PADE APPROXIMATION IN ONE AND MORE VARIABLES.

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Abstract.

We first recall results from univariate Padé approximation theory (UPA). The recursive ϵ-algorithm and the continued fraction representation obtained from the qd-algorithm are given for the normal case as well as for a non-normal table composed of square blocks. Convergence of UPA for meromorphic functions and continuity of the univariate Padé operator are discussed.

The same approximation problem is considered in the multivariate case. General order multivariate Padé approximants (MPA) are defined and a recursive computation scheme and a continued fraction representation are given, both for the normal case and for the case of a table of MPA with degenerate solutions. A de Montessus de Ballore convergence theorem is presented and the continuity of the multivariate Padé operator is considered.

1. Notations and definitions for UPA.

Consider a formal power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots \tag{1}$$

with $c_0 \neq 0$. In the sequel of the text we shall write ∂p for the exact degree of a polynomial p and ωp for the order of a power series p (i.e. the degree of the first nonzero term). The Padé approximation problem of order (n, m) for f consists in finding polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

and

$$q(x) = \sum_{i=0}^{m} b_i x^i$$

such that in the power series $(fq - p)(x)$ the coefficients of x^i for $i = 0, \dots, n + m$ disappear, in other words

$$\begin{aligned} &\partial p \leq n \\ &\partial q \leq m \\ &\omega(fq - p) \geq n + m + 1 \end{aligned} \tag{2}$$

S. P. Singh (ed.), Approximation Theory, Spline Functions and Applications, 41–68.

Condition (2) is equivalent with the following two linear systems of equations

$$\begin{cases} c_0 b_0 = a_0 \\ c_1 b_0 + c_0 b_1 = a_1 \\ \vdots \\ c_n b_0 + c_{n-1} b_1 + \ldots + c_{n-m} b_m = a_n \end{cases} \tag{3a}$$

$$\begin{cases} c_{n+1} b_0 + c_n b_1 + \ldots + c_{n-m+1} b_m = 0 \\ \vdots \\ c_{n+m} b_0 + c_{n+m-1} b_1 + \ldots + c_n b_m = 0 \end{cases} \tag{3b}$$

with $c_i = 0$ for $i < 0$. In general a solution for the coefficients a_i is known after substitution of a solution for the b_i in the left hand side of (3a). So the crucial point is to solve the homogeneous system (3b) of m equations in the $m + 1$ unknowns b_i. This system has at least one nontrivial solution because one of the unknowns can be chosen freely. Moreover all nontrivial solutions of (3) supply the same irreducible form. If $p(x)$ and $q(x)$ satisfy (3) we shall denote by $r_{n,m}(x) = (p_{n,m}/q_{n,m})(x)$ the irreducible form of p/q normalized such that $q_{n,m}(0) = 1$. This rational function $r_{n,m}(x)$ is called the Padé approximant of order (n, m) for f. By calculating the irreducible form, a polynomial may be cancelled in numerator and denominator of p/q. We shall therefore denote the exact degrees of $p_{n,m}$ and $q_{n,m}$ in $r_{n,m}$ respectively by n' and m'. Although $p_{n,m}$ and $q_{n,m}$ are computed from polynomials p and q that satisfy (2), it is not necessarily so that $p_{n,m}$ and $q_{n,m}$ satisfy (2) themselves. A simple example will illustrate this. Consider $f(x) = 1 + x^2$ and take $n = 1 = m$. A solution is given by $b_0 = 0 = a_0$ and $b_1 = 1 = a_1$. So $p(x) = x = q(x)$. Consequently $p_{n,m} = 1 = q_{n,m}$ with $\omega(fq_{n,m} - p_{n,m}) = 2 < n + m + 1$ and the corresponding equations (2) do not hold.

The Padé approximants $r_{n,m}$ for f can be ordered in a table for different values of n and m:

$$\begin{matrix} r_{0,0} & r_{0,1} & r_{0,2} & r_{0,3} & \cdots \\ r_{1,0} & r_{1,1} & r_{1,2} & r_{1,3} & \cdots \\ r_{2,0} & r_{2,1} & r_{2,2} & r_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{matrix}$$

This table is called the Padé table of f. The first column consists of the partial sums of f. The first row contains the reciprocals of the partial sums of $1/f$. We call a Padé approximant normal if it occurs only once in the Padé table. A criterion for the normality of an approximant is given in the next theorem.

THEOREM 1:
The Padé approximant $r_{n,m} = (p_{n,m}/q_{n,m})$ for f is normal if and only if the following three conditions are satisfied simultaneously:
(a) $n' = n$
(b) $m' = m$
(c) $\omega(fq_{n,m} - p_{n,m}) = n + m + 1$

Normality of a Padé approximant can also be guaranteed by the nonvanishing of certain determinants. We introduce the notation

$$D_{n,n+m} = \begin{vmatrix} c_n & c_{n-1} & \cdots & c_{n-m} \\ c_{n+1} & c_n & \cdots & c_{n-m+1} \\ \vdots & \ddots & & \vdots \\ c_{n+m} & c_{n+m-1} & \cdots & c_n \end{vmatrix}$$

The following result can be proved [PERR p. 243].

THEOREM 2:
The Padé approximant $r_{n,m} = (p_{n,m}/q_{n,m})$ for f is normal if and only if

$$D_{n,n+m-1} \neq 0$$
$$D_{n+1,n+m} \neq 0$$
$$D_{n,n+m} \neq 0$$
$$D_{n+1,n+m+1} \neq 0$$

2. Methods to compute normal Padé approximants.

In this section we suppose that every Padé approximant in the Padé table does at least itself satisfy condition (2). This is the case if for instance $\min(n-n', m-m') = 0$ for all n and m. A survey of algorithms for computing Padé approximants is given in [WUYT] and [BULT]. We discuss a limited number of them, mainly with the aim to generalize them to the multivariate case in the following sections.

2.1. Determinant formulas.

One can solve the system of equations (3b) explicitly and thus get a determinant representation for the Padé approximant. The following determinant formula for $q_{n,m}(x)$ can very easily be proved by solving (3b) using Cramer's rule after choosing $b_0 = D_{n,n+m-1}$. For $f(x) = \sum_{i=0}^{\infty} c_i x^i$ we write

$$F_k(x) = \sum_{i=0}^{k} c_i x^i$$

with $F_k(x) = 0$ for $k < 0$.

THEOREM 3:
If the Padé approximant of order (n, m) for f is given by $r_{n,m}(x) = (p_{n,m}/q_{n,m})(x)$ and if $D_{n,n+m-1} \neq 0$, then

$$p_{n,m}(x) = \frac{1}{D_{n,n+m-1}} \begin{vmatrix} F_n(x) & xF_{n-1}(x) & \cdots & x^m F_{n-m}(x) \\ c_{n+1} & c_n & \cdots & c_{n-m+1} \\ \vdots & & \ddots & \\ c_{n+m} & c_{n+m-1} & \cdots & c_n \end{vmatrix}$$

and

$$q_{n,m}(x) = \frac{1}{D_{n,n+m-1}} \begin{vmatrix} 1 & x & \cdots & x^m \\ c_{n+1} & c_n & \cdots & c_{n-m+1} \\ \vdots & & \ddots & \\ c_{n+m} & c_{n+m-1} & \cdots & c_n \end{vmatrix}$$

These determinant expressions are of course only useful for small values of n and m because the calculation of a determinant involves a lot of additions, multiplications and possible round-off. They merely exhibit closed form formulas for the solution.

