J. OPERATOR THEORY
6(1981), 195–216

ON THE PROPERTIES OF ABSTRACT RATIONAL (1-POINT) APPROXIMANTS

ANNIE A. M. CUYT

1. DEFINITION OF APA AND ARA

Let $F:X \to Y$ be an operator analytic in 0 [5, pp. 113] where $X \supsetneqq \{0\}$ is a Banach space and $Y \supseteq \{0, I\}$ is a commutative Banach algebra (0 is the unit for addition and I is the unit for multiplication; X and Y have the same scalar field Λ where Λ is **R** or **C**). We always work in the norm-topology.

Write $D(F) = \{x \in X \mid F(x)$ is regular in Y, i.e. there exists $y \in Y : F(x)\cdot y = I = y\cdot F(x)\}$.

For every positive integer k we consider the spaces $\mathrm{L}(X^k, Y) = \{L \mid L$ is a k-linear bounded operator $L:X \to \mathrm{L}(X^{k-1}, Y)\}$, where $\mathrm{L}(X^0, Y) = Y$. So $Lx_1 \ldots x_k = (Lx_1)x_2 \ldots x_k$ with $(x_1, \ldots, x_k) \in X^k$ and $Lx_1 \in \mathrm{L}(X^{k-1}, Y)$. The operator $L \in \mathrm{L}(X^k, Y)$ is called *symmetric* if $Lx_1 \ldots x_k = Lx_{i_1} \ldots x_{i_k}$ for all $(x^1, \ldots, x_k) \in X^k$ and all permutations $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$. An example of a symmetric k-linear bounded operator is $F^{(k)}(x_0)$, the k^{th}(Fréchet-) derivative of F in x_0[5, pp. 100–110].

An *abstract polynomial* is a non-linear operator $P:X \to Y$ such that $P(x) = A_n x^n + \ldots + A_0 \in Y$ with $A_i \in \mathrm{L}(X^i, Y)$ and A_i symmetric. The degree of $P(x)$ is n.

We also introduce the following notations.

If there exists a positive integer j_1 such that $\forall k, 0 \leqslant k < j_1 : A_k x^k \equiv 0$ and $A_{j_1} x^{j_1} \not\equiv 0$, then $\partial_0 P = j_1$.

If there exists a positive integer j_2 such that $\forall k, j_2 < k \leqslant n : A_k x^k \equiv 0$ and $A_{j_2} x^{j_2} \not\equiv 0$, then $\partial P = j_2$.

∂P is the exact degree of the abstract polynomial P; $\partial_0 P$ is the order of the abstract polynomial P.

We have the following important lemma for abstract polynomials.

LEMMA 1.1. *Let U, V be abstract polynomials*: $X \to Y$. *If* $U(x)\cdot V(x) \equiv 0$ *and if* $D(V) \neq \emptyset$, *then* $U \equiv 0$.

The proof is given in [3].

Since F is analytic in 0, there exists an open ball $B(0, r)$ with centre $0 \in X$ and radius $r > 0$ such that

$$F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(0)x^k \quad \text{for } \|x\| < r$$

where $\frac{1}{0!} F^{(0)}(0)x^0 = F(0)$.

We say that $F(x) = 0(x^j)$ if there exist $J \in \mathbf{R}_0^+$ and an open ball $B(0, r)$ with $0 < r < 1$ such that $\forall x \in B(0, r)$

$$\| F(x) \| \leqslant J \cdot \|x\|^j \, (j \in \mathbf{N}).$$

Write $\frac{1}{k!} F^{(k)}(0) = C_k \in \mathrm{L}(X^k, Y)$.

Definition 1.1. The couple of abstract polynomials

$$(P(x), Q(x)) = (A_{nm+n}x^{nm+n} + \ldots + A_{nm}x^{nm}, B_{nm+m}x^{nm+m} + \ldots + B_{nm}x^{nm})$$

is called a *solution of the Padé approximation problem* if the abstract power series

$$(1) \qquad (F \cdot Q - P)(x) = 0(x^{nm+n+m+1}).$$

The choice of $(P(x), Q(x))$ is justified in [1].

The notations $P(x) = \sum_{i=0}^{n} A_{nm+i}x^{nm+i}$ and $Q(x) = \sum_{j=0}^{m} B_{nm+j}x^{nm+j}$ will be used throughout this paper to indicate a solution of (1).

For every non-negative integers n and m a solution of the problem described in Definition 1.1 exists (see [1]).

We define $\frac{1}{Q} : D(Q) \to Y$ by $x \to [Q(x)]^{-1}$ (the inverse element of $Q(x)$ for the multiplication in Y).

We call the abstract rational operator $\frac{1}{Q} \cdot P$, the quotient of two abstract polynomials, *reducible* if there exist abstract polynomials T, R, S such that $P = T \cdot R$, $Q = T \cdot S$, $\partial T \geqslant 1$, and T is not a unit in the ring of abstract polynomials $\left(\text{i.e. } \frac{1}{T} \text{ is not an abstract polynomial}\right)$.

From now on we impose some conditions on the considered Banach space X and the commutative Banach algebra Y:

(i) Every abstract polynomial $T: X \to Y$ with $D(T) \neq \emptyset$ must have a unique prime factorization in the ring of abstract polynomials [4, p. 155].

(ii) For couples (T, U) and (V, W) of abstract polynomials with

$$D(U) \neq \emptyset \text{ or } D(T) \neq \emptyset,$$
$$D(W) \neq \emptyset \text{ or } D(V) \neq \emptyset,$$
$$\frac{1}{U} \cdot T \text{ irreducible,}$$
$$\frac{1}{W} \cdot V \text{ irreducible,}$$

the equation $T(x) \cdot W(x) = V(x) \cdot U(x)$ must imply that there exists a unit I_{AP} in the ring of abstract polynomials such that

$$T(x) = V(x) \cdot I_{AP}(x)$$
(2)
$$U(x) = W(x) \cdot I_{AP}(x).$$

We give some examples of such spaces X and Y that satisfy these restrictive conditions.

(3) $X = \mathbf{R}^p$ or $\mathbf{C}^p$ and $Y = \mathbf{R}^q$ or $\mathbf{C}^q$ for $p, q \in \mathbf{N}$, with the componentwise multiplication in Y.

(4) Let Z be a Banach algebra with unit I, not necessarily commutative. Take $a \in Z$, a regular. $X = \{\lambda a \mid \lambda \in \Lambda\}$ and $Y = \left\{ \sum_{i=0}^{\infty} \lambda_i a^i \mid \lambda_i \in \Lambda \right\}$ are proper spaces.

Condition (2) sees to it that the irreducible form of an abstract rational operator is unique and will also see to the uniqueness of the abstract Padé-approximant (Definition 1.2) and the uniqueness of the abstract rational approximant (Definition 1.3).

It will also see to it that, for n and m fixed, if a solution (P, Q) of (1) provides an abstract Padé-approximant then all solutions (R, S) of (1) with $D(R) \neq \emptyset$ or $D(S) \neq \emptyset$ do so, and if a solution (P, Q) of (1) provides an abstract rational approximant then all solutions (R, S) of (1) with $D(R) \neq \emptyset$ or $D(S) \neq \emptyset$ do so.

We mention the following theorems about solutions of (1).

