x_1^{j_1}\ldots x_p^{j_p}=\sum_{j=\ell m}^{\ell m+m} b_j(x)$$

such that

$$(f\cdot q_{[\ell,m]}-p_{[\ell,m]})(x)=\sum_{k_1+\ldots+k_p\geqq \ell m+\ell+m+1} d_{k_1\ldots k_p}x_1^{k_1}\ldots x_p^{k_p}. \qquad (1)$$

If we denote by ∂_0 the order of a power series, i.e. the degree of the first nonzero term (where a term $x_1^{k_1}\ldots x_p^{k_p}$ is said to be of degree $k_1+\ldots+k_p$), then condition (1) can be reformulated as

$$\partial_0(f\cdot q_{[\ell,m]}-p_{[\ell,m]})\geqq \ell m+\ell+m+1.$$

In the sequel of the text ∂ will denote the degree of a polynomial.

The shift of the degrees of $p_{[\ell,m]}(x)$ and $q_{[\ell,m]}(x)$ by ℓm is necessary to obtain a nontrivial solution for the $b_{j_1\ldots j_p}$. The following simple example will illustrate this.

Consider

$$f(x)=f(x_1,x_2)=1+x_1+\sin(x_1x_2)=1+x_1+\sum_{k=0}^{\infty}(-1)^k\frac{(x_1x_2)^{2k+1}}{(2k+1)!}.$$

Now take $\ell=1$, $m=3$ and calculate

$$p(x_1,x_2)=a_{00}+a_{10}x_1+a_{01}x_2,$$
$$q(x_1,x_2)=b_{00}+b_{10}x_1+b_{01}x_2+b_{20}x_1^2+b_{11}x_1x_2+b_{02}x_2^2+b_{30}x_1^3 +b_{21}x_1^2x_2+b_{12}x_1x_2^2+b_{03}x_2^3$$

such that

$$\partial_0(f\cdot q-p)\geqq \ell+m+1$$

i.e. without performing a shift in p, q and $f\cdot q-p$.

The only solution for $p(x_1,x_2)$ and $q(x_1,x_2)$ is

$$p(x_1,x_2)\equiv 0,$$
$$q(x_1,x_2)\equiv 0$$

which is useless.

A representation of $p_{[\ell,m]}(x)$ and $q_{[\ell,m]}(x)$ satisfying $\partial_0(f\cdot q_{[\ell,m]}-p_{[\ell,m]})\geqq \ell m+\ell+m+1$, is

$$p_{[\ell,m]}(x)=\begin{vmatrix} f_\ell(x) & f_{\ell-1}(x) & \ldots & f_{\ell-m}(x)\\ c_{\ell+1}(x) & c_\ell(x) & \ldots & c_{\ell-m+1}(x)\\ \vdots & & & \\ c_{\ell+m}(x) & & & c_\ell(x)\end{vmatrix},$$

$$q_{[\ell,m]}(x)=\begin{vmatrix} 1 & \ldots & \ldots & 1\\ c_{\ell+1}(x) & c_\ell(x) & \ldots & c_{\ell-m+1}(x)\\ \vdots & & & \\ c_{\ell+m}(x) & \ldots & \ldots & c_\ell(x)\end{vmatrix}$$

where $f_\ell(x)=\sum_{k=0}^{\ell} c_k(x)$.

This is a direct generalization of the existing univariate formulas [10]. Observe that here the term

$$c_k(x)=\sum_{k_1+\ldots+k_p=k} c_{k_1\ldots k_p} x_1^{k_1}\ldots x_p^{k_p}$$

takes the place of the coefficient c_k. This is precisely the difference between the usual definition of the QD-table and the one given in Sect. 1. More about this type of multivariate Padé-approximants can be found in [3].

3. The Multivariate QD-Scheme

We proceed exactly as in the univariate case.