2.2. The ϵ-algorithm.

Using these determinant representations for the Padé approximant, it can be proved that the elements in a normal Padé table satisfy the relationship [WYNN]

$$(r_{n,m+1} - r_{n,m})^{-1} + (r_{n,m} - r_{n-1,m})^{-1} = (r_{n+1,m} - r_{n,m})^{-1} + (r_{n,m} - r_{n,m-1})^{-1} \tag{4}$$

where we have defined

$$r_{n,-1} = \infty$$
$$r_{-1,m} = 0$$

The identity (4) is a star identity which relates

$$r_{n-1,m}(x) = N$$

$$r_{n,m-1}(x) = W \qquad r_{n,m}(x) = C \qquad r_{n,m+1}(x) = E$$

$$r_{n+1,m}(x) = S$$

and is often written as

$$(N - C)^{-1} + (S - C)^{-1} = (E - C)^{-1} + (W - C)^{-1}$$

If we introduce the following new notation for our Padé approximants

$$r_{n,m}(x) = \epsilon_{2m}^{(n-m)}$$

we obtain a table of ϵ-values where the subscript indicates a column and the superscript indicates a diagonal:

$$\begin{array}{cccc}
\epsilon_0^{(0)} & \epsilon_2^{(-1)} & \epsilon_4^{(-2)} & \dots \\
\epsilon_0^{(1)} & \epsilon_2^{(0)} & \epsilon_4^{(-1)} & \dots \\
\epsilon_0^{(2)} & \epsilon_2^{(1)} & \epsilon_4^{(0)} & \dots \\
\epsilon_0^{(3)} & \epsilon_2^{(2)} & \epsilon_4^{(1)} & \dots \\
\vdots & \vdots & \vdots &
\end{array}$$

The $\epsilon_0^{(n)}$ are the partial sums $F_n(x)$ of the Taylor series $f(x)$. Remark the fact that only even column-indices occur. The table can be completed with odd-numbered columns in the following way. We define elements

$$\epsilon_{2m+1}^{(n-m-1)} = \epsilon_{2m-1}^{(n-m)} + \frac{1}{\epsilon_{2m}^{(n-m)} - \epsilon_{2m}^{(n-m-1)}} \qquad n = 0,1,\dots \qquad m = 0,1,\dots \tag{5a}$$

From the star-identity (4) and with the aid of (5a) we can conclude by induction that also for the even-numbered columns [WYNN]

$$\epsilon_{2m}^{(n-m)} = \epsilon_{2m-2}^{(n-m+1)} + \frac{1}{\epsilon_{2m-1}^{(n-m+1)} - \epsilon_{2m-1}^{(n-m)}} \tag{5b}$$

The relations (5a) and (5b) are a means to calculate all the elements in the Padé table. This algorithm is very handy when one needs the value of a Padé approximant for a given x and one does not want to compute the coefficients of the Padé approximant explicitly. Computational difficulties can occur when the Padé table is not normal. Reformulations of the ϵ-algorithm in this case can be found in the next section.

2.3. The qd-algorithm.

Let us now consider the following sequence of elements on a descending staircase in the Padé table

$$T_k = \{r_{k,0}, r_{k+1,0}, r_{k+1,1}, r_{k+2,1}, \dots\} \qquad k \geq 0$$

and the following continued fraction

$$d_0 + d_1 x + \dots + d_k x^k + \frac{d_{k+1}x^{k+1}|}{|\ 1} + \frac{d_{k+2}x|}{|\ 1} + \frac{d_{k+3}x|}{|\ 1} + \dots \tag{6}$$

Theorem 4:
If every three consecutive elements in T_k are different, then a continued fraction of the form (6) exists with $d_{k+i} \neq 0$ for $i \geq 1$ and such that the n^{th} convergent equals the $(n+1)^{th}$ element of T_k.

In this way we are able to construct corresponding continued fractions for functions f analytic in the origin: if the n^{th} convergent of (6) equals the $(n+1)^{th}$ element of T_0 then (6) is the corresponding continued fraction to the power series (2). By continued fractions of the form (6) one can only compute Padé approximants below the main diagonal in the Padé table. For the right upper half of the table one can use the reciprocal covariance property of Padé approximants. We now turn to the problem of the calculation of the coefficients d_{k+i} in (6) starting from the coefficients c_i of f. Consider the continued fraction $g_k(x)$, which is of the form (6), and which is given by

$$c_0+\ldots+c_k\,x^k+\left.\frac{c_{k+1}x^{k+1}}{1}\right|+\left.\frac{-q_1^{(k+1)}x}{1}\right|+\left.\frac{-e_1^{(k+1)}x}{1}\right|+\left.\frac{-q_2^{(k+1)}x}{1}\right|+\left.\frac{-e_2^{(k+1)}x}{1}\right|+\ldots \tag{7}$$

If the coefficients $q_\ell^{(k+1)}$ and $e_\ell^{(k+1)}$ are computed in order to satisfy theorem 4 then the convergents of g_k equal the successive elements of T_k. If we calculate the even part of $g_k(x)$ we get

$$c_0+\ldots+c_k\,x^k+\left.\frac{c_{k+1}x^{k+1}}{1-q_1^{(k+1)}x}\right|+\left.\frac{-q_1^{(k+1)}e_1^{(k+1)}x^2}{1-(q_2^{(k+1)}+e_1^{(k+1)})x}\right|+\left.\frac{-q_2^{(k+1)}e_2^{(k+1)}x^2}{1-(q_3^{(k+1)}+e_2^{(k+1)})x}\right|+\ldots$$

If we calculate the odd part of $g_{k-1}(x)$ we get

$$c_0+\ldots+c_k\,x^k+\left.\frac{c_kq_1^{(k)}x^{k+1}}{1-(q_1^{(k)}+e_1^{(k)})x}\right|+\left.\frac{-e_1^{(k)}q_2^{(k)}x^2}{1-(q_2^{(k)}+e_2^{(k)})x}\right|+\left.\frac{-e_2^{(k)}q_3^{(k)}x^2}{1-(q_3^{(k)}+e_3^{(k)})x}\right|-\ldots$$

The even part of $g_k(x)$ and the odd part of $g_{k-1}(x)$ are two continued fractions which have the same convergents $r_{k,0}, r_{k+1,1}, r_{k+2,2}, \ldots$ and which also have the same form. Hence the partial numerators and denominators must be equal, and we obtain for $k \geq 1$ and $\ell \geq 1$ [RUTI]

$$e_0^{(k)} = 0$$

$$q_1^{(k)} = \frac{c_{k+1}}{c_k}$$

$$e_\ell^{(k)} = e_{\ell-1}^{(k+1)} + q_\ell^{(k+1)} - q_\ell^{(k)} \tag{8a}$$

$$q_{\ell+1}^{(k)} = q_\ell^{(k+1)}\frac{e_\ell^{(k+1)}}{e_\ell^{(k)}} \tag{8b}$$

The numbers $q_\ell^{(k)}$ and $e_\ell^{(k)}$ are usually arranged in a table, where the superscript (k) indicates a diagonal and the subscript ℓ indicates a column. This table is called the *qd*-table:

$$
\begin{array}{cccccc}
e_0^{(1)} & & & & & \\
 & q_1^{(1)} & & & & \\
e_0^{(2)} & & e_1^{(1)} & & & \\
 & q_1^{(2)} & & q_2^{(1)} & & \\
e_0^{(3)} & & e_1^{(2)} & & e_2^{(1)} & \\
 & q_1^{(3)} & & q_2^{(2)} & & \ddots \\
e_0^{(4)} & \vdots & e_1^{(3)} & \vdots & e_2^{(2)} & \\
\vdots & & \vdots & & \vdots &
\end{array}
$$