THEOREM 1.1. a) *Let (P, Q) be a solution of* (1). *Then* $\partial_0 P \geqslant \partial_0 Q$.

b) *Let* $\frac{1}{Q_*} \cdot P_*$ *be the irreducible form of* $\frac{1}{Q} \cdot P = \frac{1}{Q_* \cdot T} \cdot P_* \cdot T$ *with* $T(x) = \sum_{k=t_0}^{\partial T} T_k x^k$, $t_0 = \partial_0 T$. *Let* $P_*(x) = \sum_{i=0}^{\partial P_*} A_{*i} x^i$, $Q_*(x) = \sum_{j=0}^{\partial Q_*} B_{*j} x^j$, *and* $p_0 = \partial_0 P_*$. *If* $D(T_{t_0}) \neq \emptyset$ *or* $D(A_{*p_0}) \neq \emptyset$ *or* $T_{t_0} x^{t_0} \cdot A_{*p_0} x^{p_0} \not\equiv 0$, *then* $\partial_0 P_* \geqslant \partial_0 Q_*$.

Proof. The proof of a) is very simple because of the equivalence of (1) with the following systems (1a) and (1b):

$$\text{(1a)} \quad \begin{cases} C_0 \cdot B_{nm} x^{nm} = A_{nm} x^{nm} & \forall x \in X \\ \vdots \\ C_n x^n \cdot B_{nm} x^{nm} + \ldots + C_0 \cdot B_{nm+n} x^{nm+n} = A_{nm+n} x^{nm+n} & \forall x \in X \end{cases}$$

with $B_{nm+j}x^{nm+j} \equiv 0$ if $j > m$.

$$\text{(1b)}\quad \begin{cases} C_{n+1}x^{n+1}\cdot B_{nm}x^{nm} + \ldots + C_{n+1-m}x^{n+1-m}\cdot B_{nm+m}x^{nm+m} = 0 & \forall x \in X \\ \vdots \\ C_{n+m}x^{n+m}\cdot B_{nm}x^{nm} + \ldots + C_n x^n \cdot B_{nm+m}x^{nm+m} = 0 & \forall x \in X \end{cases}$$

with $C_k x^k \equiv 0$ if $k < 0$.

The proof of b) is very similar.

Let P and Q be abstract polynomials that satisfy (1) and supply P_* and Q_*. Because $\frac{1}{Q_*}\cdot P_*$ is the irreducible form of $\frac{1}{Q}\cdot P$ there exists an abstract polynomial T such that $P = P_*\cdot T$ and $Q = Q_*\cdot T$, and $T(x) = \sum_{k=t_0}^{\partial T} T_k x^k$ with $t_0 = \partial_0 T$.

Suppose $\partial_0 P_* < \partial_0 Q_* = l$. Then

$$P(x) = A_{*p_0}x^{p_0}\cdot T_{t_0}x^{t_0} + \ldots \quad \text{with } t_0 + p_0 < t_0 + l,$$

$$Q(x) = B_{*l}x^l \cdot T_{t_0}x^{t_0} + \ldots .$$

This is a contradiction with a).

The notations $P_*(x) = \sum_{i=l}^{\partial P_*} A_{*i}x^i$ and $Q_*(x) = \sum_{j=l}^{\partial Q_*} B_{*j}x^j$ with $l = \partial_0 Q_*$ will be used from here on throughout this paper to indicate the irreducible form $\frac{1}{Q_*}\cdot P_*$ of $\frac{1}{Q}\cdot P$ whenever the conditions of Theorem 1.1 b) are satisfied.

Let $T(x) = \sum_{k=t_0}^{\partial T} T_k x^k$ with $t_0 = \partial_0 T$ be the abstract polynomial such that $P = P_*\cdot T$ and $Q = Q_*\cdot T$. If

$$\text{(5)}\qquad D(B_{*l}) \neq \emptyset$$

or if

$$\text{(6)}\qquad D(T_{t_0}) \neq \emptyset$$

then

$$t_0 \geqslant nm - l.$$

Also $t_0 = \partial_0 Q - l$ under the assumption of (5) or (6).

THEOREM 1.2. *Let* (P, Q) *and* (R, S) *be solutions of* (1) *(for* n *and* m *fixed). Then* $P(x)\cdot S(x) = Q(x)\cdot R(x)$.

The proof is given in [1] and is based on Lemma 1.1. This property is called the "*equivalence-property*" of solutions of (1).

From now on we also assume that

$$\{y \in Y \mid \text{there exists } n \in \mathbf{N} : y^n = 0\} = \{0\}. \tag{7}$$

The examples (3) and (4) both satisfy (7).

We can now formulate the next lemma. The proof is given in [3].

LEMMA 1.2. *Let U, V be abstract polynomials: $X \to Y$. If $D(U) \neq \emptyset$ and either* (7) *holds or $\partial_0 U = \partial U$, then $\partial V \leqslant \partial(U \cdot V) - \partial_0 U$.*

Due to Lemma 1.2 we can make the following conclusion. If I_{AP} is a unit in the ring of abstract polynomials, then $\partial I_{\text{AP}} \leqslant \partial\left(I_{\text{AP}} \cdot \frac{1}{I_{\text{AP}}}\right) = 0$. This implies that $I_{\text{AP}}(x) = y$, for a fixed $y \in Y$.

In the ring of abstract polynomials the only units are the regular elements in Y. This justifies the following definition.

DEFINITION 1.2. Let (P, Q) be a couple of abstract polynomials satisfying (1) and suppose $D(Q) \neq \emptyset$ or $D(P) \neq \emptyset$. Possibly $\frac{1}{Q} \cdot P$ is reducible. Let $\frac{1}{Q_*} \cdot P_*$ be the irreducible form of $\frac{1}{Q} \cdot P$ such that $0 \in D(Q_*)$ and $Q_*(0) = I$, if it exists. We then call $\frac{1}{Q_*} \cdot P_*$ *the abstract Padé-approximant* (APA) *of order* (n, m) *for* F.

Uniqueness of the (n, m)-APA is based on Theorem 1.2 and on the conditions imposed on X and Y (the units are fixed by the normalization $Q_*(0) = I$).

For the (n, m)-APA deduced from (P, Q) which is a solution of (1) it follows that the polynomial T such that $(P, Q) = (P_* \cdot T, Q_* \cdot T)$ is $T(x) = \sum_{k=nm}^{\partial T} T_k x^k$ $(t_0 \geqslant nm)$ because $l = 0$ and $D(B_{*0}) = D(I) = X \neq \emptyset$.

Because (7) holds, Lemma 1.2 implies that for the (n, m)-APA for F (since $D(T) \neq \emptyset$):

$$\partial P_* \leqslant \partial(P_* \cdot T) - nm = \partial P - nm \leqslant n$$

$$\partial Q_* \leqslant \partial(Q_* \cdot T) - nm = \partial Q - nm \leqslant m.$$

DEFINITION 1.3. If for all the solutions (P, Q) of (1) with $D(P) \neq \emptyset$ or $D(Q) \neq \emptyset$, the irreducible form $\frac{1}{Q_*} \cdot P_*$ is such that $0 \notin D(Q_*)$, then we call $\frac{1}{Q_*} \cdot P_*$ *the abstract rational approximant* (ARA) *of order* (n, m) *for* F.

Uniqueness of the (n, m)-ARA up to units is also based on Theorem 1.2 and on the conditions imposed on X and Y.

For an abstract rational approximant $\frac{1}{Q_*} \cdot P_*: \mathbf{R}^p \to \mathbf{R}^q$ a condition such as $\|B_{*l}\| = 1$ could be imposed to fix the units; but generally spoken such a condition is not restrictive enough.

If for all the solutions (P, Q) of (1) $0 \notin D(Q_*)$ or $D(Q) = \emptyset = D(P)$, we shall call the abstract Padé-approximant *undefined* (the ARA belong to the undefined ones).

2. RELATIONS BETWEEN APA, ARA, AND SOLUTIONS OF (1)

Let (P, Q) be a solution of (1). Because of the Definitions 1.2 and 1.3 it is very well possible that (P_*, Q_*) itself does not satisfy (1).

We need some new definitions to prove the important Theorem 2.1.

