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PADÉ-APPROXIMANTS IN OPERATOR THEORY FOR THE SOLUTION OF NONLINEAR DIFFERENTIAL AND INTEGRAL EQUATIONS

ANNIE A. M. CUYT†
University of Antwerp, Department of Mathematics, Universiteitsplein 1, B-2610 Wilrijk, Belgium

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Abstract—The Padé approximation problem in operator theory and its solution will be repeated briefly together with some properties of the Padé approximants. Effective methods for the solution of differential and integral equations are of great practical importance and there is a vast literature devoted to this subject.

Here a few methods, resulting from the use of Padé approximants in operator theory, are introduced and illustrated by means of some typical examples (initial value problems, boundary value problems, partial differential equations, nonlinear integral equations).

Among the new methods is an iterative scheme which we will call Halley's method and which proves to be very useful in the neighbourhood of singularities.

Well known methods such as Newton's and Chebyshev's method prove to be special cases of the class of iterative procedures.

1. INTRODUCTION

Let X be a Banach space and Y a commutative Banach algebra without nilpotent elements. We shall denote the scalar field by Λ (where Λ is $\mathbf{R}$ or $\mathbf{C}$), the unit for the addition in the Banach spaces by 0 and the unit for the multiplication in the Banach algebra by 1. Let F: $X \to Y$ be a nonlinear operator analytic at 0([10], p. 113). In other words, there exists an open ball $B(0, r)$ with centre 0 and radius $r > 0$ such that:

$$F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(0)x^k \text{ for } \|x\| < r$$

with

$$\frac{1}{0!} F^{(0)}(0)x^0 = F(0)$$

and

$$F^{(k)}(0) \text{ the } k^{\text{th}} \text{ Fréchet derivative of } F \text{ at } 0.$$

Write $(1/k!)F^{(k)}(0) = C_k$. The C_k are symmetric k-linear bounded operators ([10], pp. 100–110).

We say that $F(x) = 0(x^j)$ if there exists an open ball $B(0, r)$ with $0 < r < 1$ such that:

$$\|F(x)\| \le J\|x\|^j \text{ for all } x \text{ in } B(0, r)$$

$$\text{and with } j \in \mathbf{N} \text{ and } J \in \mathbf{R}_0^+.$$

Write $D(F) = \{x \in X | F(x) \text{ is regular in } Y, \text{ i.e. there exists } y \in Y : F(x) \cdot y = 1 = y \cdot F(x)\}$.

An abstract polynomial is a nonlinear operator $P : X \to Y$ such that $P(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_0$ with A_i a symmetric i-linear bounded operator and A_0 an element of Y. The degree of $P(x)$ is n.

We also introduce the following notations.

If there exists a positive integer j_1, such that for all $0 \le k < j_1$: $A_k x^k \equiv 0$ and $A_{j_1} x^{j_1} \not\equiv 0$ then $\partial_0 P = j_1$ is called the order of the abstract polynomial P.

†Research Assistant of the National Fund for Scientific Research, Belgium.

If there exists a positive integer j_2, such that for all $j_2 < k \le n$: $A_k x^k \equiv 0$ and $A_{j_1} x^{j_1} \not\equiv 0$ then $\partial P = j_2$ is called the exact degree of the abstract polynomial P.

We can easily prove the following important lemmas for abstract polynomials [5].

LEMMA 1.1

Let U, V be abstract polynomials. If $U(x) \cdot V(x) \equiv 0$ and if $D(V) \neq \emptyset$, then $U(x) \equiv 0$.

LEMMA 1.2

Let U, V be abstract polynomials. If $D(U) \neq \emptyset$ then $\partial V \le \partial(U \cdot V) - \partial_0(U)$ (because Y contains no nilpotent elements).

2. THE PADÉ APPROXIMATION PROBLEM

Definition 2.1. The couple of abstract polynomials $(P(x), Q(x)) = (A_{nm+n}x^{nm+n} + \ldots + A_{nm}x^{nm}, B_{nm-m}x^{nm+m} + \ldots + B_{nm}x^{nm})$ is called a solution of the Padé approximation problem of order (n, m) for F if the abstract power series

$$(F \cdot Q - P)(x) = 0(x^{nm+n+m+1}). \tag{1}$$

The choice of $P(x)$ and $Q(x)$ is in [4] justified by the fact that for all non-negative integers n and m a solution of the problem described in definition 2.1 exists.

We shall restrict ourselves now to those n and m for which a solution $(P(x), Q(x))$ with $D(P) \neq \emptyset$ or $D(0) \neq \emptyset$ can be found.

We define $(1/Q)$: $D(Q) \to Y : x \to [Q(x)]^{-1}$, the inverse element of $Q(x)$ for the multiplication in Y.

We call the abstract rational operator $(1/Q) \cdot P$, the quotient of two abstract polynomials, reducible if there exist abstract polynomials T, P_*, Q_*, such that $P = T \cdot P_*$ and $Q = T \cdot Q_*$ and $\partial T \ge 1$.

For the solutions (P, Q) of the Padé approximation problem and for the reduced rational operators $(1/Q_*) \cdot P_*$ we can prove the following properties. The proofs of those properties can be found in [5] except some small modifications.

THEOREM 2.1

Let (P, Q) satisfy (1) and $(1/Q_*) \cdot P_*$ be a reduced form of $(1/Q) \cdot P$. Let $P = P_* \cdot T$ and $Q = Q_* \cdot T$ with $T(x) = \sum_{k=t_0}^{\partial T} T_k x^k$ and $t_0 = \partial_0 T$. If $D(T_{t_0}) \neq \emptyset$ or $\partial_0 Q_* = 0$ then $\partial_0 P_* \ge \partial_0 Q_*$.