We call the series f normal if

$$\begin{vmatrix} c_n(x) & c_{n+1}(x) & \ldots & c_{n+k-1}(x) \\ c_{n+1}(x) & & & \vdots \\ \vdots & & & \vdots \\ c_{n+k-1}(x) & & & c_{n+2k-2}(x) \end{vmatrix} \neq 0$$

for all $n \geqq 0$ and $k \geqq 1$, where now

$$c_k(x)=\sum_{k_1+\ldots+k_p=k} c_{k_1\ldots k_p} x_1^{k_1}\ldots x_p^{k_p}.$$

For a normal series the irreducible form $\frac{p_\Delta}{q_\Delta}(x)$ of $\frac{p_{[\ell,m]}}{q_{[\ell,m]}}(x)$ satisfies

$$\begin{aligned} \partial_1 p_\Delta &= \partial p_\Delta - \partial_0 q_\Delta = \ell, \\ \partial_1 q_\Delta &= \partial q_\Delta - \partial_0 q_\Delta = m, \\ \partial_0(f\cdot q_\Delta - p_\Delta) &= \ell m + \ell + m + 1 \end{aligned}$$

and thus $\frac{p_\Delta}{q_\Delta}(x)$ occurs only once in the Padé-table [4, pp. 61–62].

Define the table with double entry as follows:

$$e_0^{(n)}=0 \qquad n=0,1,\ldots,$$

$$q_1^{(n)}=\frac{c_{n+1}(x)}{c_n(x)}=\frac{\sum_{k_1+\ldots+k_p=n+1} c_{k_1\ldots k_p} x_1^{k_1}\ldots x_p^{k_p}}{\sum_{k_1+\ldots+k_p=n} c_{k_1\ldots k_p} x_1^{k_1}\ldots x_p^{k_p}} \qquad n=0,1,\ldots,$$

$$e_k^{(n)}=q_k^{(n+1)}+e_{k-1}^{(n+1)}-q_k^{(n)} \qquad n=0,1,2,\ldots \quad k=1,2,\ldots,$$

$$q_{k+1}^{(n)}=q_k^{(n+1)}e_k^{(n+1)}/e_k^{(n)} \qquad n=0,1,2,\ldots \quad k=1,2,\ldots$$

and construct the following continued fractions:
(all the $q_k^{(n)}$ and $e_k^{(n)}$ exist because f is normal [5, pp. 610])

$$\sum_{k=0}^{\ell-m} c_k(x)+\frac{c_{\ell-m+1}(x)|}{|1}-\frac{q_1^{(\ell-m+1)}|}{|1}-\frac{e_1^{(\ell-m+1)}|}{|1}-\frac{q_2^{(\ell-m+1)}|}{|1}-\frac{e_2^{(\ell-m+1)}|}{|1}-\ldots, \tag{2}$$

$$\sum_{k=0}^{\ell-m-1} c_k(x)+\frac{c_{\ell-m}(x)|}{|1}-\frac{q_1^{(\ell-m)}|}{|1}-\frac{e_1^{(\ell-m)}|}{|1}-\frac{q_2^{(\ell-m)}|}{|1}-\frac{e_2^{(\ell-m)}|}{|1}-\ldots \tag{3}$$

We shall now prove that these continued fractions are of the same form as in the univariate case where $q_k^{(n)}$ and $e_k^{(n)}$ contain a factor x, and also that the convergents yield the multivariate Padé-approximants.

Theorem 1. *If we write* $q_k^{(n)}=\dfrac{N_{q,k,n}}{D_{q,k,n}}$ *and* $e_k^{(n)}=\dfrac{N_{e,k,n}}{D_{e,k,n}}$ *then* $\partial N_{q,k,n}=\partial D_{q,k,n}+1$ *and* $\partial N_{e,k,n}=\partial D_{e,k,n}+1$