We introduce the notion of *pseudo-degree* for polynomials without tail like the ones considered in Definition 1.1 if $n > 0$ and $m > 0$, and like the numerator and denominator of an ARA if $l > 0$.

DEFINITION 2.1. a) Let (P, Q) be a solution of (1). We call $\partial P - \partial_0 Q = \partial_1 P$ the *pseudo-degree* of P and $\partial Q - \partial_0 Q = \partial_1 Q$ the *pseudo-degree* of Q. (Theorem 1.1 a) justifies the term $-\partial_0 Q$.)

b) Let $\frac{1}{Q_*} \cdot P_*$ be the irreducible form of $\frac{1}{Q} \cdot P$ and let the conditions of Theorem 1.1 b) be satisfied. We call $\partial P_* - l = \partial_1 P_*$ the *pseudo-degree* of P_* and $\partial Q_* - l = \partial_1 Q_*$ the *pseudo-degree* of Q_*. (Theorem 1.1 b) justifies the term $-l$.)

Remark the fact that if $\frac{1}{Q_*} \cdot P_*$ is the (n, m)-APA for F then

$$\partial P_* = \partial_1 P_*$$

$$\partial Q_* = \partial_1 Q_* .$$

When (7) holds together with (5) or (6), Lemma 1.2 implies that for the (n, m)-ARA for F (since $D(T) \neq \emptyset$):

$$\partial_1 P_* \leqslant [\partial(P_* \cdot T) - (nm - l)] - l \leqslant n$$

$$\partial_1 Q_* \leqslant [\partial(Q_* \cdot T) - (nm - l)] - l \leqslant m.$$

The demand $D(B_{*l}) \neq \emptyset$ for the ARA is the analogue of the demand $0 \in D(Q_*)$ (i.e. Q_{*0} regular) for the APA; condition (5) is always satisfied for the APA.

The following example proves the need of the conditions (5) or (6) to conclude that $\partial_1 Q_* \leqslant m$ and $\partial_1 P_* \leqslant n$.

Take

$$F\colon \mathbf{R}^2 \to \mathbf{R}^2 \text{ defined by } (x, y) \to \begin{pmatrix} \cos(\alpha - x + y) \\ \\ \dfrac{xe^x - ye^y}{x - y} \end{pmatrix} \quad (\alpha \neq k\pi).$$

The (1,1)-ARA: $\mathbf{R}^2 \to \mathbf{R}^2$

$$(x, y) \to \begin{pmatrix} \dfrac{\cos\alpha + (x - y)(\sin\alpha + 0.5 \operatorname{cotg} \alpha \cos \alpha)}{1 + 0.5(x - y)\operatorname{cotg} \alpha} \\ \\ \dfrac{x + y + 0.5(x^2 + 3xy + y^2)}{x + y - 0.5(x^2 + xy + y^2)} \end{pmatrix}.$$

Here $l = 0$, and $Q_{*0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $T(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} y - x \\ 0 \end{pmatrix}$.

This example also illustrates that if $l = 0$ we do not necessarily have an (n,m)-APA. We remark that it would be sufficient that $B_{*l}x^l \cdot T_{t_0}x^{t_0} \not\equiv 0$ (cf. Theorem 1.1 b)) to conclude that $t_0 \geqslant nm - l$ and thus $\partial_1 Q_* \leqslant m$ and $\partial_1 P_* \leqslant n$, which is a weaker condition than (5) or (6), but we shall need (6) further on too.

THEOREM 2.1. *Let* $\dfrac{1}{Q_*} \cdot P_*$ *be the* (n, m)-ARA *or* (n, m)-APA *for* F.

a) *Let* (6) *be satisfied in the case of an* (n, m)-ARA. *Then there exist an integer* s, $0 \leqslant s \leqslant \max(n,m)$ *and an abstract polynomial* $T(x) = \sum_{k=nm-l}^{nm-l+s} T_k x^k$ *with* $T_{nm-l+s}x^{nm-l+s} \not\equiv 0$ *and* $D(T) \neq \emptyset$ *such that* $(P_* \cdot T, Q_* \cdot T)$ *satisfies* (1) ($l = 0$ *for* APA, $T_k x^k \equiv 0$ *for* $k < 0$ i.e. $l > nm$).

b) *Let* (6) *be satisfied. Also for* $\partial_0 T = nm - l + r$ *with* $r \geqslant 0$, $(P_* \cdot T_{nm-l+r}, Q_* \cdot T_{nm-l+r})$ *satisfies* (1) *and* $0 \leqslant r \leqslant \min(n - \partial_1 P_*, m - \partial_1 Q_*)$ ($l = 0$ *for* APA).

Proof. Because (1) is solvable for every $n, m \in \mathbf{N}$ we may consider abstract polynomials P and Q that satisfy (1) and supply P_* and Q_*.

Because of Definition 1.3 or 1.2, there exists an abstract polynomial T with $D(T) \neq \emptyset$ such that $P = P_* \cdot T$, $Q = Q_* \cdot T$, and $T(x) = \sum_{k=t_0}^{\partial T} T_k x^k$.

Because of (6) for the ARA and (5) for the APA, $t_0 \geqslant nm - l$ for APA as well as for ARA.

Because of Lemma 1.2 and Theorem 1.1:

$$\begin{cases} \partial T \leqslant \partial(P_* \cdot T) - l = \partial P - l \leqslant (nm - l) + n \\ \text{or} \\ \partial T \leqslant \partial(Q_* \cdot T) - l = \partial Q - l \leqslant (nm - l) + m \end{cases} \quad \text{since} \quad \begin{cases} D(P_*) \neq \emptyset \\ \text{or} \\ D(Q_*) \neq \emptyset. \end{cases}$$

If we write $\partial T = (nm - l) + s$ then $0 \leqslant s \leqslant \max(n, m)$.

Now

$$F(x)\cdot Q(x) - P(x) = T(x)\cdot[F(x)\cdot Q_*(x) - P_*(x)] = 0(x^{nm+n+m+1}).$$

If $T(x) = T_{nm-l+r}x^{nm-l+r} + \ldots$ with $D(T_{nm-l+r}) \neq \emptyset$ due to (6) then also

$$T_{nm-l+r}x^{nm-l+r}\cdot[F(x)Q_*(x) - P_*(x)] = 0(x^{nm+n+m+1})$$

because of the equivalence of (1) with (1a) and (1b).

Clearly $0 \leqslant r \leqslant \min(n - \partial_1 P_*, m - \partial_1 Q_*)$ because according to Lemma 1.2 (since $D(T) \neq \emptyset$):

$$\partial_1 P_* + l = \partial P_* \leqslant (nm + n) - (nm - l + r)$$

$$\partial_1 Q_* + l = \partial Q_* \leqslant (nm + m) - (nm - l + r)$$

and thus

$$r \leqslant n - \partial_1 P_*$$

$$r \leqslant m - \partial_1 Q_*.$$

We give some illustrative examples.

Let $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$(x, y) \to \begin{pmatrix} \dfrac{1}{1-x} \\ e^{x+y} \end{pmatrix}.$$

The (1,2)-APA is

$$\begin{pmatrix} \dfrac{1}{1-x} \\ \dfrac{1 + \frac{1}{3}(x+y)}{1 - \frac{2}{3}(x+y) + \frac{1}{6}(x+y)^2} \end{pmatrix}.$$

Theorem 2.1 holds with

$$T(x, y) = \begin{pmatrix} (1+x)\,L_2\begin{pmatrix} x \\ y \end{pmatrix}^2 + L_3\begin{pmatrix} x \\ y \end{pmatrix}^3 \\ \dfrac{(x+y)^2}{2} \end{pmatrix}$$

where $L_2 \in \mathrm{L}(X^2, Y)$ and $L_3 \in \mathrm{L}(X^3, Y)$. So if $D(L_2) \neq \emptyset$ then

$$\left(P_*\cdot\begin{pmatrix} L_2 \\ \dfrac{(x+y)^2}{2} \end{pmatrix},\ Q_*\cdot\begin{pmatrix} L_2 \\ \dfrac{(x+y)^2}{2} \end{pmatrix}\right)$$

is a solution of (1); $l = 0$, $nm = 2$, $s = 1$ and $r = 0$ since $\partial_1 P_* = 1$ and $\partial_1 Q_* = 2$.