We write $n' = \partial P_* - \partial_0 Q_*$ and $m' = \partial Q_* - \partial_0 Q_*$ if $D(T_{t_0}) \neq \emptyset$ or $\partial_0 Q_* = 0$.
The term $-\partial_0 Q_*$ for n' is justified by the preceding theorem.

THEOREM 2.2

If $D(T_{t_0}) \neq \emptyset$ then $n' \le n$ and $m' \le m$ and $(F \cdot Q_* - P_*)(x) = 0(x^{\partial_0 Q_* + n' + m' + 1})$

The fact that $(F \cdot Q_* - P_*)(x) = 0(x^{\partial_0 Q_* + n' + m' + 1})$ implies that $(F \cdot Q_* - P_*)^{(i)}(0) = 0$ for $i = 0, \ldots, \partial_0 Q_* + n' + m'$ at least. For polynomials P_* and Q_* with $\partial_0 P_* \ge \partial_0 Q_*$ we know that always

$$(F \cdot Q_* - P_*)^{(i)}(0) \equiv 0 \text{ for } i = 0, \ldots, \partial_0 Q_* - 1.$$

So the meaningful relations are:

$$(F \cdot Q_* - P_*)^{(i)}(0) \equiv 0 \text{ for } i = \partial_0 Q_*, \ldots, \partial_0 Q_* + n' + m' \text{ at least.} \tag{2}$$

When $0 \in D(Q_*)$ and thus $\partial_0 Q_* = 0$, the relations can be rewritten as:

$$F^{(i)}(0) = \left(\frac{1}{Q_*} \cdot P_*\right)^{(i)}(0) \text{ for } i = 0, \ldots, n' + m' \text{ at least.}$$

So (2) clearly has an interpolatory meaning at 0 and $(1/Q_*) \cdot P_*$ is a local approximation for F.

3. ITERATIVE METHODS FOR THE SOLUTION OF OPERATOR EQUATIONS

Consider the nonlinear operator $F: X \to Y$. Suppose we want to find x^* in X such that $F(x^*) = 0$.

Let F be analytic in a neighbourhood of the simple root x^*. Since x^* is a simple root, $F'(x^*)^{-1}$ is a linear bounded operator. If the inverse operator $G: Y \to X$ exists, it is analytic in a neighbourhood of 0 ([2], p. 301).

By means of solutions of the Padé approximation problem for the inverse operator G we can construct iterative methods to find x^*, i.e. starting from an approximation x_0 for x^* a sequence of further approximations $\{x_i\}$ is constructed in such a way that x_{i+1} is computed by means of x_i. By F_i' and F_i'' we mean respectively the first and second Fréchet derivative of F at x_i. Let $F_i = F(x_i) = y_i$ and $G(y_i) = x_i$.

We know that $G(0) = x^*$ and we can write:

$$G(y) = G(y_i) + F_i'^{-1}(y - y_i) - \frac{1}{2}F_i'^{-1}(F_i''F_i'^{-1})(y - y_i)^2 + \cdots \tag{3}$$

where $(F_i''F_i'^{-1})(y - y_i)^2$ is the bilinear operator F_i'' evaluated in

$$(F_i'^{-1}(y - y_i),\ F_i'^{-1}(y - y_i)).$$

If we calculate a solution (P_i, Q_i) of the Padé approximation problem of order (n, m) for G in y_i we could iterate:

$$x_{i+1} = \frac{1}{Q_i} \cdot P_i(0) \text{ or } \frac{1}{Q_{i*}} \cdot P_{i*}(0)$$

where $(1/Q_{i*}) \cdot P_{i*}$ is a reduced form of $(1/Q_i) \cdot P_i$

We can expect that iterative procedures where $m > 0$ will be more suitable than those were $m = 0$ if the operator G has singularities in the neighbourhood of 0.

An example has been given in [6] for the solution of nonlinear systems of equations. Other examples will be given here.

Observe that the Newton-iteration results from approximating the series (3) by its first two terms, i.e. the solution of the Padé approximation problem of order (1, 0) for G:

$$x_{i+1} = x_i + a_i \text{ where } a_i = -F_i'^{-1}F_i. \tag{4}$$

The (0, 1) Padé approximation problem gives the following iterative method:

$$x_{i+1} = \frac{x_i^2}{x_i - a_i}. \tag{5}$$

The first three terms in (3), which form in fact a solution for the (2, 0) Padé approximation problem, could also be used to approximate x^*, giving the next iteration:

$$x_{i+1} = x_i + a_i - \frac{1}{2}F_i'^{-1}F_i''a_i^2 \tag{6}$$

The iteration (6) is known as Chebyshev's method for the solution of operator equations ([10], p. 205).