The formulas (8) can also be memorized as follows: $e_\ell^{(k)}$ is calculated such that in the following rhombus the sum of the two elements on the upper diagonal equals the sum of the two elements on the lower diagonal

$$
\begin{array}{ccccc}
 & & q_\ell^{(k)} & & \\
 & & & + & \\
e_{\ell-1}^{(k+1)} & & & & e_\ell^{(k)} \\
 & + & & & \\
 & & q_\ell^{(k+1)} & &
\end{array}
$$

and $q_{\ell+1}^{(k)}$ is computed such that in the next rhombus the product of the two elements on the upper diagonal equals the product of the two elements on the lower diagonal

$$
\begin{array}{ccccc}
 & & e_\ell^{(k)} & & \\
 & & & * & \\
q_\ell^{(k+1)} & & & & q_{\ell+1}^{(k)} \\
 & * & & & \\
 & & e_\ell^{(k+1)} & &
\end{array}
$$

Since the *qd*-algorithm computes the coefficients in (7), it can be used to compute the Padé approximants below the main diagonal in the Padé table. To calculate the Padé approximants in the right upper half of the table, the *qd*-algorithm itself can also be extended above the diagonal and the following results can be proved [HENR pp. 615-617]:

$$\begin{array}{ccccccccc}
 & q_1^{(0)} & & q_2^{(-1)} & & q_3^{(-2)} & \cdots \\
e_0^{(1)} & & e_1^{(0)} & & e_2^{(-1)} & & \\
 & q_1^{(1)} & & q_2^{(0)} & & q_3^{(-1)} & \cdots \\
e_0^{(2)} & & e_1^{(1)} & & e_2^{(0)} & & \\
 & q_1^{(2)} & & q_2^{(1)} & & q_3^{(0)} & \cdots \\
e_0^{(3)} & & e_1^{(2)} & & e_2^{(1)} & & \\
 & q_1^{(3)} & & q_2^{(2)} & & q_3^{(1)} & \cdots \\
e_0^{(4)} & \vdots & e_1^{(3)} & \vdots & e_2^{(2)} & \vdots & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Let $(1/f)(x) = w_0 + w_1 x + w_2 x^2 + \ldots$ and put

$$q_1^{(0)} = \frac{-w_1}{w_0}$$
$$e_1^{(0)} = \frac{w_2}{w_1}$$

and for $k \geq 1$

$$q_{k+1}^{(-k)} = 0$$
$$e_{k+1}^{(-k)} = \frac{w_{k+2}}{w_{k+1}}$$

If the elements in the extended *qd*-table are all calculated by the use of (8) using the above starting values, then the continued fraction $h_k(x)$ given by

$$\frac{1}{\lvert w_0 + \ldots + w_k x^k} + \frac{w_{k+1}x^{k+1}\rvert}{\lvert 1} + \frac{-e_{k+1}^{(-k)}x\rvert}{\lvert 1} + \frac{-q_{k+2}^{(-k)}x\rvert}{\lvert 1} + \frac{-e_{k+2}^{(-k)}x\rvert}{\lvert 1} + \frac{-q_{k+3}^{(-k)}x\rvert}{\lvert 1} + \ldots$$

supplies the Padé approximants on the staircase

$$U_k = \{r_{0,k}, r_{0,k+1}, r_{1,k+1}, r_{1,k+2}, r_{2,k+2}, \ldots\}$$

It is obvious that difficulties can arise if the division in (8b) cannot be performed by the fact that $e_\ell^{(k)} = 0$. This is the case if the Padé table is not normal since consecutive elements in T_k or U_k can then be equal. Reformulations of the *qd*-algorithm in this case are given in section 3. Other algorithms exist for the computation of Padé approximants in a row, column, diagonal, sawtooth or ascending staircase in the Padé table. We do not mention them here, but we refer to [BULTb], [LONG], [PIND] and [MCCA].

3. Block structure of the Padé table.

It is often the case that certain Padé approximants $r_{n,m}$ in the Padé table coincide. Then the table is not normal. In general the following result can be proved.

THEOREM 5:
Let the Padé approximant of order (n, m) for f be given by $r_{n,m} = p_{n,m}/q_{n,m}$. Let $n' = \partial p_{n,m}$ and $m' = \partial q_{n,m}$. Then
(a) $\omega(fq_{n,m} - p_{n,m}) = n' + m' + t + 1$ with $t \geq 0$
(b) for k and ℓ satisfying $n' \leq k \leq n' + t$ and $m' \leq \ell \leq m' + t : r_{k,\ell}(x) = r_{n,m}(x)$
(c) $n \leq n' + t$ and $m \leq m' + t$

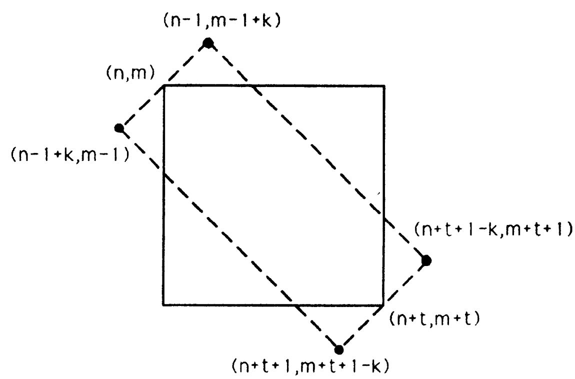
The previous property is called the block structure of the Padé table: the table consists of square blocks of size $t+1$ containing equal Padé approximants. Theorem 5c in fact says that the same Padé approximant does not reappear outside the block. In a non-normal Padé table the algorithms presented in the previous section do not hold anymore. However, singular rules to deal with the square blocks, have been developed.

3.1. The ϵ-algorithm.

The star-identity (4) can be extended to the non-normal case as follows [CORD].

THEOREM 6:
Let the Padé table for f contain a block of size $t + 1$ with corners $r_{n,m}$, $r_{n,m+t}$, $r_{n+t,m}$ and $r_{n+t,m+t}$. Then for $k = 1, 2, \ldots, t + 1$:

$$(r_{n-1,m-1+k} - r_{n,m})^{-1} + (r_{n+t+1,m+t+1-k} - r_{n,m})^{-1} = (r_{n-1+k,m-1} - r_{n,m})^{-1} + (r_{n+t+1-k,m+t+1} - r_{n,m})^{-1}$$

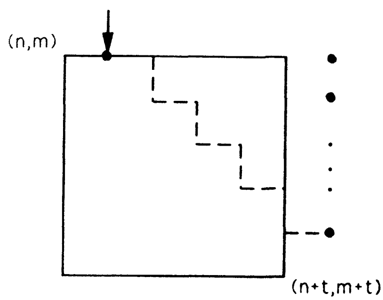


Starting from this generalized star-identity instead of from (4), it is possible to set up an algorithm generalizing the ϵ-algorithm, which makes it possible to compute the elements in a non-normal Padé table recursively.

3.2. The qd-algorithm.

Since several elements on a staircase in a non-normal Padé table might be equal, the usual representation (7) does not hold anymore. It is however possible to give other types of staircases, jumping over square blocks, whose elements can again be represented as the convergents of a continued fraction [CLAE]. Suppose there is a block of size $t+1$ having as corner elements $r_{n,m}$, $r_{n,m+t}$, $r_{n+t,m}$ and $r_{n+t,m+t}$. Consider for $k = 1, \ldots, t$ the perturbed staircase

$$\begin{aligned} T^*_{n-m-t+k-1} = \{ & r_{n-m-t+k-1,0}, r_{n-m-t+k,0}, \ldots, r_{n,m+t-k}, \\ & r_{n,m+t+1}, r_{n+1,m+t+1}, r_{n+2,m+t+1}, \ldots, r_{n+k+1,m+t-1}, r_{n+k+1,m+t}, \ldots \} \end{aligned}$$



and its corresponding continued fraction

$$\begin{aligned} g_{n-m-t+k-1}(x) = {} & c_0 + \ldots + c_s x^s + \frac{c_{s+1}x^{s+1}|}{|1} + \sum_{i=1}^{n-s-1} \left(\frac{-q_i^{(s+1)}x|}{|1} + \frac{-e_i^{(s+1)}x|}{|1} \right) + \\ & \frac{-v_{k,1}^{(t+1)}x^{k+1}|}{|1 - v_{k,k+1}^{(t+1)}x - \ldots - v_{k,2}^{(t+1)}x^k} + \frac{-v_{k,k+2}^{(t+1)}x|}{|1} + \sum_{i=1}^{k} \frac{-v_{k,k+i+2}^{(t+1)}x|}{|1 + v_{k,k+i+2}^{(t+1)}x} + \\ & \sum_{i=m+t+2}^{\infty} \left(\frac{-q_i^{(s+1)}x|}{|1} + \frac{-e_i^{(s+1)}x|}{|1} \right) \qquad s = n-m-t+k-1 \end{aligned}$$