Let $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$(x, y) \to \begin{pmatrix} 1 + \sin(x + xy) \\ 1 + \dfrac{x}{0.1 - y} + \sin(xy) \end{pmatrix}.$$

The (1,2)-ARA is

$$\begin{pmatrix} \dfrac{x - y + \frac{5}{6}x^2 - 2xy}{x - y - xy - \frac{x^2}{6} + xy^2 + \frac{x^3}{6}} \\ \dfrac{x - 1.01y + 10y^2 + 10x^2 - 20.2xy}{x - 1.01y + 10y^2 - 10.1xy + 2.01xy^2} \end{pmatrix}.$$

Theorem 2.1 holds with $T(x, y) = \begin{pmatrix} x \\ 100x \end{pmatrix}$. So $D(T_{t_0}) = D(T) \neq \emptyset$ and $l = 1$, $nm = 2$, $s = 0 = r$, $\partial_1 P_* = 1, \partial_1 Q_* = 2$.

Let $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$(x, y) \to \begin{pmatrix} 1 + \dfrac{y}{1 + y^2} \\ 1 - \cos x \end{pmatrix}.$$

The (2,1)-APA is

$$\begin{pmatrix} 1 + y \\ \dfrac{x^2}{2} \end{pmatrix}$$

where

$$T(x, y) = \begin{pmatrix} 0 \\ -\dfrac{x^2}{2} \end{pmatrix} + \begin{pmatrix} -y^3 \\ 0 \end{pmatrix}.$$

So $D(T_{t_0}) = \emptyset$. We can prove that no $T_2 \in L(X^2, Y)$ and no $T_3 \in L(X^3, Y)$ exist such that $(P_* \cdot T_2, Q_* \cdot T_2)$ or $(P_* \cdot T_3, Q_* \cdot T_3)$ is a solution of (1) with $D(T_2) \neq \emptyset$ or $D(T_3) \neq \emptyset$. Theorem 2.1 b) can by no means be satisfied. Nevertheless $l = 0$ $nm = 2$ and $s = 1$ because of Theorem 2.1 a).

3. COVARIANCE-PROPERTIES FOR ABSTRACT RATIONAL APPROXIMANTS

The proofs of the following properties are very similar to the ones given in [1] for the same covariance-properties of abstract Padé-approximants. But in the proofs mentioned here no use is made of Theorem 2.1.

We consider the (n, m)-APA to be a special case of the (n, m)-ARA.

THEOREM 3.1. *Suppose* $F(0)$ *is regular in* Y *and* $\frac{1}{Q_*} \cdot P_*$ *is the* (n, m)-ARA *for* F. *Then* $\frac{1}{P_*} \cdot Q_*$ *is the* (m, n)-ARA *for* $\frac{1}{F}$.

Proof. Since $F(0)$ is regular in Y there exists $B(0, r)$ such that $\frac{1}{F}$ is defined in $B(0, r)$. Since $\frac{1}{Q_*} \cdot P_*$ is the (n, m)-ARA for F there exists an abstract polynomial T with $D(T) \neq \emptyset$ such that $(P, Q) = (P_* \cdot T, Q_* \cdot T)$ satisfies (1) for F and provides the irreducible form $\frac{1}{Q_*} \cdot P_*$ of $\frac{1}{Q} \cdot P$. So $(F \cdot Q - P)(x) = 0(x^{nm+n+m+1})$. This implies that $\left(\frac{1}{F} \cdot P - Q\right) = 0(x^{nm+n+m+1})$ since $\frac{1}{F}$ is abstract analytic in the neighbourhood of 0. So $(Q, P) = (Q_* \cdot T, P_* \cdot T)$ satisfies (1) for $\frac{1}{F}$ $(D(Q) \neq \emptyset$ or $D(P) \neq \emptyset)$ and provides the irreducible form $\frac{1}{P_*} \cdot Q_*$ of $\frac{1}{P} \cdot Q$.

Remark the fact that if (6) holds, $F(0)$ being regular implies:

$Q_{*l}x^l \not\equiv 0 \Rightarrow P_{*l}x^l \not\equiv 0$;

$l > 0 \Rightarrow 0 \notin D(Q_*)$ and $0 \notin D(P_*)$;

$l = 0 \Rightarrow$ if Q_{*0} is not regular in Y then P_{*0} is not regular in Y, since $C_0 \cdot (T_{t_0} x^{t_0} \cdot Q_{*0}) = T_{t_0} x^{t_0} \cdot P_{*0}$ implies $C_0 Q_{*0} = P_{*0}$ $(t_0 \geqslant nm)$.

THEOREM 3.2. *Suppose* $a, b, c, d \in Y$, $c \cdot F(0) + d$ *is regular in* Y, $a \cdot d - b \cdot c$ *is regular in* Y, $\frac{1}{Q_*} \cdot P_*$ *is the* (n, n)-ARA *for* F *and* $D(c \cdot P + d \cdot Q) \neq \emptyset$ *or* $D(a \cdot P + b \cdot Q) \neq \emptyset$, *then* $\frac{1}{c \cdot P_* + d \cdot Q_*} \cdot (a \cdot P_* + b \cdot Q_*)$ *is the* (n, n)-ARA *for* $\frac{1}{c \cdot F + d} \cdot (a \cdot F + b)$.

Proof. Since $c \cdot F(0) + d$ is regular in Y there exists $B(0, r)$ such that

$\frac{1}{c\cdot F+d}\cdot(a\cdot F+b)$ is defined in $B(0, r)$. Since $\frac{1}{Q_*}\cdot P_*$ is the (n, n)-ARA for F there exists an abstract polynomial T with $D(T)\neq\emptyset$ such that $(P, Q)=(P_*\cdot T, Q_*\cdot T)$ satisfies (1) and provides the irreducible form $\frac{1}{Q_*}\cdot P_*$ of $\frac{1}{Q}\cdot P$. In other words

$$[(FQ_* - P_*)\cdot T](x) = (F\cdot Q - P)(x) = O(x^{n^2+2n+1}).$$

Now

$$\partial_0(a\cdot P + b\cdot Q) \geqslant n^2 \quad \text{since} \quad \partial_0 P \geqslant n^2 \text{ and } \partial_0 Q \geqslant n^2$$

$$\partial(a\cdot P + b\cdot Q) \leqslant n^2 + n \quad \text{since} \quad \partial P \leqslant n^2 + n \text{ and } \partial Q \leqslant n^2 + n.$$

Also

$$\partial_0(c\cdot P + d\cdot Q) \geqslant n^2$$

$$\partial(c\cdot P + d\cdot Q) \leqslant n^2 + n.$$

Since $(F\cdot Q - P)(x) = 0(x^{n^2+2n+1})$ and $c\cdot F(0) + d$ is regular

$$\left[(a\cdot d - b\cdot c)\cdot \frac{1}{c\cdot F + d}\cdot(F\cdot Q - P)\right](x) = 0(x^{n^2+2n+1}).$$

Now

$$\frac{1}{c\cdot F+d}\cdot(a\cdot F + b)\cdot(c\cdot P + d\cdot Q) - (a\cdot P + b\cdot Q) =$$

$$= \frac{1}{c\cdot F + d}\cdot(F\cdot Q - P)\cdot(a\cdot d - b\cdot c)$$

and

$$\left[(a\cdot d - b\cdot c)\cdot \frac{1}{c\cdot F + d}\cdot(F\cdot Q - P)\right](x) = 0(x^{n^2+2n+1}).$$

We now search the irreducible form of $\frac{1}{(c\cdot P_* + d\cdot Q_*)\cdot T}\cdot(a\cdot P_* + b\cdot Q_*)\cdot T.$

It is $\frac{1}{c\cdot P_* + d\cdot Q_*}\cdot(a\cdot P_* + b\cdot Q_*)$.