Another way to approximate x^* is to use a solution of the (1, 1) Padé approximation problem for the series in (3):

$$x_{i+1} = x_i + \frac{a_i^2}{a_i + (1/2)F_i'^{-1}F_i''a_i^2} \tag{7}$$

which is a generalisation of a formula of Frame [7] and a rediscovery of the Halley-correction, now for operator equations. If $F_i'^{-1}F_i''a_i^2 = a_i \cdot La_i$ for a bounded linear operator L, then (7) reduces to:

$$x_{i+1} = x_i + \frac{a_i}{1+(1/2)La_i}.$$

Using a solution of the (0,2) Padé approximation problem we get

$$x_{i+1} = \frac{x_i^3}{x_i^2 - x_i a_i + a_i^2 + (1/2)x_i F_i'^{-1}F_i''a_i^2}. \tag{8}$$

We will now use these methods for the solution of a few typical problems. The considered Banach spaces and Banach algebras will be $C^{(l)}(B)$ where: $B \subset \mathbf{R}^m$, B subset of closure of its interior,

$$C^{(l)}(B) = \left\{ f: B \to \mathbf{R} \mid \frac{\partial^k f(z_1, \ldots, z_n)}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}} \text{ exist for } 0 \le k \le l \text{ and } (z_1, \ldots, z_n) \text{ in the interior of } B, \text{ and} \right.$$

$$\left. \text{are continuous and bounded} \right\}.$$

So the successive approximations x_i in an iterative procedure will be real-valued functions.

4. INITIAL VALUE PROBLEMS

Consider the equation
$$\frac{dy}{dt} - f(t, y) = 0 \qquad y(0) = c \tag{9}$$
for $t \in [0, T]$.

Let $C'([0, T])$ and $C([0, T])$ denote the set of all real-valued functions that are respectively continuously differentiable and continuous on the real interval $[0, T]$. In fact we could restrict ourselves to the space $C_c'([0, T]) = \{y \in C'([0, T]) | y(0) = c\}$ and try to find a zero $y^*(t)$ of the following operator

$$F: C_c'([0, T]) \to C([0, T]): y \to \frac{dy}{dt} - f(t, y)$$

starting from an initial approximation $y_0(t)$ that satisfies $y_0(0) = c$, and compute corrections $(y_{i+1} - y_i)(t)$ that satisfy $(y_{i+1} - y_i)(0) = 0$. We calculate the necessary derivatives:

$$F'(y_0): C_c'([0, T]) \to C([0, T]): y \to \left(\frac{d}{dt} - \left.\frac{\partial f(t, y)}{\partial y}\right|_{y=y_0(t)}\right) y$$

$$F''(y_0): C_c'([0, T]) \times C_c'([0, T]) \to C([0, T]): (y, y) \to \left.\frac{-\partial^2 f(t, y)}{\partial y^2}\right|_{y=y_0(t)} \cdot y^2$$

For the calculation of the Newton-correction $a_0(t)$ we have to solve the linear problem:

$$F'(y_0)a_0 = -F(y_0) \tag{10}$$

and iterate

$$y_1(t) = y_0(t) - F'(y_0)^{-1}F(y_0) = y_0(t) + a_0(t).$$

One can prove that the solution of (10) is ([10] p. 170):

$$y_1(t) = y_0(t) - \int_0^t e^{A_0(s) - A_0(t)} F(y_0)(s)\, ds$$

where

$$A_0(t) = -\int_0^t \frac{\partial f(s, y(s))}{\partial y}\bigg|_{y=y_0(s)} \mathrm{d}s$$

The whole procedure can be repeated to calculate the next iteration steps. For the Chebyshev- or Halley-iteration one has to solve two linear problems:

$$\begin{aligned} F'(y_0)a_0 &= -F(y_0) \\ F'(y_0)b_0 &= F''(y_0)a_0^2 \end{aligned} \tag{11}$$

and iterate respectively:

$$y_1(t) = y_0(t) + a_0(t) - \frac{1}{2}b_0(t) \text{ or } y_1(t) = y_0(t) + \frac{a_0^2(t)}{a_0(t) + \frac{1}{2}b_0(t)}.$$

We now turn to some examples.

Consider the nonlinear initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} - (1 + y^2) = 0$$
$$y(0) = 0$$

for $t \in [0, T]$.

We will calculate $y_1(t)$ starting from $y_0(t) = t$ for the Newton-, Chebyshev- and Halley-iteration.

Observe that:

$$A_0(t) = -t^2$$

$$F''(y_0)y^2 = -2y^2$$

$$-F(y_0) = t^2$$

$$a_0(t) = \int_0^t \mathrm{e}^{t^2-s^2} s^2 \,\mathrm{d}s = \frac{t^3}{3} + \frac{2t^5}{15} + \frac{4t^7}{105} + \frac{8t^9}{945} + \frac{16t^{11}}{10395} + \ldots$$

(term by term integration)

$$b_0(t) = -\int_0^t 2\,\mathrm{e}^{t^2-s^2}[a_0(s)]^2 \,\mathrm{d}s = (-2)\left(\frac{t^7}{63} + \frac{38t^9}{2835} + \frac{992t^{11}}{155925} + \ldots\right).$$

The next iteration steps are:

$$y_1(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{4}{105}t^7 + \frac{8}{945}t^8 + \ldots \text{(Newton)}$$

$$y_1(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{16}{2025}t^{11} + \ldots \text{(Chebyshev)}$$

$$y_1(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \frac{91369}{81\,860\,625}t^{11} + \ldots \text{(Halley)}.$$

For $T < (\Pi/2)$ the exact solution is

$$y^*(t) = \mathrm{tg}\,t = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{4146}{467775}t^{11} + \ldots.$$

Initial value problems correspond to Volterra integral equations. So equation (9) can be transformed into the following nonlinear integral equation:

$$F(y) = y(t) - c - \int_0^t f(s, y(s))\,\mathrm{d}s.$$

Now $F'(y_0) = I - V_0$ and

$$F''(y_0)y^2 = -\int_0^t \left.\frac{\partial^2 f(s, y(s))}{\partial y^2}\right|_{y=y_0(s)} y^2(s)\,\mathrm{d}s$$

where $I: y \to y$ is the identity-operator and

$$V_0 y = \int_0^t \frac{\partial f}{\partial y}(s, y(s))|_{y=y_0(s)} y(s)\,\mathrm{d}s.$$