This staircase passes above the block of equal elements in the Padé table and goes down column $m+t+1$ to recapture the old staircase. It is this vertical movement down column $m+t+1$ that introduces the v-values (v from "vertical"). Similar continued fractions can be constructed whose convergents are the elements on a special staircase passing below the block of equal elements and moving horizontally along row $n+t+1$ while introducing h-values. In general, when the Padé table consists of several blocks of different size each with equal elements, it remains possible using the same technique to construct continued fractions whose convergents are the elements of certain staircases in the Padé table. Let us now try to identify the new values $v_{k,i}^{(t+1)}$ for the continued fraction above.

THEOREM 7:
Let the Padé table for f contain a block of size $t+1$ with corners $r_{n,m}$, $r_{n,m+t}$, $r_{n+t,m}$ and $r_{n+t,m+t}$. Then the following relations hold:
(a)

$$\begin{aligned} v_{1,1}^{(t+1)} &= e_{m+t}^{(n-m-t)} q_{m+t+1}^{(n-m-t)} \\ v_{1,2}^{(t+1)} &= q_{m+t+1}^{(n-m-t)} \\ v_{1,3}^{(t+1)} &= e_{m+t+1}^{(n-m-t)} \\ v_{1,4}^{(t+1)} &= q_{m+t+2}^{(n-m-t)} \end{aligned}$$

(b) for $k>1$ and $i=2,3,\ldots,2k+1$:

$$\begin{aligned} v_{k,1}^{(t+1)} &= e_{m+t-k+1}^{(n-m-t+k-1)} v_{k-1,1}^{(t+1)} \\ v_{k,i}^{(t+1)} &= v_{k-1,i-1}^{(t+1)} \\ v_{k,2k+2}^{(t+1)} &= q_{m+t+2}^{(n-m-t+k-1)} \end{aligned}$$

(c)

$$q_{m+t+1}^{(n-m-t)} \prod_{i=1}^{t+1} e_{m+i-1}^{(n-m-i+1)} = e_m^{(n-m+t+1)} \prod_{i=1}^{t+1} q_m^{(n-m+i)}$$

(d)

$$q_{m+t+1}^{(n-m-t)} + e_{m+t+1}^{(n-m-t)} = e_m^{(n-m+t+1)} + q_{m+1}^{(n-m+t+1)}$$

(e) for $k=1,2,\ldots,t$:

$$q_{m+t+1}^{(n-m-t)} \prod_{i=1}^{k} e_{m+t-i+1}^{(n-m-t+i-1)} + e_{m+t+1}^{(n-m+t+2)} \prod_{i=1}^{k} q_{m+t+2}^{(n-m-t+i-1)} =$$

$$e_m^{(n-m+t+1)} \prod_{i=1}^{k} q_m^{(n-m+t-i+2)} + q_{m+1}^{(n-m+t+1)} \prod_{i=1}^{k} e_{m+i}^{(n-m+t-i+2)}$$

(f)

$$e_{m+t+1}^{(n-m+t+2)} \prod_{i=1}^{t+1} q_{m+t+2}^{(n-m-t+i-1)} = q_{m+1}^{(n-m+t+1)} \prod_{i=1}^{t+1} e_{m+i}^{(n-m+t-i+2)}$$

Using these rules it is possible to compute $q_{m+t}^{(n-m-t)}$, $e_{m+t}^{(n-m-t)}$ and $q_{m+t+2}^{(n-m-t+i-1)}$ for $i = 1, 2, \ldots, t+1$.

4. Convergence and continuity.

It is clear that the poles of the Padé approximants in a sequence of elements from the Padé table will play an important role when studying the convergence of that sequence. A lot of information on the convergence of Padé approximants can be found in [BAKE]. We restrict ourselves to the convergence problem of Padé approximants for meromorphic functions f [MONT].

THEOREM 8:
Let f be analytic in $B(0,R) = \{z \in \mathbb{C} : |z| < R\}$ except in the poles $z_1, \ldots, z_k$ of f with

$$0 < |z_1| \leq \ldots \leq |z_k| < R$$

Let the pole at z_i have multiplicity μ_i and let the total multiplicity be $m = \sum_{i=1}^{k} \mu_i$. Then

$$lim_{n\to\infty} q_{n,m}(z) = \prod_{i=1}^{k} \left(1 - \frac{z}{z_i}\right)^{\mu_i}$$

and the column $(r_{n,m})_{n\in\mathbb{N}}$ of the Padé table converges uniformly to f on every closed and bounded subset of $B(0,r) \setminus \{z_1, \ldots, z_k\}$.

For the proof we refer to Saff's short and elegant proof which can for instance be found in [BAKE, pp. 252–254].

When we compute $r_{n,m}$ in finite precision arithmetic the computed result is not exactly the (n,m) Padé approximant, but it differs slightly from it by rounding errors and data perturbations. Since we can consider the computed result as the exact (n,m) Padé approximant of a slightly perturbed input power series, it is important to study the effect of such small perturbations on the operator $\mathcal{P}_{n,m}$ that associates with f its (n,m) Padé approximant. Since n and m are fixed here, we can adopt the notations

$$\mathcal{P} := \mathcal{P}_{n,m}$$
$$\mathcal{P}f := r_{n,m}$$

To measure the small perturbations we introduce a pseudo-norm for formal power series:

$$||c||_{n+m} = \max_{0\leq i\leq n+m} |c_i|$$

with $c = (c_0, \ldots, c_{n+m})$, and the supremum norm for continuous functions on an interval $[a,b]$:

$$||q|| = \sup_{a\leq x\leq b} |q(x)|$$

The Padé approximants $r_{n,m}(x) = (p_{n,m}/q_{n,m})(x)$ were normalized such that $q_{n,m}(0) = 1$. This implies the existence of an interval $[a,b]$ around the origin where $q_{n,m}(x)$ is strictly positive. For given $f(x) = \sum_{i=0}^{\infty} c_i x^i$ we call a neighbourhood U_δ of f, the set of power series $g(x) = \sum_{i=0}^{\infty} d_i x^i$ such that the pseudo-norm $||c-d||_{n+m} \leq \delta$. Under weaker conditions than normality it is possible to obtain a necessary and sufficient condition for the continuity of $\mathcal{P}$ [WERN].

THEOREM 9:
The Padé operator $\mathcal{P}$ is continuous in f, if and only if $\min(n-n', m-m') = 0$ *where n' and m' are respectively the exact degrees of numerator and denominator of $\mathcal{P}f$.*

5. Multivariate Padé approximants.

We have seen in the previous sections that univariate Padé approximants can be obtained in several equivalent ways: one can solve the system of defining equations explicitly and thus obtain a determinant expression, one can set up a recursive scheme such as the ϵ-algorithm or one can construct a continued fraction whose convergents lie on a descending staircase in the Padé table. In the past few years all these approaches have been generalized to the multivariate case [CHIS, HUGH, KARL, LEVI, LUTTb, MURP, SIEM] but mostly the equivalence between the different techniques was lost. However, for the following definition a lot of properties of the univariate Padé approximant remain valid, also the recursive computation and the continued fraction representation. We shall describe here a general framework that includes all types of definitions based on the use of a linear system of defining equations for the numerator and denominator coefficients of the multivariate Padé approximant. Definitions of this type can be found in [CHIS, CUYTd, HUGH, KARL, LEVI, LUTTa, LUTTb]. Other generalizations based on symbolic manipulation [CHAF] or using branched continued fractions instead of ordinary continued fractions [CUYTh] are not treated here. We restrict ourselves to the case of two variables because the generalization to functions of more variables is only notationally more difficult.