Proof. The proof is by induction.
For $k=1$ we have

$$N_{q,k,n}=c_{n+1}(x) \quad\text{and}\quad D_{q,k,n}=c_n(x),$$
$$N_{e,k,n}=(c_{n+1}(x))^2-c_n(x)\cdot c_{n+2}(x),$$
$$D_{e,k,n}=c_n(x)\cdot c_{n+1}(x)$$

so that

$$\partial N_{q,k,n}=n+1=\partial D_{q,k,n}+1,$$
$$\partial N_{e,k,n}=2n+2=\partial D_{e,k,n}+1.$$

Suppose the theorem holds for $q_1^{(n)},\ldots,q_k^{(n)}$, $e_1^{(n)},\ldots,e_k^{(n)}$; we shall prove it then for $q_{k+1}^{(n)}$ and $e_{k+1}^{(n)}$.
Since $q_{k+1}^{(n)}=q_k^{(n+1)}e_k^{(n+1)}/e_k^{(n)}$, we have

$$q_{k+1}^{(n)}=\frac{N_{q,k,n+1}N_{e,k,n+1}D_{e,k,n}}{N_{e,k,n}D_{q,k,n+1}D_{e,k,n+1}}=\frac{N_{q,k+1,n}}{D_{q,k+1,n}}.$$

Thus $\partial N_{q,k+1,n}=\partial N_{q,k,n+1}+\partial N_{e,k,n+1}+\partial D_{e,k,n}=\partial D_{q,k+1,n}+1.$

Now $\quad e_{k+1}^{(n)}=q_{k+1}^{(n+1)}+e_k^{(n+1)}-q_{k+1}^{(n)}$

$$=\frac{N_{q,k+1,n+1}D_{e,k,n+1}D_{q,k+1,n}+\ldots-\ldots}{D_{q,k+1,n+1}D_{e,k,n+1}D_{q,k+1,n}}=\frac{N_{e,k+1,n}}{D_{e,k+1,n}}$$

which proves that $\partial N_{e,k+1,n}=\partial D_{e,k+1,n}+1$.

This already generalizes the univariate case, where only a factor x remains in $q_k^{(n)}$ and $e_k^{(n)}$ after division of numerator and denominator. Now consider the following descending staircase of distinct multivariate Padé-approximants:

$$\begin{array}{lll}
\dfrac{p_{[\ell-m,0]}(x)}{q_{[\ell-m,0]}(x)}, & & \\
\dfrac{p_{[\ell-m+1,0]}(x)}{q_{[\ell-m+1,0]}(x)} & \dfrac{p_{[\ell-m+1,1]}(x)}{q_{[\ell-m+1,1]}(x)}, & \\
 & \dfrac{p_{[\ell-m+2,1]}(x)}{q_{[\ell-m+2,1]}(x)} & \ldots \\
 & & \vdots
\end{array}$$

Theorem 2. $\frac{p_{[\ell,m]}(x)}{q_{[\ell,m]}(x)}$ *is the* $(2m)^{\text{th}}$ *convergent of the continued fraction* (2).

Proof. Let $\frac{p_{[\ell-m+i,j]}(x)}{q_{[\ell-m+i,j]}(x)}=K_{i+j},\ i+j=0,1,\dots$.

It is possible to construct a continued fraction with convergents $K_0, K_1, K_2, \dots$.

This continued fraction has the form

$$K_0+\frac{K_1-K_0|}{|\ 1}+\sum_{n=2}^{\infty}\frac{\frac{K_{n-1}-K_n}{K_{n-1}-K_{n-2}}\Big|}{\Big|\frac{K_n-K_{n-2}}{K_{n-1}-K_{n-2}}}$$

which can be written as

$$\sum_{k=0}^{\ell-m}c_k(x)+\frac{c_{\ell-m+1}(x)|}{|\ 1}+\frac{\frac{K_1-K_2}{K_2-K_0}\Big|}{|\ 1}+\sum_{n=3}^{\infty}\frac{\frac{(K_{n-1}-K_n)(K_{n-2}-K_{n-3})}{(K_n-K_{n-2})(K_{n-1}-K_{n-3})}\Big|}{|\ 1}. \tag{4}$$

It is easy to verify that

$$\frac{K_1-K_2}{K_2-K_0}=q_1^{(\ell-m+1)} \quad\text{and}\quad \frac{(K_2-K_3)(K_1-K_0)}{(K_3-K_1)(K_2-K_0)}=e_1^{(\ell-m+1)}$$

using the representation of $p_{[\ell-m,0]}(x)$, $q_{[\ell-m,0]}(x)$, $p_{[\ell-m+1,0]}(x)$, $q_{[\ell-m+1,0]}(x)$, ... given in the previous section.