So

$$F'(y_0)^{-1} = \sum_{n=0}^{\infty} V_0^n \text{ if } \|V_0\| < 1.$$

If we rewrite

$$F'(y_0)^{-1}y = \left(I + \sum_{n=1}^{\infty} V_0^n\right)y = y + V_0(F'(y_0)^{-1}y)$$

the eqns (10) and (11) can be solved iteratively:

$$a_0^{(0)}(t) = 0$$

$$a_0^{(i)}(t) = -F(y_0)(t) + V_0 a_0^{(i-1)}(t)$$

$$= -y_0(t) + c + \int_0^t f(s, y_0(s))\,\mathrm{d}s + \int_0^t \frac{\partial f}{\partial y}(s, y(s))|_{y=y_0(s)}\, a_0^{(i-1)}(s)\,\mathrm{d}s$$

$$b_0^{(0)}(t) = 0$$

$$b_0^{(i)}(t) = F''(y_0)a_0^2(t) + V_0 b_0^{(i-1)}(t)$$

$$= -\int_0^t \left.\frac{\partial^2 f(s, y(s))}{\partial y^2}\right|_{y=y_0(s)} a_0^2(s)\,\mathrm{d}s + \int_0^t \left.\frac{\partial f}{\partial y}(s, y(s))\right|_{y=y_0(s)} b_0^{(i-1)}(s)\,\mathrm{d}s$$

where $a_0(t)$ is the last approximation $a_0^{(i)}(t)$ for the Newton-correction.

For our example where $f(t, y) = 1 + y^2$ and $c = 0$, we get the next iteration steps:

$$y_1(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{4}{105}t^7 + \frac{8}{945}t^9 + \ldots \text{(Newton)}$$

$$y_1(t) = t = \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{16}{2025}t^{11} + \ldots \text{(Chebyshev)}$$

$$y_1(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \frac{91369}{81\,860\,625}t^{11} + \ldots \text{(Halley).}$$

Let us now turn to an example where the method of Halley, which is newly introduced here in (7), proves to be much better than the methods resulting from the Padé approximation problem of order $(n, 0)$ for G.

Consider the equation $e^{y(t)}(dy/dt) - (0.1 + \epsilon) = 0$

$$y(1) = \ln \epsilon$$

for $t \in [1, T]$ with ϵ a small nonzero positive number and T large. We are looking for a zero $y^*(t)$ of the nonlinear operator:

$$F: y \to e^{y} \frac{dy}{dt} - (0.1 + \epsilon) = z.$$

The inverse operator:

$$G: z \to \ln\left(\epsilon t + \int_1^t (0.1 + z)\, ds\right) = y$$

comes nearby a singularity for $z = -0.1$, thus in the neighbourhood of $z = 0$.
The exact solution is $y^*(t) = \ln(\epsilon t + 0.1(t-1))$. Let us take our initial approximation $y_0(t) = \ln \epsilon t$.
The derivatives at y_0 are:

$$F'(y_0)y = e^{y_0(t)}\left(\frac{dy_0}{dt} y + \frac{dy}{dt}\right)$$

$$F''(y_0)y^2 = e^{y_0(t)} \cdot y \cdot \left(2\frac{dy}{dt} + y\frac{dy_0}{dt}\right).$$

For $y_0(t) = \ln \in t$:

$$F'(y_0)y = \epsilon t\left(\frac{dy}{dt} + \frac{1}{t} y\right)$$

$$F''(y_0)y^2 = \epsilon t \cdot y \cdot \left(2\frac{dy}{dt} + \frac{1}{t} y\right).$$

For the Newton-correction we have to solve the linear equation:

$$\frac{da_0}{dt} + \frac{1}{t} a_0(t) = \frac{0.1}{\epsilon t}.$$

The solution is constructed in the same way as for (10):

$$a_0(t) = \int_1^t e^{A_0(s) - A_0(t)} \frac{0.1}{\epsilon s}\, ds$$

where

$$A_0(t) = \int_1^t \frac{1}{s}\, ds = \ln t.$$

So

$$a_0(t) = \frac{0.1}{\epsilon t}(t-1).$$

For the Chebyshev- and Halley-iteration we need the $b_0(t)$:

$$b_0(t) = \int_1^t \frac{s}{t}\left(\frac{0.1}{\epsilon}\right)^2 \frac{s^2-1}{s^3}\, ds = [a_0(t)]^2$$

because

$$F''(y_0)a_0^2 = \left(\frac{0.1}{\epsilon}\right)^2 \frac{t^2-1}{t^2}\epsilon$$

The next iteration step is:

$$y_1(t) = \ln \epsilon t + \frac{0.1}{\epsilon}\frac{t-1}{t} \text{ (Newton)}$$

$$y_1(t) = \ln\ \epsilon t + \frac{0.1}{\epsilon}\frac{t-1}{t}\left(1 - \frac{0.1}{2\epsilon}\frac{t-1}{t}\right) \text{ (Chebyshev)}$$

$$y_1(t) = \ln \epsilon t + \frac{0.1}{\epsilon}\frac{t-1}{t}\Big/\left(1 + \frac{0.1}{2\epsilon}\frac{t-1}{t}\right) \text{ (Halley)}$$

$$y_1(t) = (\ln \epsilon t)^2/\left(\ln \epsilon t - \frac{0.1}{\epsilon}\frac{t-1}{t}\right) \text{ (iteration (5)).}$$