Remark the fact that if (6) holds, then $l > 0 \Rightarrow 0 \notin D(c\cdot P_* + d\cdot Q_*)$ since $c\cdot P_*(0) + d\cdot Q_*(0) = 0$.

THEOREM 3.3. *Let* $A \in L(X, X)$, A^{-1} *exists and* $\frac{1}{Q^*} \cdot P_*$ *be the* (n, m)-ARA *for* F. *If* $S_*(x) := Q_*(Ax)$, $R_*(x) := P_*(Ax)$, $G(x) := F(Ax)$, *then* $\frac{1}{S_*} \cdot R_*$ *is the* (n, m)-ARA *for* G.

Proof. We remark that if $L \in L(X^i, Y)$, then for every $A \in L(X, X)$ also $L \circ A \in L(X^i, Y)$ when defined by $(L \circ A)x^i = L(Ax)^i$ [6, p. 289]. Because $\frac{1}{Q_*} \cdot P_*$ is the (n, m)-ARA for F, there exists an abstract polynomial T with $D(T) \neq \emptyset$ such that $(P, Q) = (P_* \cdot T, Q_* \cdot T)$ satisfies (1) and provides the irreducible form $\frac{1}{Q_*} \cdot P_*$ of $\frac{1}{Q} \cdot P$. So there exists $K \in \mathbf{R}_0^+$ such that

$$\|(F \cdot Q - P)(x)\| \leqslant K \cdot \|x\|^{nm+n+m+1}.$$

Let

$$S(x) := S_*(x) \cdot T(Ax)$$

$$R(x) := R_*(x) \cdot T(Ax).$$

This implies

$$\|(G \cdot S - R)(x)\| = \|(F \cdot Q - P)(Ax)\| \leqslant K \cdot \|Ax\|^{nm+n+m+1} \leqslant$$

$$\leqslant K \cdot (\|A\| \cdot \|x\|)^{nm+n+m+1}.$$

And thus

$$(G \cdot S - R)(x) = 0(x^{nm+n+m+1}).$$

Since

$$D(S) = \{x \in X \mid Q(Ax) \text{ regular in } Y\} = A^{-1}(D(Q)) = \{A^{-1}x \mid x \in D(Q)\}$$

$$D(R) = A^{-1}(D(P))$$

$$\frac{1}{S_*} \cdot R_* \text{ is the irreducible form of } \frac{1}{S} \cdot R$$

the proof is completed.

Remark that if $0 \notin D(Q_*)$ then $0 \notin D(S_*)$, since $S_*(0) = Q_*(0)$.

4. BLOCK-STRUCTURE OF THE COMPLETED ABSTRACT PADÉ-TABLE

Let $R_{n,m}$ denote the (n, m)-APA for F if it is not undefined. The $R_{n,m}$ can be ordered for different values of n and m in a table:

$$\begin{matrix} R_{0,0} & R_{0,1} & R_{0,2} & \cdots \\ R_{1,0} & R_{1,1} & R_{1,2} & \cdots \\ R_{2,0} & R_{2,1} & \cdots & \\ R_{3,0} & \vdots & & \\ \vdots & & & \end{matrix}$$

Gaps can occur in this table because of undefined elements. Let us now fill up these gaps with the (n, m)-ARA. We call the new table the "*completed*" *abstract Padé-table*. Gaps can still occur if for certain (n, m) all solutions (P, Q) are such that $D(P) = \emptyset = D(Q)$. For $F: \mathbf{R}^p \to \mathbf{R}^q$ e.g. the completed table has no gaps any more.

From now on $R_{n,m}$ denotes the $(n, m)^{\text{th}}$ element of the completed abstract Padé-table.

Let us repeat some results about the $R_{n,m}$.

SUMMARY. *For every non-negative integers n and m the systems* (1a) *and* (1b) *are solvable. The abstract Padé-approximant of order* (n, m) *is unique and the abstract rational approximant of order* (n, m) *is unique up to a multiplicative regular element of* Y.

For $R_{n,m} = \frac{1}{Q_*} \cdot P_*$ *we know that* P_* *and* Q_* *are abstract polynomials, respectively of pseudo-degree at most n and at most m (when* (5) *or* (6) *holds for the* (n, m)-ARA).

The usual abstract Padé-table as well as the completed abstract Padé-table consist of squares of equal elements under the following condition.

Let (P, Q) be a solution of (1). Let $\frac{1}{Q_*} \cdot P_*$ be the (n, m)-ARA and $T(x) = \sum_{k=t_0}^{\partial T} T_k x^k$ such that

$$P_* \cdot T = P,$$

$$Q_* \cdot T = Q.$$

We need a solution (P, Q) where (6) is satisfied to be able to proof the block-structure of the Padé-table (this property has been proved for the usual table in [1]). For every (n, m) where this condition is not satisfied the block-structure may be disturbed. An interesting example of this phenomenon is given in [3] for the usual abstract Padé-table.

We now prove the block structure of the completed table.

LEMMA 4.1. *Take* $x \in X \setminus \{0\}$; $\forall n \in \mathbf{N}$, *there exists* $D_n \in \mathrm{L}(X^n, Y)$ *such that* $D_n x^n = I$.

The proof is given in [3].

This lemma implies that for every $n \in \mathbf{N}$, for x chosen in $X \setminus \{0\}$, there exists $D_n \in \mathrm{L}(X^n, Y)$ such that $x \in D(D_n)$. We shall use this result frequently.

THEOREM 4.1. *Let* $\frac{1}{Q_*} \cdot P_* = R_{n,m}$ *for* F. *Let* (6) *be satisfied and let* $l \leqslant \partial_1 P_* \cdot \partial_1 Q_*$. *Then:*

a) $$(F \cdot Q_* - P_*)(x) = \sum_{i=0}^{\infty} J_i x^{l+\partial_1 P_* + \partial_1 Q_* + t + i + 1}$$

with $J_i \in \mathrm{L}(X^{l+\partial_1 P_* + \partial_1 Q_* + t + i + 1}, Y)$, $t \geqslant 0$ *and* $J_0 x^{l+\partial_1 P_* + \partial_1 Q_* + t + 1} \not\equiv 0$;

b) $$n \leqslant \partial_1 P_* + t$$
$$m \leqslant \partial_1 Q_* + t\,;$$

c) *for all integers* i, j *satisfying*

$$\partial_1 P_* \leqslant i \leqslant \partial_1 P_* + t$$
$$\partial_1 Q_* \leqslant j \leqslant \partial_1 Q_* + t$$

we have $R_{i,j} = R_{n,m}$.