Let us denote

$$\frac{(K_{n-1}-K_n)(K_{n-2}-K_{n-3})}{(K_n-K_{n-2})(K_{n-1}-K_{n-3})}$$

by $A_{\frac{n}{2}}^{(\ell-m+1)}$ if n is even and by $B_{\frac{n-1}{2}}^{(\ell-m+1)}$ if n is odd. We write also $A_1^{(\ell-m+1)}=q_1^{(\ell-m+1)}$. If we write down the continued fraction that is the even contraction of (4) (i.e. a continued fraction having as convergents the K_{2n} for $n=0,1,\dots$), we get

$$\sum_{k=0}^{\ell-m}c_k(x)+\frac{c_{\ell-m+1}(x)\ |}{|1-A_1^{(\ell-m+1)}}-\frac{A_1^{(\ell-m+1)}B_1^{(\ell-m+1)}\ |}{|1-B_1^{(\ell-m+1)}-A_2^{(\ell-m+1)}}-\dots. \tag{5}$$

If we write down the continued fraction that is the odd contraction of (4) with $\ell-m$ replaced by $\ell-m-1$ (i.e. a continued fraction having as convergents the

$$\frac{p_{[\ell-m,0]}(x)}{q_{[\ell-m,0]}(x)},\quad \frac{p_{[\ell-m+1,1]}(x)}{q_{[\ell-m+1,1]}(x)},\quad \frac{p_{[\ell-m+2,2]}(x)}{q_{[\ell-m+2,2]}(x)},\quad \dots$$

on the descending staircase (7)), we get

$$\sum_{k=0}^{\ell-m-1}c_k(x)+\frac{c_{\ell-m}(x)A_1^{(\ell-m)}\ |}{|1-A_1^{(\ell-m)}-B_1^{(\ell-m)}}-\frac{B_1^{(\ell-m)}A_2^{(\ell-m)}\ |}{|1-A_2^{(\ell-m)}-B_2^{(\ell-m)}}-\dots. \tag{6}$$

Because (5) and (6) have the same convergents we have

$$A_k^{(\ell-m+1)}B_k^{(\ell-m+1)}=B_k^{(\ell-m)}A_{k+1}^{(\ell-m)} \qquad k=1,2,\ldots,$$

$$B_{k-1}^{(\ell-m+1)}+A_k^{(\ell-m+1)}=A_k^{(\ell-m)}+B_k^{(\ell-m)} \qquad k=1,2,\ldots$$

if we put $B_0^{(\ell-m+1)}=0$.

So

$$\begin{array}{l} A_k^{(\ell-m+1)}=q_k^{(\ell-m+1)} \\ B_k^{(\ell-m+1)}=e_k^{(\ell-m+1)} \end{array} \quad k=1,2,\ldots.$$

This completes the proof.

Analogously we can formulate and prove the next theorem.

Theorem 3. $\dfrac{p_{[\ell,m]}(x)}{q_{[\ell,m]}(x)}$ *is the* $(2m+1)^{\text{th}}$ *convergent of the continued fraction* (3).