Given a Taylor series expansion

$$f(x,y) = \sum_{(i,j)\in \mathbb{N}^2} c_{ij} x^i y^j$$

with

$$c_{ij} = \frac{1}{i!}\frac{1}{j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\Big|_{(0,0)}$$

we shall compute an approximant $(p/q)(x,y)$ to $f(x,y)$ where $p(x,y)$ and $q(x,y)$ are determined by an accuracy–through–order principle. The polynomials $p(x,y)$

and $q(x,y)$ are of the form

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}x^i y^j$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}x^i y^j$$

where N and D are finite subsets of $\mathbb{N}^2$. The sets N and D indicate the degree of the polynomials $p(x,y)$ and $q(x,y)$. Let us denote

$$\partial p = N \qquad \#N = n+1$$

$$\partial q = D \qquad \#D = m+1$$

It is now possible to let $p(x,y)$ and $q(x,y)$ satisfy the following condition for the power series $(fq-p)(x,y)$, namely

$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\backslash E} d_{ij}x^i y^j \tag{9}$$

if, in analogy with the univariate case, the index set E is such that

$$N \subseteq E \tag{10a}$$

$$\#(E\backslash N) = m = \#D - 1 \tag{10b}$$

$$E \text{ satisfies the inclusion property} \tag{10c}$$

where (10c) means that when a point belongs to the index set E, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in E. Condition (10a) enables us to split the system of equations

$$d_{ij} = 0 \qquad (i,j) \in E$$

in an inhomogeneous part defining the numerator coefficients

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}b_{i-\mu,j-\nu} = a_{ij} \qquad (i,j) \in N$$

and a homogeneous part defining the denominator coefficients

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}b_{i-\mu,j-\nu} = 0 \qquad (i,j) \in E\backslash N \tag{11}$$

By convention $b_{k\ell} = 0$ if $(k, \ell) \notin D$. Condition (10b) guarantees the existence of a nontrivial denominator $q(x, y)$ because the homogeneous system has one equation less than the number of unknowns and so one unknown coefficient can be chosen freely. Condition (10c) finally takes care of the Padé approximation property, namely

$$q(0,0) \neq 0 \Longrightarrow (f - \frac{p}{q})(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus E} e_{ij} x^i y^j$$

For more information we refer to [CUYTc, CUYTd]. We denote this multivariate Padé approximant by $[N/D]_E$ and we can arrange successive Padé approximants in a table after fixing an enumeration of the points in the degree sets N and D and the equation set E. After numbering the points in $\mathbb{N}^2$, for instance as (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), ..., we can carry this enumeration over to the index sets N, D and E which are finite subsets of $\mathbb{N}^2$, to get:

$$N = \{(i_0, j_0), \ldots, (i_n, j_n)\} \tag{12a}$$

$$D = \{(d_0, e_0), \ldots, (d_m, e_m)\} \tag{12b}$$

$$E = N \cup \{(i_{n+1}, j_{n+1}), \ldots, (i_{n+m}, j_{n+m})\} \tag{12c}$$

With (12) we can set up descending chains of index sets, defining bivariate polynomials of lower degree and bivariate Padé approximation problems of lower order:

$$N = N_n \supset \ldots \supset N_k = \{(i_0, j_0), \ldots, (i_k, j_k)\} \supset \ldots \supset N_0 = \{(i_0, j_0)\} \qquad k = 0, \ldots, n \tag{13a}$$

$$D = D_m \supset \ldots \supset D_\ell = \{(d_0, e_0), \ldots, (d_\ell, e_\ell)\} \supset \ldots \supset D_0 = \{(d_0, e_0)\} \qquad \ell = 0, \ldots, m \tag{13b}$$

$$E = E_{n+m} \supset \ldots \supset E_{k+\ell} = \{(i_0, j_0), \ldots, (i_{k+\ell}, j_{k+\ell})\} \supset \ldots \supset E_0 = \{(i_0, j_0)\} \qquad k + \ell = 0, \ldots, n + m \tag{13c}$$

$$E_{k+1,k+\ell} = \{(i_{k+1}, j_{k+1}), \ldots, (i_{k+\ell}, j_{k+\ell})\} \qquad E \setminus N = E_{n+1,n+m}$$

We assume that the enumeration was such that each set $E_{k+\ell} \subset E_{n+m}$ satisfies the inclusion property in its turn. It was shown in [LEVIa] that a determinant representation for

$$p_k(x, y) = \sum_{(i,j) \in N_k} a_{ij} x^i y^j \qquad 0 \leq k \leq n$$

and

$$q_\ell(x,y) = \sum_{(i,j)\in D_\ell} b_{ij}x^i y^j \qquad 0 \le \ell \le m$$

satisfying

$$(fq_\ell - p_k)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus E_{k+\ell}} d_{ij}x^i y^j$$

is given by

$$p_k(x,y) = \begin{vmatrix} \sum_{(i,j)\in N_k} c_{i-d_0,j-e_0}x^i y^j & \cdots & \sum_{(i,j)\in N_k} c_{i-d_\ell,j-e_\ell}x^i y^j \\ c_{i_{k+1}-d_0,j_{k+1}-e_0} & \cdots & c_{i_{k+1}-d_\ell,j_{k+1}-e_\ell} \\ \vdots & & \vdots \\ c_{i_{k+\ell}-d_0,j_{k+\ell}-e_0} & \cdots & c_{i_{k+\ell}-d_\ell,j_{k+\ell}-e_\ell} \end{vmatrix} \tag{14a}$$

$$q_\ell(x,y) = \begin{vmatrix} x^{d_0}y^{e_0} & \cdots & x^{d_\ell}y^{e_\ell} \\ c_{i_{k+1}-d_0,j_{k+1}-e_0} & \cdots & c_{i_{k+1}-d_\ell,j_{k+1}-e_\ell} \\ \vdots & & \vdots \\ c_{i_{k+\ell}-d_0,j_{k+\ell}-e_0} & \cdots & c_{i_{k+\ell}-d_\ell,j_{k+\ell}-e_\ell} \end{vmatrix} \tag{14b}$$

where $c_{ij} = 0$ if $i < 0$ or $j < 0$. A solution of the original problem (10) is then given by $(p_n/q_m)(x,y)$ because $N_n = N$, $D_m = D$ and $E_{n+m} = E$. This formula is very analogous to the univariate formula given in section 3.

6. Computation of nondegenerate MPA.

From now on we assume that the homogeneous system of equations (11) has maximal rank. Then the Padé approximant $[N/D]_E$ is called nondegenerate.