When we compare $\|y^*(t) - y_1(t)\|_\infty = \sup|y^*(t) - y_1(t)|$ for $t \in [1, T]$ in the different procedures (see also Figs. 4.1–4.4) we see that $\epsilon = 0.01$ and T very large:

$$\|y^* - y_1\|_\infty \simeq 10 - \ln 11 \simeq 7.60 \text{ (Newton)}$$

$$\|y^* - y_1\|_\infty \simeq 40 + \ln 11 \simeq 42.40 \text{ (Chebyshev)}$$

$$\|y^* - y_1\|_\infty \simeq -\frac{10}{6} + \ln 11 \simeq 0.73 \text{ (Halley)}$$

$$\|y^* - y_1\|_\infty \simeq 10 - \ln 11 \simeq 7.60 \text{ (iteration (5)).}$$

Also the function-values for $t = 2$ and $\epsilon = 0.01$ illustrate that the iterative procedures that take into account the singularity of the operator G in the neighbourhood of 0, are much more accurate;

$$y^*(2) = -2.12026354$$

$$y_1(2) = 1.08797700 \text{ (Newton)}$$

$$y_1(2) = -11.4120230 \text{ (Chebyshev)}$$

$$y_1(2) = -2.48345158 \text{ (Halley)}$$

$$y_1(2) = -1.71722223 \text{ (iteration (5)).}$$

Fig. 4.1. $|y^*(t) - y_1(t)|$(Newton).

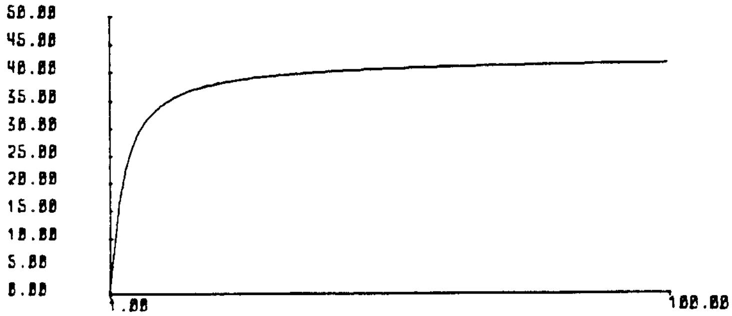

Fig. 4.2. $|y^*(t) - y_1(t)|$ (Chebyshev).

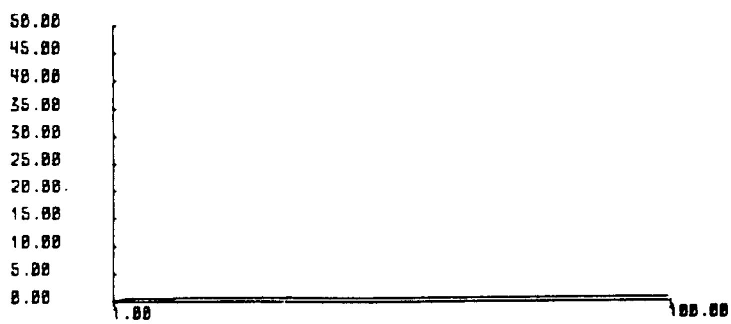

Fig. 4.3. $|y^*(t) - y_1(t)|$ (Halley).

Fig. 4.4. $|y^*(t) - y_1(t)|$ (iteration (5)).

An iterative method resulting from the solution of the Padé approximation problem of order (n, m) for G with $m > 0$, is also very useful when there are several singularities in the solution $y^*(t)$ itself, because the rational approximations $y_i(t)$ can simulate certain singularities. We emphasis the fact that discontinuities cause difficulties when discretisation techniques are used.

We will illustrate the advantage of the use of Halley's method and iteration (5) by an example.

Suppose we want to solve

$$F(y) = \frac{\mathrm{d}y}{\mathrm{d}t} + y^2 = 0$$

$$y(0) = -1$$

for $t \in [0, (1/2)] \cup [(3/2), T]$ with T large.

The solution

$$y^*(t) = \frac{1}{t-1}.$$

As an initial approximation we take $y_0(t) = -1$ and we calculate

$$F(y_0) = 1$$

$$F'(y_0)y = \frac{dy}{dt} - 2y$$

$$F''(y_0)y^2 = 2y^2.$$

For the Newton-correction we have to solve the linear problem:

$$\frac{da_0}{dt} - 2a_0(t) = -1.$$

The solution is constructed in the same way as previously:

$$a_0(t) = -\int_0^t e^{A_0(s)-A_0(t)}\, ds$$

with

$$A_0(t) = -\int_0^t 2\, ds = -2t.$$

So

$$a_0(t) = \frac{1}{2}(1 - e^{2t}).$$

Now we calculate the $b_0(t)$ for the Chebyshev- and Halley-iteration

$$\frac{db_0}{dt} - 2b_0(t) = \frac{1}{2}(1 - e^{2t})^2$$

So

$$b_0(t) = \int_0^t e^{A_0(s)-A_0(t)} \frac{1}{2}(1 - e^{2s})^2\, ds$$

$$= \frac{1}{4}(e^{4t} - 1) - te^{2t}.$$

The next iteration steps are:

$$y_1(t) = -\frac{1}{2}(1 + e^{2t}) \text{ (Newton)}$$

$$y_1(t) = -\frac{1}{2}e^{2t}(1 - t) - \frac{1}{8}(e^{4t} + 3) \text{ (Chebyshev)}$$

$$y_1(t) = \frac{t\, e^{2t} + (1/4)(e^{4t} - 1)}{-(1 + t)\, e^{2t} + (1/4)(e^{4t} + 3)} \text{ (Halley)}$$