This can easily be seen by writing down the continued fraction (4) with $\ell-m$ replaced by $\ell-m-1$; the convergents of this continued fraction are the multivariate Padé-approximants on the following descending staircase:

$$\begin{array}{lll} \dfrac{p_{[\ell-m-1,0]}(x)}{q_{[\ell-m-1,0]}(x)} & & \\ \dfrac{p_{[\ell-m,0]}(x)}{q_{[\ell-m,0]}(x)} & \dfrac{p_{[\ell-m,1]}(x)}{q_{[\ell-m,1]}(x)} & \\ & \dfrac{p_{[\ell-m+1,1]}(x)}{q_{[\ell-m+1,1]}(x)} & \cdots \\ & & \vdots \end{array} \tag{7}$$

The following example both illustrates Theorems 2 and 3. Consider

$$f(x)=f(x_1,x_2)=\frac{x_1e^{x_1}-x_2e^{x_2}}{x_1-x_2}=\sum_{i_1,i_2=0}^{\infty}\frac{x_1^{i_1}x_2^{i_2}}{(i_1+i_2)!}$$
$$=1+x_1+x_2+\tfrac{1}{2}(x_1^2+x_1x_2+x_2^2)+\ldots.$$

Take $\ell=2$ and $m=1$. The Padé-approximant $\dfrac{p_{[2,1]}}{q_{[2,1]}}(x_1,x_2)$ is given by

$$p_{[2,1]}(x_1,x_2)=\begin{vmatrix} 1+x_1+x_2+\frac{1}{2}(x_1^2+x_1x_2+x_2^2) & 1+x_1+x_2 \\ \frac{1}{6}(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3) & \frac{1}{2}(x_1^2+x_1x_2+x_2^2) \end{vmatrix}$$
$$=\tfrac{1}{2}(x_1^2+x_1x_2+x_2^2)+\tfrac{1}{3}(x_1^3+\tfrac{5}{2}(x_1^2x_2+x_1x_2^2)+x_2^3)$$
$$+\tfrac{1}{12}(x_1^4+2(x_1^3x_2+x_1x_2^3)+5x_1^2x_2^2+x_2^4)$$

$$q_{[2,1]}(x_1,x_2)=\begin{vmatrix} 1 & 1 \\ \frac{1}{6}(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3) & \frac{1}{2}(x_1^2+x_1x_2+x_2^2) \end{vmatrix}$$
$$=\tfrac{1}{2}(x_1^2+x_1x_2+x_2^2)-\tfrac{1}{6}(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3).$$

According to Theorem 2, this is also the second convergent of the continued fraction (2), i.e.

$$1+x_1+x_2+\frac{\frac{1}{2}(x_1^2+x_1x_2+x_2^2)|}{|\quad 1}-\frac{\dfrac{(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3)}{3(x_1^2+x_1x_2+x_2^2)}\Big|}{|\quad 1}.$$

According to Theorem 3 it is also the third convergent of the continued fraction (3), i.e.

$$1+\frac{x_1+x_2|}{|\quad 1}-\frac{\dfrac{(x_1^2+x_1x_2+x_2^2)}{2(x_1+x_2)}\Big|}{|\quad 1}-\frac{\dfrac{-(x_1^4+2x_1^3x_2+5x_1^2x_2^2+2x_1x_2^3+x_2^4)}{6(x_1^2+x_1x_2+x_2^2)(x_1+x_2)}\Big|}{|\quad 1}.$$

In both cases the QD-scheme is started with

$$q_1^{(n)}=\sum_{i_1+i_2=n+1}\frac{x_1^{i_1}x_2^{i_2}}{(i_1+i_2)!}\Bigg/\sum_{i_1+i_2=n}\frac{x_1^{i_1}x_2^{i_2}}{(i_1+i_2)!}.$$

4. Numerical Example

The beta function is defined by

$$B(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

where Γ is the gamma function.

Singularities occur for $x=-n$ and $y=-n$ ($n=0, 1, 2, \ldots$) and zeros for $y=-x-n$ ($n=0, 1, 2, \ldots$). We write

$$B(x,y)=\frac{A(x-1,y-1)}{xy}$$

with

$$A(u,v)=1+uvf(u,v).$$

We will calculate the Padé-approximant $p_{[\ell,m]}(u,v)/q_{[\ell,m]}(u,v)$ for $f(u,v)$ by means of the multivariate QD-algorithm and compute

$$\frac{q_{[\ell,m]}(x-1,y-1)+(x-1)(y-1)\,p_{[\ell,m]}(x-1,y-1)}{xy\,q_{[\ell,m]}(x-1,y-1)} \tag{8}$$

as an approximation for $B(x,y)$.