Proof. a) Suppose $\partial_0(FQ_* - P_*) = j$ with $j < l + \partial_1 P_* + \partial_1 Q_* + 1$. Then for every $r \in \mathbf{N}$, $0 \leqslant r \leqslant \min(n - \partial_1 P_*, m - \partial_1 Q_*)$:

$$l + \partial_1 P_* + \partial_1 Q_* + 1 + (nm - l + r) > j + (nm - l + r).$$

This is in contradiction with Theorem 2.1 since $\partial_1 P_* \leqslant n$ and $\partial_1 Q_* \leqslant m$, because of (7) and (6), and $l + \partial_1 P_* + \partial_1 Q_* + 1 + (nm - l + r) \leqslant nm + n + m + 1$.

b) Suppose $n > \partial_1 P_* + t$ or $m > \partial_1 Q_* + t$. Then for every $r \in \mathbf{N}$, $0 \leqslant r \leqslant \min(n - \partial_1 P_*, m - \partial_1 Q_*)$ and for every $T_{nm-l+r} \in \mathrm{L}(X^{nm-l+r}, Y)$ with $D(T_{nm-l+r}) \neq \emptyset$, we know that $(F \cdot Q_* \cdot T_{nm-l+r} - P_* \cdot T_{nm-l+r})(x)$ is not $0(x^{nm+n+m+1})$ since $(FQ_* - P_*)(x) = \sum_{i=0}^{\infty} J_i x^{l+\partial_1 P_* + \partial_1 Q_* + t + i + 1}$ with $J_0 x^{l+\partial_1 P_* + \partial_1 Q_* + t + 1} \not\equiv 0$ and $(nm - l + r) + (l + \partial_1 P_* + \partial_1 Q_* + t + 1) < nm + n + m + 1$.

This is in contradiction with Theorem 2.1.

c) Let $s = \min(i - \partial_1 P_*, j - \partial_1 Q_*)$. So $s \geqslant 0$ and since $l \leqslant \partial_1 P_* \cdot \partial_1 Q_*$ it follows that $i \cdot j + s - l \geqslant 0$. Take $D_s \in \mathrm{L}(X^{i \cdot j + s - l}, Y)$ with $D(D_s) \cap D(P_*) \neq \emptyset$ or $D(D_s) \cap D(Q_*) \neq \emptyset$, which is possible because of Lemma 4.1.

For $P_1 = P_* \cdot D_s$ and $Q_1 = Q_* \cdot D_s$

$$\partial P_1 \leqslant (\partial_1 P_* + l) + (i \cdot j + s - l) \leqslant i \cdot j + i$$
$$\partial Q_1 \leqslant (\partial_1 Q_* + l) + (i \cdot j + s - l) \leqslant i \cdot j + j$$

and

$$\partial_0 P_1 \geqslant l + (i \cdot j + s - l) \geqslant i \cdot j$$
$$\partial_0 Q_1 \geqslant l + (i \cdot j + s - l) \geqslant i \cdot j$$

and $(F\cdot Q_1 - P_1)(x) = 0(x^{\partial_1 P_* + \partial_1 Q_* + t + 1 + s + i\cdot j})$ because of a). For $i \leqslant \partial_1 P_* + t$ and $j \leqslant \partial_1 Q_* + t$ we know that $i\cdot j + i + j + 1 \leqslant i\cdot j + \partial_1 P_* + \partial_1 Q_* + t + s + 1$. So $(F\cdot Q_1 - P_1)(x) = 0(x^{i\cdot j + i + j + 1})$ and $D(P_1) \neq \emptyset$ or $D(Q_1) \neq \emptyset$.

Remark the fact that if one element of a square in the completed abstract Padé-table is an APA, then all the elements of the same square are, and if one element is an ARA then all the elements of the same square are.

A lot of the following properties are a generalization for the "completed" abstract Padé-table of theorems mentioned in [2].

DEFINITION 4.1. The $R_{n,m}$ for F is called *regular* if $(F\cdot Q_* - P_*)(x) = 0(x^{l+n+m+1+t})$ with $t \geqslant 0$.

DEFINITION 4.2. The $R_{n,m}$ for F is called *normal* if it occurs only once in the completed abstract Padé-table. The completed abstract Padé-table is called *normal* if each of its elements is normal.

The following theorem makes clear that under the assumption of (6) normality is stronger than regularity.

THEOREM 4.2. *Let* (6) *be satisfied and let* $l \leqslant \partial_1 P_* \cdot \partial_1 Q_*$. *The* $R_{n,m}$ *for* F *is normal if and only if*

a) $\partial_1 P_* = n$ *and* $\partial_1 Q_* = m$,

and

b) $(FQ_* - P_*)(x) = \sum_{i=0}^{\infty} J_i x^{n+m+l+i+1}$ *with* $J_i \in L(X^{n+m+l+i+1}, Y)$ *and* $J_0 x^{n+m+l+1} \not\equiv 0$.

Proof. "$\Rightarrow$"

Since $R_{n,m}$ is normal, $t = 0$ in Theorem 4.1c). And then according to Theorem 4.1b)

$$n \leqslant \partial_1 P_*$$

$$m \leqslant \partial_1 Q_*.$$

Already

$$\partial_1 P_* \leqslant n$$

$$\partial_1 Q_* \leqslant m$$

because of (7) and (6).

So $\partial_1 P_* = n$ and $\partial_1 Q_* = m$. Theorem 4.1a) then implies

$$(FQ_* - P_*)(x) = \sum_{i=0}^{\infty} J_i x^{l+n+m+i+1}$$

with $J_0 x^{l+n+m+1} \not\equiv 0$.

"$\Leftarrow$"

The proof goes by contraposition. Suppose there exist integers i, j with $i > n$ or $j > m$ and such that $R_{i,j} = R_{n,m}$ (in any case $n = \partial_1 P_* \leq i$ and $m = \partial_1 Q_* \leq j$).

Now b) implies that $(FQ_* - P_*)(x) = 0(x^{l+n+m+1})$. If $s \in \mathbf{N}$, $D_s \in L(X^{i \cdot j+s-l}, Y)$, $D(D_s) \cap D(P_*) \neq \emptyset$ or $D(D_s) \cap D(Q_*) \neq \emptyset$ such that $[(F \cdot Q_* - P_*)D_s](x) = 0(x^{i \cdot j+i+j+1})$ then $i \cdot j + i + j + 1 \leq n + m + l + 1 + (i \cdot j + s - l)$ and thus $s > i - n$ or $s > j - m$.

This is a contradiction with Theorem 2.1.

THEOREM 4.3. *Let* $\frac{1}{Q_*} \cdot P_* = R_{n,m}$ *for* F, *let* (6) *be valid and let* $l \leq \partial_1 P_* \cdot \partial_1 Q_*$. *Then* $\frac{1}{Q_*} \cdot P_*$ *is normal if and only if for every solution*

$$(P, Q) = (P_* \cdot T, Q_* \cdot T) = \left(\sum_{i=0}^{n} A_{nm+i} x^{nm+i}, \sum_{j=0}^{m} B_{nm+j} x^{nm+j} \right)$$

of (1) *with* $D(T) \neq \emptyset$:

a) $\partial_0(F \cdot Q - P) = nm + n + m + 1$,

b) $B_{nm} x^{nm} \not\equiv 0$,

c) $B_{nm+m} x^{nm+m} \not\equiv 0$,

d) $A_{nm+n} x^{nm+m} \not\equiv 0$.

Proof. "$\Rightarrow$"

If $\frac{1}{Q_*} \cdot P_*$ is normal then for every solution $(P, Q) = (P_* \cdot T, Q_* \cdot T)$ with $D(T) \neq \emptyset$, $\partial_0 T = t_0 = nm - l$ because for $t_0 > nm - l$ we would have $\partial_1 Q_* = \partial Q_* - l \leq \partial(Q_* \cdot T) - t_0 - l < m$ and analogously $\partial_1 P_* < n$ which contradicts Theorem 4.2 a).