$$y_1(t) = \frac{-2}{3 - e^{2t}} \text{ (iteration (5)).}$$

The exact solution $y^*(t)$ has a pole in $t = 1$. The iteration steps $y_1(t)$, obtained by making use of the solution of the Padé approximation problem of order (1, 1) (i.e. Halley's method) and (0, 1) (i.e. iteration (5)), are more accurate than the Newton- and Chebyshev-iteration steps, because they approximate the pole of $y^*(t)$ respectively by a pole in $t = 1.01993442$ (Halley's method) and $t = 0.54930615$ (iteration (5)). So they also approximate $y^*(t)$ well beyond the discontinuity while for the Newton- and Chebyshev-iteration steps $\lim_{t\to+\infty} y_1(t) = -\infty$.

To illustrate this we compare the function-values for $t = (3/2)$:

$$y^*\left(\frac{3}{2}\right) = 2.000$$

$$y_1\left(\frac{3}{2}\right) = -10.54 \text{ (Newton)}$$

$$y_1\left(\frac{3}{2}\right) = -45.78 \text{ (Chebyshev)}$$

$$y_1\left(\frac{3}{2}\right) = 2.544 \text{ (Halley)}$$

$$y_1\left(\frac{3}{2}\right) = 0.117 \text{ (iteration (5)).}$$

5. BOUNDARY VALUE PROBLEMS

Consider the equation $(d^2y/dx^2) - f(x, y) = 0$

$$y(0) = 0 = y(1)$$

for $x \in [0, 1]$.

Let C'' ([0, 1]) denote the set of all real-valued functions that are twice continuously differentiable. Then we seek for a zero of the operator

$$F:\{y \in C''([0,1]) | y(0) = 0 = y(1)\} \to C([0,1]): y \to \frac{d^2y}{dx^2} - f(x, y).$$

The Newton-correction $a_0(x)$ is the solution of the following boundary value-problem:

$$\frac{d^2a_0}{dx^2} - \frac{\partial f}{\partial y}\bigg|_{y=y_0(x)} \cdot a_0(x) = -\frac{d^2y_0}{dx^2} + f(x, y_0(x)) = v_0(x).$$

Since boundary value problems correspond to Fredholm integral equations the Newton-correction is also the solution of the following linear Fredholm integral equation of the second kind:

$$a_0(x) - \int_0^1 G(x,t)\frac{\partial f}{\partial y}(t, y)|_{y=y_0} \cdot a_0(t)\, dt = \int_0^1 G(x,t)v_0(t)\, dt = w_0(x) \tag{12}$$

where

$$G(x,t) = \begin{cases} t(x-1) & \text{for } 0 \le t \le x \\ x(t-1) & \text{for } x \le t \le 1 \end{cases}$$

([10], p. 176).

This linear equation can be written as:

$$(I - L)a_0(x) = w_0(x)$$

where

$$La_0(x) = \int_0^1 L(x, t)a_0(t)\,\mathrm{d}t$$

with

$$L(x, t) = \begin{cases} t(x-1)\dfrac{\partial f}{\partial y}(t, y(t))|_{y=y_0} & \text{for } 0 \le t \le x \\ x(t-1)\dfrac{\partial f}{\partial y}(t, y(t))|_{y=y_0} & \text{for } x \le t \le 1. \end{cases}$$

If this linear operator $(I - L)$ is bounded then $(I - L)^{-1}$ exists if and only if a linear bounded operator K exists with inverse K^{-1} such that $\|I - K(I - L)\| < 1$. Then $(I - L)^{-1} = \sum_{n=0}^{\infty} [I - K(I - L)]^n K$ ([10], p. 43).

Let us take $K = I$ here. Then $I - K(I - L) = L$. Now

$$\begin{aligned} \|L\| &\le \max_{[0,1]} \int_0^1 |L(x, t)|\,\mathrm{d}t \\ &\le \left\| \frac{\partial f}{\partial y}\Big|_{y=y_0(x)} \right\| \cdot \max_{[0,1]} \int_0^1 |G(x, t)|\,\mathrm{d}t \\ &= \frac{1}{8}\left\| \frac{\partial f}{\partial y}\Big|_{y=y_0(x)} \right\| \end{aligned}$$

So if $\left\| \frac{\partial f}{\partial y}\Big|_{y=y_0} \right\|$ is small enough then $(I - L)^{-1} = \sum_{n=0}^{\infty} L^n$.

Again the Newton-correction can be computed iteratively:

$$a_0^{(0)}(x) = 0$$

$$a_0^{(j)}(x) = w_0(x) + \int_0^1 L(x, t)a_0^{(j-1)}(t)\,\mathrm{d}t$$

where

$$\|a_0(x) - a_0^{(j)}(x)\| \le \frac{\|L\|^{j+1}\|w_0\|}{1 - \|L\|}.$$

The correction $b_0(x)$ can be calculated analogously, and the whole procedure can be repeated for the next iteration steps.

As an example we will solve the equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - -(xy^2 - 1) = 0$$

$$y(0) = 0 = y(1)$$

for $x \in [0, 1]$.

Let us take $y_0(x) = 0$. For this $f(x, y)$, $\|L\| = (1/54) < 1$.