The trajectories of the poles and zeros of the beta function are shown in Fig. 1, while the poles of (8) for $\ell=7$ and $m=1$ and the zeros of (8) for $\ell=2$ and $m=2$ can respectively be found in Fig. 2 and Fig. 3.

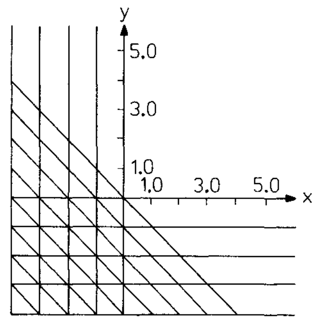


Fig. 1

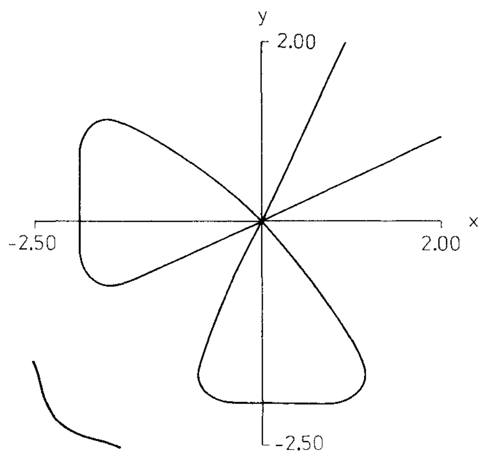


Fig. 2

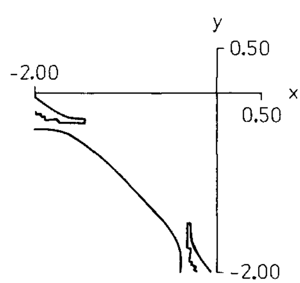


Fig. 3

In both cases we remark that the vertical, horizontal and diagonal lines are nicely simulated, and that the drawings are symmetric because the symmetry of $B(x, y)$ is preserved by the multivariate Padé-approximants [4].

References

1. Brezinski, C.: Padé-type approximation and general orthogonal polynomials. ISNM 50. Basel: Birkhäuser Verlag 1980
2. Chisholm, J.S.R.: N-variable rational approximants. Padé and rational approximation: theory and applications. Saff, E.B., Varga, R.S. (eds.). London: Academic Press, pp. 23–42, 1977
3. Cuyt, A.A.M.: Multivariate Padé-approximants. (to appear in J. Math. Anal. Appl.)
4. Cuyt, A.A.M.: Abstract Padé Approximants for operators: theory and applications. Ph. D., Universitaire Instelling Antwerpen, 1982
5. Henrici, P.: Applied and computational complex analysis I. New York: John Wiley 1974
6. Hughes Jones, R.: General rational approximants in n variables. J. Approximation Theory **16**, 201–233 (1976)
7. Karlsson, J., Wallin, H.: Rational approximants by an interpolation procedure in several variables. Padé and rational approximation: theory and applications. Saff, E.B., Varga, R.S. (eds.). London: Academic Press, pp. 83–100, 1977
8. Lutterodt, C.H.: A two-dimensional analogue of Padé approximant theory. J. Phys. A: Math. **7** (9), 1027–1037 (1974)
9. Lutterodt, C.H.: Rational approximants to holomorphic functions in n dimensions. J. Math. Anal. Appl. **53**, 89–98 (1976)
10. Perron, O.: Die Lehre von den Kettenbrüchen. Stuttgart: Teubner, 1977
11. Siemaszko, W.: Thiele-type branched continued fractions for two-variable functions. (to appear in J. Comp. Appl. Maths.)

Received September 6, 1982 / April 18, 1983