6.1. The E-algorithm.

Let us rewrite the determinant formulas (14) as

$$p_k(x,y) = \begin{vmatrix} t_0(k) & \cdots & t_\ell(k) \\ \Delta\, t_0(k) & \cdots & \Delta\, t_\ell(k) \\ \vdots & & \vdots \\ \Delta\, t_0(k+\ell-1) & \cdots & \Delta\, t_\ell(k+\ell-1) \end{vmatrix} \tag{15a}$$

$$q_\ell(x,y) = \begin{vmatrix} 1 & \cdots & 1 \\ \Delta\, t_0(k) & \cdots & \Delta\, t_\ell(k) \\ \vdots & & \vdots \\ \Delta\, t_0(k+\ell-1) & \cdots & \Delta\, t_\ell(k+\ell-1) \end{vmatrix} \tag{15b}$$

with the series $t_0, \ldots, t_\ell$ defined by:

$$t_0(0) = c_{i_0-d_0,j_0-e_0} x^{i_0-d_0} y^{j_0-e_0}$$

$$\Delta t_0(s-1) = t_0(s) - t_0(s-1) = c_{i_s-d_0,j_s-e_0} x^{i_s-d_0} y^{j_s-e_0} \qquad s = 1, \ldots, k+\ell \quad (16a)$$

$$\Delta t_0(s-1) = 0 \qquad i_s < d_0 \text{ or } j_s < e_0$$

and for $r = 1, \ldots, \ell$

$$t_r(0) = c_{i_0-d_r,j_0-e_r} x^{i_0-d_r} y^{j_0-e_r}$$

$$\Delta t_r(s-1) = t_r(s) - t_r(s-1) = c_{i_s-d_r,j_s-e_r} x^{i_s-d_r} y^{j_s-e_r} \qquad s = 1, \ldots, k+\ell \quad (16b)$$

$$\Delta t_r(s-1) = 0 \qquad i_s < d_r \text{ or } j_s < e_r$$

This quotient of determinants can be computed using the E-algorithm given in [BREZb]:

$$E_0^{(k)} = t_0(k) \qquad k = 0, \ldots, n+m$$

$$g_{0,\ell}^{(k)} = t_\ell(k) - t_{\ell-1}(k) \qquad \ell = 1, \ldots, m \qquad k = 0, \ldots, n+m$$

$$E_\ell^{(k)} = \frac{E_{\ell-1}^{(k)} g_{\ell-1,\ell}^{(k+1)} - E_{\ell-1}^{(k+1)} g_{\ell-1,\ell}^{(k)}}{g_{\ell-1,\ell}^{(k+1)} - g_{\ell-1,\ell}^{(k)}} \qquad k = 0, 1, \ldots, n \qquad \ell = 1, 2, \ldots, m$$

$$g_{\ell,r}^{(k)} = \frac{g_{\ell-1,r}^{(k)} g_{\ell-1,\ell}^{(k+1)} - g_{\ell-1,r}^{(k+1)} g_{\ell-1,\ell}^{(k)}}{g_{\ell-1,\ell}^{(k+1)} - g_{\ell-1,\ell}^{(k)}} \qquad r = \ell+1, \ell+2, \ldots$$

The values $E_\ell^{(k)}$ and $g_{\ell,r}^{(k)}$ are stored as in the tables below.

$$\begin{array}{ccccc}
E_0^{(0)} & & & & \\
 & E_1^{(0)} & & & \\
E_0^{(1)} & & \ddots & & \\
 & E_1^{(1)} & & E_m^{(0)} & \\
E_0^{(2)} & \vdots & & \vdots & \ddots \\
\vdots & & & & E_{n+m}^{(0)} \\
 & & & & \cdot^{\cdot^{\cdot}} \\
 & & & E_m^{(n)} & \\
 & E_1^{(n+m-1)} & & & \\
E_0^{(n+m)} & & & &
\end{array}$$

$$
\begin{array}{c|cc|c|cccc|c|ccc}
g_{0,1}^{(0)} & g_{0,2}^{(0)} & & & g_{0,r}^{(0)} & & & & & g_{0,m}^{(0)} & & \\
& & g_{1,2}^{(0)} & & & g_{1,r}^{(0)} & & & & & \ddots & \\
g_{0,1}^{(1)} & g_{0,2}^{(1)} & & & g_{0,r}^{(1)} & & \ddots & & & g_{0,m}^{(1)} & & \\
& & g_{1,2}^{(1)} & & & g_{1,r}^{(1)} & & g_{r-1,r}^{(0)} & & \vdots & & g_{m-1,m}^{(0)} \\
g_{0,1}^{(2)} & g_{0,2}^{(2)} & \vdots & \cdots & g_{0,r}^{(2)} & \vdots & & \vdots & \cdots & & & \vdots \\
\vdots & \vdots & & & \vdots & & & g_{r-1,r}^{(n+m-r+1)} & & & & g_{m-1,m}^{(n+1)} \\
& & & & & & \iddots & & & & & \\
& & g_{1,2}^{(n+m-1)} & & & g_{1,r}^{(n+m-1)} & & & & & \iddots & \\
g_{0,1}^{(n+m)} & g_{0,2}^{(n+m)} & & & g_{0,r}^{(n+m)} & & & & & g_{0,m}^{(n+m)} & &
\end{array}
$$

Finally with $k = n$ and $\ell = m$, this is with $N_k = N$, $D_\ell = D$ and $E_{k+\ell} = E$ we get $(p_n/q_m)(x, y) = E_m^{(n)}$ while intermediate values in the computation scheme are also multivariate Padé approximants since

$$E_\ell^{(k)} = \frac{p_k(x,y)}{q_\ell(x,y)}$$

and thus

$$q_\ell(0,0) \neq 0 \Longrightarrow f - E_\ell^{(k)} = \sum_{(i,j)\in \mathbb{N}^2 \setminus E_{k+\ell}} e_{ij} x^i y^j$$

6.2. The qdg-algorithm.

In the same way as for UPA, these intermediate values can be used to build a table of multivariate Padé approximants:

$$\begin{array}{cccc}
[N_0/D_0]_{E_0} & [N_0/D_1]_{E_1} & [N_0/D_2]_{E_2} & \dots \\
[N_1/D_0]_{E_1} & [N_1/D_1]_{E_2} & [N_1/D_2]_{E_3} & \dots \\
[N_2/D_0]_{E_2} & [N_2/D_1]_{E_3} & [N_2/D_2]_{E_4} & \dots \\
\vdots & \vdots & \vdots &
\end{array}$$

Let us consider descending staircases

$$\begin{array}{cccc}
[N_s/D_0]_{E_s} & & & \\
[N_{s+1}/D_0]_{E_{s+1}} & [N_{s+1}/D_1]_{E_{s+2}} & & \\
 & [N_{s+2}/D_1]_{E_{s+3}} & [N_{s+2}/D_2]_{E_{s+4}} & \\
 & & \vdots & \dots
\end{array}$$

in this table of MPA. It was proved in [CUYTn] that continued fractions of the form

$$[N_s/D_0]_{E_s} + \left. \frac{[N_{s+1}/D_0]_{E_{s+1}} - [N_s/D_0]_{E_s}}{1} \right| + \left. \frac{-q_1^{(s+1)}}{1 + q_1^{(s+1)}} \right| + \left. \frac{-e_1^{(s+1)}}{1 + e_1^{(s+1)}} \right| + \left. \frac{-q_2^{(s+1)}}{1 + q_2^{(s+1)}} \right| + \left. \frac{-e_2^{(s+1)}}{1 + e_2^{(s+1)}} \right| + \dots \qquad (17)$$

can be constructed of which the successive convergents are the multivariate Padé approximants on this descending staircase. Here

$$[N_s/D_0]_{E_s} = \sum_{(i,j)\in N_s} c_{ij}x^iy^j$$

$$[N_{s+1}/D_0]_{E_{s+1}} = \sum_{(i,j)\in N_{s+1}} c_{ij}x^iy^j$$

and the partial numerators and denominators are obtained from an algorithm which is very qd-like: for $\ell \geq 2$

$$q_\ell^{(s+1)} = \frac{e_{\ell-1}^{(s+2)}q_{\ell-1}^{(s+2)}}{e_{\ell-1}^{(s+1)}} \; \frac{g_{\ell-2,\ell-1}^{(s+\ell-1)} - g_{\ell-2,\ell-1}^{(s+\ell)}}{g_{\ell-2,\ell-1}^{(s+\ell-1)}} \; \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell)} - g_{\ell-1,\ell}^{(s+\ell+1)}} \tag{18a}$$

and for $\ell \geq 1$

$$e_\ell^{(s+1)} + 1 = \frac{g_{\ell-1,\ell}^{(s+\ell)} - g_{\ell-1,\ell}^{(s+\ell+1)}}{g_{\ell-1,\ell}^{(s+\ell)}} \left(q_\ell^{(s+2)} + 1\right) \tag{18b}$$