Since $D(T_{t_0}) = D(T_{nm-l}) \neq \emptyset$ and since $B_{nm} x^{nm} = Q_{*l} x^l \cdot T_{nm-l} x^{nm-l}$ with $Q_{*l} x^l \not\equiv 0$ we have $B_{nm} x^{nm} \not\equiv 0$. Suppose there exists a solution (P, Q) such that

$$A_{nm+n} x^{nm+n} \equiv 0,$$

or

$$B_{nm+m} x^{nm+m} \equiv 0;$$

then

$$\partial P_* \leq \partial P - (nm - l) < n + l$$

or

$$\partial Q_* \leq \partial Q - (nm - l) < m + l$$

and so $\partial_1 P_* < n$ or $\partial_1 Q_* < m$ which contradicts Theorem 4.2 a). In $(F \cdot Q - P)(x)$ the term of lowest degree equals $T_{nm-l} x^{nm-l} \cdot$[term of lowest degree in $(F \cdot Q_* - P_*)(x)$]. According to Theorem 4.2 b) then $\partial_0(F \cdot Q - P) = nm + n + m + 1$.

"$\Leftarrow$"

For a solution (P, Q) we know that $\partial_0 Q = nm$ and so $\partial_0 T = \partial_0 Q - l = nm - l$ because of (6). Now $(P_* \cdot T_{nm-l}, Q_* \cdot T_{nm-l})$ is also a solution of (1) according to Theorem 2.1. Thus

$$\partial P_* = nm + n - (nm - l)$$

$$\partial Q_* = nm + m - (nm - l)$$

and so

$$\partial_1 P_* = n$$

$$\partial_1 Q_* = m.$$

Also $\partial_0(F \cdot Q_* - P_*) = n + m + l + 1$ because $\partial_0[(F \cdot Q_* - P_*) \cdot T_{nm-l}] = nm + n + m + 1$.

We illustrate the preceding theorems by some examples.

Let F: $\mathbf{R}^2 \to \mathbf{R}$ defined by

$$(x, y) \to 1 + \frac{x}{0.1 - y} + \sin(xy) = 1 + 10x + 101xy +$$

$$+ \sum_{k=3}^{\infty} 10^k xy^{k-1} + \sum_{k=1}^{\infty} (-1)^k \frac{(xy)^{2k+1}}{(2k+1)!}.$$

Clearly $D(T_{t_0}) \neq \emptyset$ for all $R_{n,m}$.

The (1,1)-APA is

$$\frac{1 + 10x - 10.1y}{1 - 10.1y}$$

and

$$(F \cdot Q_* - P_*)(x, y) = 0(xy^2) = 0((x, y)^3) \text{ exactly,}$$

$$l = 0,$$

$$\partial_1 P_* = 1,$$

$$\partial_1 Q_* = 1.$$

So it is a normal element in the abstract Padé-table.

The (3,1)-APA is

$$\frac{1+10x-10y+xy-10xy^2}{1-10y}$$

and

$$(F\cdot Q_* - P_*)(x,y) = 0(x^3y^3) = 0((x,y)^6) \text{ exactly,}$$

$$l=0,$$

$$\partial_1 P_* = 3,$$

$$\partial_1 Q_* = 1.$$

So it is a regular element in the abstract Padé-table and we have the following square of equal elements: $R_{3,1} = R_{3,2} = R_{4,1} = R_{4,2}$.

The (1,2)-ARA is

$$\frac{x-1.01y+10y^2+10x^2-20.2xy}{x-1.01y+10y^2-10.1xy+2.01xy^2}$$

and

$$(F\cdot Q_* - P_*)(x,y) = 0(x^2y^3, xy^4) = 0((x,y)^5) \text{ exactly,}$$

$$l=1,$$

$$\partial_1 P_* = 1,$$

$$\partial_1 Q_* = 2.$$

So it is a normal element in the completed abstract Padé-table.

The (3,3)-ARA is

$$\frac{y+\frac{201}{6}\cdot 10^{-5}x^2+10y(x-y)+xy^2(1-10y)+\frac{1}{600}x^2(2.01x-y)+\frac{1}{60}x^2y(1.0301x-y)}{y+\frac{201}{6}10^{-5}x^2-10y^2-\frac{1}{600}x^2y-\frac{1}{60}x^2y^2}$$

and

$$(F\cdot Q_* - P_*)(x,y) = 0(x^5y^3) = 0((x,y)^8) \text{ exactly,}$$

$$l=1,$$

$$\partial_1 P_* = 3,$$

$$\partial_1 Q_* = 3.$$

It is a normal element in the completed abstract Padé-table.

5. INTERPOLATION-PROPERTY AND PRODUCT-PROPERTY

Theorem 4.1 a) and 4.2 b) allow us to write down the following conclusions. If $\frac{1}{Q_*} \cdot P_*$ is the $R_{n,m}$ for F then $(F \cdot Q_* - P_*)(x) = 0(x^{l+\partial_1 P_* + \partial_1 Q_* + t+1})$ exactly, with $t \geqslant 0$, under the assumption of (6). In other words $(F \cdot Q_* - P_*)^{(i)}(0) \equiv \equiv 0 \in \mathrm{L}(X^i, Y)$ for $i = 0, \ldots, l + \partial_1 P_* + \partial_1 Q_* + t$. For polynomials P_* and Q_* with $\partial_0 P_* \geqslant l$ and $\partial_0 Q_* = l$ we know that $(F \cdot Q_* - P_*)^{(i)}(0) \equiv 0$ for $i = = 0, \ldots, l-1$ always. So the meaningful relations are:

(8) $(F \cdot Q_* - P_*)^{(i)}(0) \equiv 0 \in \mathrm{L}(X^i, Y) \quad$ for $i = l, \ldots, l + \partial_1 P_* + \partial_1 Q_* + t.$

When $0 \in D(Q_*)$ $\left(\frac{1}{Q_*} \cdot P_*\right.$ is then the APA$\left.\right)$ the relations can be rewritten as:

$$F^{(i)}(0) = \left(\frac{1}{Q_*} \cdot P_*\right)^{(i)}(0) \quad \text{for } i = l, \ldots, l + \partial_1 P_* + \partial_1 Q_* + t$$

$(l = 0, \partial_1 P_* = \partial P_*, \partial_1 Q_* = \partial Q_*)$.

The relations (8) clearly have an interpolatory meaning in 0. This explains the term *one-point* abstract rational approximant.

Let X be a Banach space and Y_i commutative Banach algebras. We consider non-linear operators $F_i \colon X \to Y_i$, $i = 1, \ldots, q < \infty$ and

$$F \colon X \to \prod_{i=1}^{q} Y_i, \quad x \to (F_i(x),\ i = 1, \ldots, q)$$

where $\prod_{i=1}^{q} Y_i$ is a commutative Banach algebra with componentwise multiplication and normed by one of the Minkowski-norms $\|(y_1, \ldots, y_q)\|_p$, $1 \leqslant p \leqslant \infty$:

$$\|(y_1, \ldots, y_q)\|_1 = \sum_{i=1}^{q} \|y_i\|_{(i)}$$

$$\|(y_1, \ldots, y_q)\|_p = \left(\sum_{i=1}^{q} \|y_i\|_{(i)}^p\right)^{1/p} \qquad 1 < p < \infty$$

$$\|(y_1, \ldots, y_q)\|_\infty = \max(\|y_1\|_{(1)}, \ldots, \|y_q\|_{(q)})$$

where $\|y_i\|_{(i)}$ is the norm of y_i in Y_i.