The solution of equation (12) is:

$$a_0(x) = \frac{1}{2}x(1-x).$$

The correction $b_0(x)$ is the solution of the boundary value problem:

$$\frac{d^2 b_0}{dx^2} - \frac{\partial f}{\partial y}\bigg|_{y=y_0(x)} \cdot b_0(x) = F''(y_0)a_0^2(x) = -2xa_0^2(x).$$

Or converted into an integral equation:

$$b_0(x) - \int_0^1 G(x,t)\frac{\partial f}{\partial y}(t, y(t))|_{y=y_0} \cdot b_0(t)\,dt = -\int_0^1 G(x,t)\frac{t^3}{2}(1-t)^2\,dt.$$

So

$$b_0(x) = -\frac{1}{2}\left(\frac{x^7}{42} - \frac{x^6}{15} + \frac{x^5}{20} - \frac{x}{140}\right) = a_0(x)\cdot\left(\frac{x^5}{42} - \frac{3}{70}x^4 + \frac{1}{140}(x^3 + x^2 + x + 1)\right).$$

The next iteration step is:

$$y_1(x) = \frac{1}{2}x(1-x) \text{ (Newton)}$$

$$y_1(x) = \frac{1}{4}x(1-x)\left[2 - \frac{x^5}{42} + \frac{3}{70}x^4 - \frac{1}{140}(x^3 + x^2 + x + 1)\right] \text{ (Chebyshev)}$$

$$y_1(x) = \frac{x(x-1)}{-2 - \frac{x^5}{42} + \frac{3}{70}x^4 - \frac{1}{140}(x^3 + x^2 + x + 1)} \text{ (Halley).}$$

If we calculate $a_1(x)$ iteratively, we get:

$$a_1^{(0)}(x) = 0$$

$$a_1^{(1)}(x) = \frac{1}{4}\left(\frac{x^7}{42} - \frac{x^6}{15} + \frac{x^5}{20} - \frac{x}{140}\right)$$

and for $y_2(x)$ in the Newton iteration:

$$y_2(x) = y_1(x) + a_1^{(1)}(x)$$

$$= a_0(x) - \frac{1}{2}a_0(x)\cdot\left(\frac{x^5}{42} - \frac{3}{70}x^4 + \frac{1}{140}(x^3 + x^2 + x + 1)\right)$$

which is precisely one Chebyshev iteration step.

The solution of the boundary value problem has been calculated for discrete values $x_i = (i/200)(i = 0, \ldots, 200)$ in the interval $[0, 1]$, by means of subroutine DD02AD of the Harwell-library and also with the initial values $y_i = y(x_i) = 0$. After interpolation through the (x_i, y_i) we get the following picture of the solution $y^*(x)$:

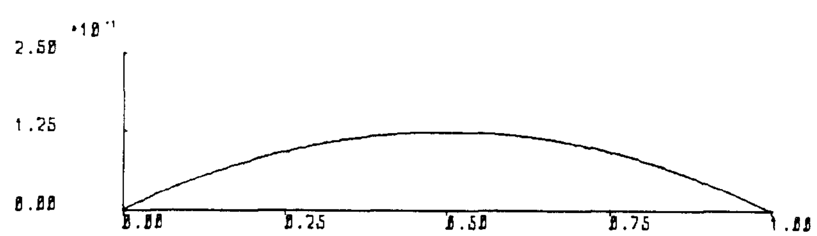

Fig. 5.1.

The different functions $y_l(x)$ mentioned above, give the same plot. We can also compare the function-values in some points (7 significant figures):

x	DD02AD	Newton	Chebyshev	Halley
0.25	0.0933169	0.0937500	0.093 3121	0.0933141
0.50	0.1242918	0.1250000	0.1242839	0.1242879
0.75	0.0932114	0.0937500	0.0932053	0.0932084

The functions $|y^*(x) - y_l(x)|$ for the different iterative schemes give the following plots:

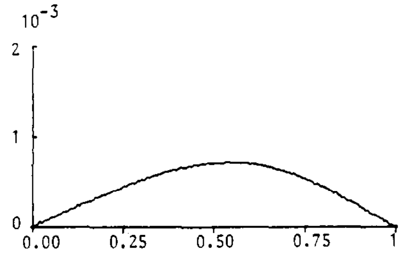

Fig. 5.2. (Newton).

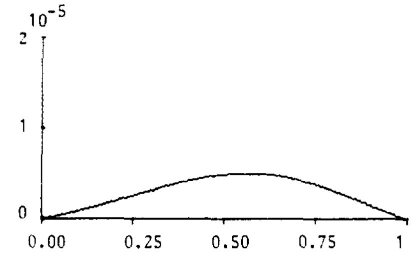

Fig. 5.3. (Chebyshev).

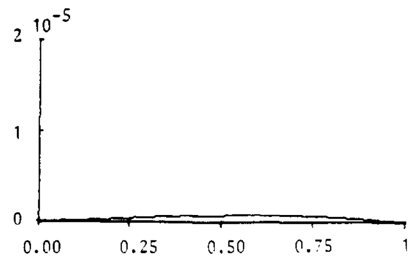

Fig. 5.4. (Halley).