If we arrange the values $q_\ell^{(s+1)}$ and $e_\ell^{(s+1)}$ in a table as follows

$$\begin{array}{ccccc}
q_1^{(1)} & & & & \\
 & e_1^{(1)} & & & \\
q_1^{(2)} & & q_2^{(1)} & & \\
 & e_1^{(2)} & & e_2^{(1)} & \\
q_1^{(3)} & & q_2^{(2)} & & \ddots \\
 & e_1^{(3)} & & e_2^{(2)} & \\
q_1^{(4)} & & q_2^{(3)} & & \ddots \\
\vdots & e_1^{(4)} & \vdots & e_2^{(3)} & \\
 & \vdots & & \vdots &
\end{array}$$

where subscripts indicate columns and superscripts indicate downward sloping diagonals, then (18a) links the elements in the rhombus

$$\begin{array}{ccc}
 & e_{\ell-1}^{(s+1)} & \\
q_{\ell-1}^{(s+2)} & & q_\ell^{(s+1)} \\
 & e_{\ell-1}^{(s+2)} &
\end{array}$$

and (18b) links two elements on an upward sloping diagonal

$$\begin{matrix} & e_\ell^{(s+1)} \\ q_\ell^{(s+2)} & \end{matrix}$$

Starting values for the algorithm are given by

$$q_1^{(s+1)} = \frac{\Delta t_0(s+1)}{\Delta t_0(s)} \quad \frac{g_{0,1}^{(s+1)}}{g_{0,1}^{(s+1)} - g_{0,1}^{(s+2)}}$$

In this way the univariate equivalence of the three main defining techniques for Padé approximants is also established for the multivariate case: algebraic relations, recurrence relations, continued fractions.

6.3. The multivariate ϵ- and qd-algorithms.

In [CUYTd, CUYTf] we introduced multivariate Padé approximants of order (ν, μ) using homogeneous polynomials. Numerator p and denominator q are of the form

$$p(x,y) = \sum_{i+j=\nu\mu}^{\nu\mu+\nu} a_{ij} x^i y^j$$

$$q(x,y) = \sum_{i+j=\nu\mu}^{\nu\mu+\mu} b_{ij} x^i y^j$$

and satisfy

$$(fq - p)(x,y) = \sum_{i+j=\nu\mu+\nu+\mu+1}^{\infty} d_{ij} x^i y^j$$

These multivariate Padé approximants satisfy a large number of the classical univariate properties but are as well a special case of the general order approximants defined above. The advantage of the "homogeneous" approximants is that they can be calculated recursively by means of the ϵ-algorithm [CUYTg] and that they can also be represented in continued fraction form using the classical *qd*-algorithm [CUYTi]. What concerns the ϵ-algorithm the starting values are again given by the partial sums of $f(x,y)$, namely

$$\epsilon_0^{(n)} = \sum_{i+j=0}^{n} c_{ij} x^i y^j$$

What concerns the multivariate *qd*-algorithm, the univariate *qd*-algorithm is first rewritten in a form such that it can immediately be generalized. If $r_{n,m}(x)$ is the $2m^{th}$ convergent of the continued fraction

$$\sum_{i=0}^{n-m} c_i x^i + \frac{c_{n-m+1} x^{n-m+1} \rfloor}{\lfloor 1} + \frac{-q_1^{(n-m+1)} x \rfloor}{\lfloor 1} + \frac{-e_1^{(n-m+1)} x \rfloor}{\lfloor 1} + \dots$$

then we can also say that $r_{n,m}(x)$ is the $2m^{th}$ convergent of the continued fraction

$$\sum_{i=0}^{n-m} c_i x^i + \frac{c_{n-m+1}x^{n-m+1}|}{|1} + \frac{-Q_1^{(n-m+1)}|}{|1} + \frac{-E_1^{(n-m+1)}|}{|1} + \dots$$

with

$$\begin{aligned} Q_1^{(k)} &= \frac{c_{k+1}x^{k+1}}{c_k x^k} \\ E_0^{(k)} &= 0 \\ E_\ell^{(k)} &= E_{\ell-1}^{(k+1)} + Q_\ell^{(k+1)} - Q_\ell^{(k)} \\ Q_{\ell+1}^{(k)} &= Q_\ell^{(k+1)} \frac{E_\ell^{(k+1)}}{E_\ell^{(k)}} \end{aligned}$$

We have simply included the factor x in $Q_\ell^{(k)}$ and $E_\ell^{(k)}$. This last continued fraction can easily be generalized for a bivariate function: replace the expression $c_k x^k$ by an expression that contains all the terms of degree k in the bivariate series $\sum_{i+j=0}^{\infty} c_{ij}x^i y^j$. The starting values are then given by

$$Q_1^{(k)} = \frac{\sum\limits_{i+j=k+1} c_{ij}x^i y^j}{\sum\limits_{i+j=k} c_{ij}x^i y^j}$$

which are very analogous to the univariate starting formulas. Explicit determinant formulas for these homogeneous approximants, involving near-Toeplitz matrices, are given in [CUYTb].

7. Structure of the table of MPA.

For the multivariate Padé approximants discussed in the last paragraph of the previous section, the square block structure which is typical for univariate Padé approximants is preserved. This result is based on the fact that for the homogeneous Padé approximants different solutions to the same Padé approximation problem are equivalent and hence result in a unique irreducible form. If this irreducible form is given by $r_{n,m}(x,y) = (p_{n,m}/q_{n,m})(x,y)$ then we define

$$\begin{aligned} n' &= \partial p_{n,m} - \omega q_{n,m} \\ m' &= \partial q_{n,m} - \omega q_{n,m} \end{aligned}$$

where ∂ and ω denote homogeneous degrees. We can prove that $n' \leq n$ and $m' \leq m$ because clearly n' and m' are an extension of the univariate definitions where $\omega q_{n,m} = 0$ because of the normalization.

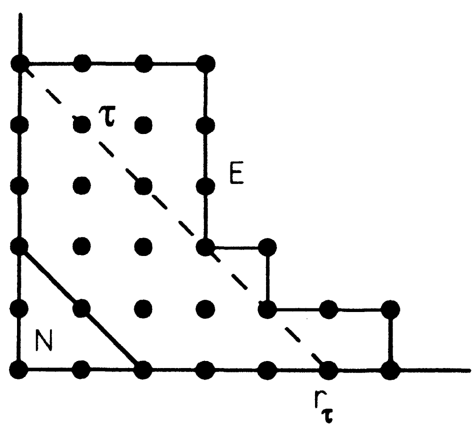
THEOREM 10:
If the homogeneous Padé approximant of order (n, m) for $f(x, y)$ is given by $r_{n,m} = p_{n,m}/q_{n,m}$ with n' and m' as defined above, then:
(a) $\omega(fq_{n,m} - p_{n,m}) = \omega q_{n,m} + n' + m' + t + 1$ with $t \geq 0$
(b) for k and ℓ satisfying $n' \leq k \leq n' + t$ and $m' \leq \ell \leq m' + t$ we have $r_{k,\ell}(x, y) = r_{n,m}(x, y)$
(c) $n \leq n' + t$ and $m \leq m' + t$

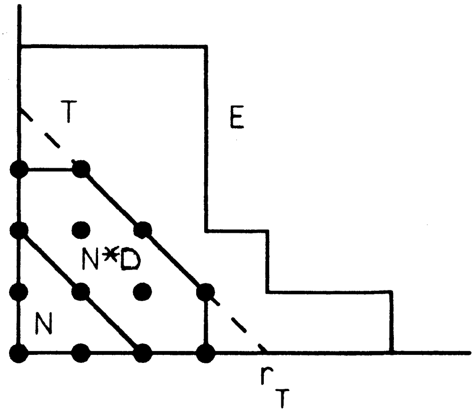
As a conclusion we take a closer look at the meaning of the numbers $\omega q_{n,m}$, n' and m'. In the solution $p(x, y)$ and $q(x, y)$ the degrees have been shifted over nm. By taking the irreducible form of p/q part of that shift can disappear, but what remains in $p_{n,m}$, $q_{n,m}$ and $fq_{n,m} - p_{n,m}$ is a shift over $\omega q_{n,m}$ [CUYTf]. Now n' and m' play the same role as in the univariate case: they measure the exact degree of a polynomial by disregarding the shift over $\omega q_{n,m}$. The singular computation rules developed by Cordellier for the ϵ-algorithm and Claessens and Wuytack for the qd-algorithm for use in a non-normal table remain valid. The multivariate version is written down in the same way as was done in section 6 for the regular ϵ- and qd-algorithm. An extension of the E- and qdg-algorithms with singular rules can be found in [CUYTa].