THEOREM 5.1. *Let* $\bigcap_{i=1}^{q} D(Q_i) \neq \emptyset$ *or* $\bigcap_{i=1}^{q} D(P_i) \neq \emptyset$ *for the solutions* (P_i, Q_i) *of* (1) *for* F_i. *Then*

$$\frac{1}{Q_{*i}} \cdot P_{*i} = R_{n,m} \quad \textit{for } F_i,\ i = 1, \ldots, q$$

if and only if

$$\frac{1}{Q_*} \cdot P_* = \left(\frac{1}{Q_{*i}} \cdot P_{*i},\ i = 1, \ldots, q\right) = R_{n,m} \quad \textit{for } F.$$

Proof. "$\Rightarrow$"

Since $\frac{1}{Q_{*i}} \cdot P_{*i} = R_{n,m}$ for F_i, there exist abstract polynomials T_i with $\bigcap_{i=1}^{q} D(T_i) \neq \emptyset$ such that $(P_i, Q_i) = (P_{*i} \cdot T_i, Q_{*i} \cdot T_i)$ satisfies (1) and provides the irreducible form $\frac{1}{Q_{*i}} \cdot P_{*i}$ of $\frac{1}{Q_i} \cdot P_i$. So there exist $K_i \in \mathbf{R}_0^+$ such that

$$\|[(F \cdot Q_{*i} - P_{*i}) \cdot T_i](x)\| \leqslant K_i \cdot \|x\|^{nm+n+m+1} \qquad \text{for } i = 1, \ldots, q.$$

We use the Minkowski-norm $\| \ \|_p$ in $\prod_{i=1}^{q} Y_i$ for some p with $1 \leqslant p \leqslant \infty$. Then

for $p = 1$, let $K = \sum_{i=1}^{q} K_i$,

for $p = \infty$, let $K = \max_i K_i$,

for $1 < p < \infty$, let $K = (\sum_i K_i^p)^{1/p}$,

and we find

$$\|([(F \cdot Q_{*i} - P_{*i}) \cdot T_i]\,(x),\ i = 1, \ldots, q)\| \leqslant K \cdot \|x\|^{nm+n+m+1}.$$

Thus $((P_{*i} \cdot T_i, Q_{*i} \cdot T_i),\ i = 1, \ldots, q)$ satisfies (1) for F.

Now the irreducible form of $\left(\frac{1}{Q_{*i} \cdot T_i} \cdot P_{*i} \cdot T_i, i = 1, \ldots, q\right)$ is $\left(\frac{1}{Q_{*i}} \cdot P_{*i}, i = 1, \ldots, q\right)$ since

$$Q = (Q_{*i} \cdot T_i,\ i = 1, \ldots, q) = (Q_{*i},\ i = 1, \ldots, q) \cdot T(_i,\ i = 1, \ldots, q)$$

$$P = (P_{*i} \cdot T_i,\ i = 1, \ldots, q) = (P_{*i}, i = 1, \ldots, q) \cdot (T_i,\ i = 1, \ldots, q)$$

with $D(Q) \neq \emptyset$ or $D(P) \neq \emptyset$.

"$\Leftarrow$"

Since $\frac{1}{Q_*} \cdot P_* = R_{n,m}$ for F there exists an abstract polynomial T with $D(T) \neq \emptyset$ such that $[(F \cdot Q_* - P_*) \cdot T](x) = 0(x^{nm+n+m+1})$. We write $(T)_i$ for the i^{th} operator-component of T. We know that

$$\|[(F_i \cdot Q_{*i} - P_{*i}) \cdot (T)_i](x)\| \leqslant \|[(F \cdot Q_* - P_*) \cdot T](x)\|$$

for $i = 1, \ldots, q$ and for whatever Minkowski-norm used in $\prod_{i=1}^{q} Y_i$. So $(P_i, Q_i) =$

$= (P_{*i}\cdot(T)_i, Q_{*i}\cdot(T)_i)$ satisfies (1) for F_i, and $D(P_i) \neq \emptyset$ or $D(Q_i) \neq \emptyset$ since $D(P_i) \supseteq \bigcap_{i=1}^{q} D(P_i)$ and $D(Q_i) \supseteq \bigcap_{i=1}^{q} D(Q_i)$, and the irreducible form of $\frac{1}{Q_{*i}\cdot(T)_i} \cdot P_{*i}\cdot(T_i)$ is $\frac{1}{Q_{*i}} \cdot P_{*i}$.

Remark the fact that if $\bigcap_{i=1}^{q} D(Q_i) = \emptyset$ and $\bigcap_{i=1}^{q} D(P_i) = \emptyset$, we cannot find $x \in X$ where the q solutions (P_i, Q_i) of (1) for F can be used simultaneously. It is useless then to consider $(P, Q) = ((P_i, Q_i),\ i = 1, \ldots, q)$ since $D(P) = \emptyset = D(Q)$.

It is very well possible that for certain i, $1 \leqslant i \leqslant q$, $\frac{1}{Q_{*i}} \cdot P_{*i}$ is the (n, m)-APA while for other i, $0 \notin D(Q_{*i})$. We now give an example that illustrates this remark.

Let

$$F_1 : \mathbf{R}^2 \to \mathbf{R} \text{ defined by } (x, y) \to 1 + \frac{x}{0.1 - y} + \sin(xy)$$

and

$$F_2 : \mathbf{R}^2 \to \mathbf{R} \text{ defined by } (x, y) \to \frac{xe^x - ye^y}{x - y}.$$

Take $n = m = 1$. Then $\frac{1}{Q_{*1}} \cdot P_{*1} : \mathbf{R}^2 \to \mathbf{R}$ is $(x, y) \to \frac{1 + 10x - 10.1y}{1 - 10.1y}$ (APA)

and $\frac{1}{Q_{*2}} \cdot P_{*2} : \mathbf{R}^2 \to \mathbf{R}$ is $(x, y) \to \frac{x + y + 0.5(x^2 + 3xy + y^2)}{x + y - 0.5(x^2 + xy + y^2)}$ (ARA).

Now

$$\frac{1}{\begin{pmatrix} Q_{*1} \\ Q_{*2} \end{pmatrix}} \cdot \begin{pmatrix} P_{*1} \\ P_{*2} \end{pmatrix} = \begin{pmatrix} \frac{1}{Q_{*1}} \cdot P_{*1} \\ \frac{1}{Q_{*2}} \cdot P_{*2} \end{pmatrix}$$

is the (1,1)-ARA for

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} : \mathbf{R}^2 \to \mathbf{R}^2 \text{ defined by } (x, y) \to \begin{pmatrix} 1 + \frac{x}{0.1 - y} + \sin(xy) \\ \frac{xe^x - ye^y}{x - y} \end{pmatrix}$$

and

$$D(Q_1) \cap D(Q_2) = \mathbf{R}^2 \setminus \left\{ (x,y) \mid x = 0 \text{ or } y = \frac{10}{101} \text{ or } -x^2 + (x+y)(2-y) = 0 \right\} \neq \emptyset.$$

We have to remark that the restrictive conditions formulated in each preceding theorem are always fulfilled in the classical theory of Padé-approximants for a function $f: \mathbf{R} \to \mathbf{R}$.

The classical properties of Padé-approximants are found back when the theorems are applied to the case $X = \mathbf{R} = Y$.

The author is "Aspirant van het Nationaal Fonds voor Wetenschappelijk Onderzoek".

REFERENCES

1. Cuyt, Annie A. M., Abstract Padé-approximants in operator theory, *Springer Lecture Notes in Mathematics*, **765** (1979), 61–87.
2. Cuyt, Annie A. M., Regularity and normality of abstract Padé-approximants. Projection-property and product-property, *Report* 79–32, University of Antwerp, October 1979.
3. Cuyt, Annie A. M., Note on the properties of abstract Padé-approximants, *Report* 80–03, University of Antwerp, February 1980.
4. Mac Lane, S.; Birkhoff, G., *Algebra*, Collier-Macmillan Ltd., London, 1967.
5. Rall, Louis B., *Computational solution of nonlinear operator equations*, John Wiley and Sons Inc., New York, 1969.
6. Tasche, M.; Bögel, K., *Analysis in normierten Räumen*, Akademie Verlag, Berlin, 1974.

ANNIE A. M. CUYT
University of Antwerp, U.I.A.,
Department of Mathematics,
Universiteitsplein 1,
B–2610 Wilrijk,
Belgium.

Received April 3, 1980; revised April 6, 1981.