8. Convergence and continuity.

The univariate theorem of de Montessus de Ballore deals with the case of simple poles as well as with the case of multiple poles. The former means that we have information on the denominator of the meromorphic function while the latter means that we also have information on the derivatives of that denominator.

By the set $N * D$ we denote the index set that results from the multiplication of a polynomial indexed by N with a polynomial indexed by D, namely $N * D = \{(i + k, j + \ell)|(i, j) \in N, (k, \ell) \in D\}$.





Since the set E satisfies the inclusion property we can inscribe isosceles triangles in E, with top in (0,0) and base along the antidiagonal. Let τ be the largest of these inscribed triangles. On the other hand, because $N * D$ is a finite subset of $\mathbb{N}^2$, we can circumscribe it with such triangles. Let T be the smallest of these circumscribing triangles. In both cases we call r_τ and r_T the "range" of the triangles τ and T respectively.

In what follows we discuss functions $f(x,y)$ which are meromorphic in a polydisc $B(0;R_1,R_2) = \{(x,y) : |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial

$$R_m(x,y) = \sum_{(d,e)\in D\subseteq \mathbb{N}^2} r_{de}x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i}$$

such that $(fR_m)(x,y)$ is analytic in the polydisc above. The denominator polynomial $R_m(x,y)$ can completely be determined by m zeros $(x_h,y_h) \in B(0;R_1,R_2)$ of $R_m(x,y)$

$$R_m(x_h,y_h) = 0 \qquad h = 1,\ldots,m \tag{19a}$$

or by a combination of zeros of R_m and some of its partial derivatives. For instance in the point (x_h,y_h) the partial derivatives

$$\frac{\partial^{i_h+j_h} R_m}{\partial x^{i_h} \partial y^{j_h}}_{|(x_h,y_h)} = 0 \qquad (i_h,j_h) \in I_h \tag{19b}$$

can be given with I_h a finite subset of $\mathbb{N}^2$ of cardinality $\mu(h)+1$ and satisfying the inclusion property. We can again enumerate the indices indicating the known and vanishing partial derivatives as follows:

$$I_h = \{(i_0^{(h)}, j_0^{(h)}), \ldots, (i_{\mu(h)}^{(h)}, j_{\mu(h)}^{(h)})\} \qquad (i_0^{(h)}, j_0^{(h)}) = (0,0)$$

THEOREM 11:
Let $f(x,y)$ be a function which is meromorphic in the polydisc $B(0;R_1,R_2) = \{(x,y) : |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial

$$R_m(x,y) = \sum_{(d,e)\in D\subseteq \mathbb{N}^2} r_{de}x^d y^e = \sum_{i=0}^{m} r_{d_i e_i} x^{d_i} y^{e_i}$$

such that $(fR_m)(x,y)$ is analytic in the polydisc above. Further, we assume that $R_m(0,0) \neq 0$ so that necessarily $(0,0) \in D$. Let there also be given k zeros $(x_h,y_h) \in B(0;R_1,R_2)$ of $R_m(x,y)$ and k sets $I_h \subset \mathbb{N}^2$ with inclusion property, satisfying

$$(fR_m)(x_h,y_h) \neq 0 \qquad h = 1,\ldots,k \tag{20a}$$

$$\begin{cases} \left.\dfrac{\partial^{i_h+j_h} R_m}{\partial x^{i_h}\partial y^{j_h}}\right|_{(x_h,y_h)} = 0 & (i_h,j_h) \in I_h \qquad h = 1,\ldots,k \\ \sum_{h=1}^{k}(\mu(h)+1) = m & \#I_h = \mu(h)+1 \end{cases} \tag{20b}$$

and producing the nonzero determinant

$$\begin{vmatrix} x_1^{d_1}y_1^{e_1} & \cdots & x_1^{d_m}y_1^{e_m} \\ \vdots & & \vdots \\ \frac{d_1!}{(d_1-\mu(1))!}\frac{e_1!}{(e_1-\mu(1))!}x_1^{d_1-\mu(1)}y_1^{e_1-\mu(1)} & \cdots & \frac{d_m!}{(d_m-\mu(1))!}\frac{e_m!}{(e_m-\mu(1))!}x_1^{d_m-\mu(1)}y_1^{e_m-\mu(1)} \\ \vdots & & \vdots \\ x_k^{d_1}y_k^{e_1} & \cdots & x_k^{d_m}y_k^{e_m} \\ \vdots & & \vdots \\ \frac{d_1!}{(d_1-\mu(k))!}\frac{e_1!}{(e_1-\mu(k))!}x_k^{d_1-\mu(k)}y_k^{e_1-\mu(k)} & \cdots & \frac{d_m!}{(d_m-\mu(k))!}\frac{e_m!}{(e_m-\mu(k))!}x_k^{d_m-\mu(k)}y_k^{e_m-\mu(k)} \end{vmatrix} \tag{20c}$$

Then the $[N/D]_E = (p/q)(x,y)$ Padé approximant with D fixed by $R_m(x,y)$ and N and E growing, converges to $f(x,y)$ uniformly on compact subsets of

$$\{(x,y) : |x| < R_1, |y| < R_2, R_m(x,y) \neq 0\}$$

and its denominator

$$q(x,y) = \sum_{i=0}^{m} b_{d_i e_i} x^{d_i} y^{e_i}$$

converges to $R_m(x,y)$ under the following conditions for N and E: the range of the largest inscribed triangle in E and the range of the smallest triangle circumscribing

$N * D$ should both tend to infinity as the sets N and E grow along the column $[N/D]_E$ in the multivariate Padé table.

For homogeneous approximants also a continuity property was proved [CUYTj]. A more general result for general order approximants is under investigation. Let us define the operator $\mathcal{P}_{n,m}$ that associates with $f(x,y)$ its homogeneous (n,m) Padé approximant. Since n and m are fixed here, we again adopt the notations

$$\mathcal{P} := \mathcal{P}_{n,m}$$
$$\mathcal{P}f := r_{n,m}$$

If we write

$$C_k(x,y) = \sum_{i+j=k} c_{ij} x^i y^j$$

then C_k is a k-linear operator and we can introduce seminorms for the power series $f(x,y)$ as follows:

$$||f||_{n+m} = \max_{0 \le k \le n+m} ||C_k||$$

where $||C_k|| = \max_{||(x,y)||=1} |C_k(x,y)|$. Bivariate functions q continuous on a poly-interval $I = I_1 \times I_2$ are normed by the Chebyshev norm

$$||q|| = \sup_{(x,y) \in I} |q(x,y)|$$

THEOREM 12:
If $\min(n-n', m-m') = 0$ and $q_{n,m}(x,y) \neq 0$ for all (x,y) in a suitably chosen poly-interval I, then the Padé operator $\mathcal{P}$ is continuous in $f(x,y)$.

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