6. PARTIAL DIFFERENTIAL EQUATIONS

Consider the following nonlinear equation which is of interest in gas dynamics:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u^2(x, y) \text{ for } (x, y) \text{ in } \Omega \subseteq \mathbf{R}^2$$

$$u(x, y) = b(x, y) \text{ on the boundary of the region } \Omega.$$

A solution $u(x, y)$ is sought in the interior of Ω. If $F(u) = \Delta u - u^2$, then

$$F'(u_0)u = \Delta u - 2u_0 \cdot u$$

$$F''(u_0)u^2 = -2u^2.$$

The Newton correction satisfies:

$$\Delta a_0(x, y) - 2a_0(x, y) \cdot u_0(x, y) = u_0^2(x, y) - \Delta u_0 \tag{13}$$

$$a_0(x, y) = 0 \text{ on the boundary of } \Omega.$$

Pohozaev has proved that [9]:

$$\Delta u = u^2$$

$$u(x, y) = b(x, y) > 0 \text{ on the boundary of } \Omega$$

has a unique positive solution $u(x, y)$, and that the Newton-iteration converges if the initial approximation u_0 is the solution of the Laplace equation with the same Dirichlet boundary conditions:

$$.\Delta u_0 = 0$$

$$u_0(x, y) = b(x, y) > 0 \text{ on the boundary of } \Omega.$$

This initial approximation cancels the term $-\Delta u_0$ in (13). Instead of solving (13) we can again rewrite it as a linear integral equation of Fredholm type and second kind by means of the Green's function $K(x, y, z, t)$ for Ω:

$$a_0(x, y) = 2 \iint_\Omega K(x, y, z, t) a_0(z, t) u_0(z, t)\, dz\, dt + \iint_\Omega K(x, y, z, t) u_0^2(z, t)\, dz\, dt. \tag{14}$$

If $\Omega = [0, 1] \times [0, 1]$ then:

$$K(x, y, z, t) = \frac{-4}{\pi^2} \sum_{\substack{j=1\\k=1}}^{\infty} \left[\frac{\sin k\pi x \sin j\pi y \sin k\pi z \sin j\pi t}{j^2 + k^2}\right] \simeq \frac{-4}{\pi^2} \sum_{\substack{j=1\\k=1}}^{n} [\ldots].$$

For $b(x, y) = 1$ the initial approximation $u_0(x, y) = 1$. We compute $a_0(x, y)$ by repeated substitution in (14), where we use the indicated approximation for $K(x, y, z, t)$:

$$a_0^{(0)}(x, y) = 0$$

$$a_0^{(1)}(x, y) = \frac{-16}{\pi^4} \sum_{\substack{j=1\\k=1\\j \text{ odd}\\k \text{ odd}}}^{n} \frac{\sin k\pi x \sin j\pi y}{(k^2 + j^2)kj} := \frac{-16}{\pi 4} \sum_{\substack{j=1\\k=1}}^{n}{}'' \frac{\sin k\pi x \sin j\pi y}{(k^2 + j^2)kj}.$$

The function $b_0(x, y)$ is the solution of:

$$b_0(x, y) = 2 \iint_\Omega K(x, y, z, t) b_0(z, t) u_0(z, t)\, dz\, dt - 2 \iint_\Omega K(x, y, z, t) a_0^2(z, t)\, dz\, dt$$

since $F''(u_0)a_0^2 = -2a_0^2$.

So for $b(x, y) = 1$ and $\Omega = [0, 1] \times [0, 1]$ we get:

$$b_0^{(0)}(x, y) = 0$$

$$b_0^{(1)}(x, y) = \frac{2^{15}}{\pi^{12}} \sum_{\substack{j,k=1\\l,m=1\\i,h=1}}^{n} \frac{ih \sin i\pi x \sin h\pi y}{(i^2 + h^2)(j^2 + k^2)(l^2 + m^2) P(i, k, l) P(h, j, m)}$$

where

$$P(i, k, l) = (i - k + l)(i + k - l)(i - k - l)(i + k + l)$$

Greenspan has proved that the solutions of the following finite systems which are the result of discretisation of (13), converge to the solution of $\Delta u = u^2$ with the given Dirichlet boundary conditions as the mesh size h approaches zero [8]:

Let

$$u_{ij} = u(x_i, y_j) = u(ih, jh)$$

construct $u_{ij}^{(k)}$ in terms of $u_{ij}^{(k-1)}$ as follows

$$u_{i,0}^{(k)} = u_{0,j}^{(k)} = u_{i,m}^{(k)} = u_{m,j}^{(k)} = 1 \text{ for } i = 0, \ldots, m;\ j = 0, \ldots, m;\ h = \frac{1}{m}$$

$$u_{ij}^{(0)} = 1$$

$$u_{ij}^{(k)}\left(-2u_{ij}^{(k-1)} - \frac{4}{h^2}\right) + \frac{1}{h^2}\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}\right) = -[u_{ij}^{(k-1)}]^2 \tag{15}$$

the procedure terminates when $\max_{i,j} |u_{ij}^{(k-1)} - u_{ij}^{(k)}| \leq \epsilon$ and this final $u_{ij}^{(k)}$ is defined to be the solution. We shall now compare the function-values of the different iteration steps $u_1(x, y)$ (Newton, Chebyshev, Halley) and the solution of (15) for $h = 1/100$ and $\epsilon = 5.(-9)$. For the calculation of $K(x, y, z, t)$ we have taken $n = 5$. The functions $u_1(x, y)$ all give the plot drawn in Fig. 6.1.

y \ x			
	Newton		
	0.25	0.50	0.75
0.25	0.954473792	0.942281229	0.954473792
0.50	0.942281229	0.925794323	0.942281229
0.75	0.954473792	0.942281229	0.954473792
	Chebyshev		
	0.25	0.50	0.75
0.25	0.954360724	0.942115745	0.954360724
0.50	0.942115745	0.925550785	0.942115745
0.75	0.954360724	0.942115269	0.954360724
	Halley		
	0.25	0.50	0.75
0.25	0.954360443	0.942115269	0.954360443
0.50	0.942115269	0.925549983	0.942115269
0.75	0.954360443	0.942115269	0.954360443
	h=1/100		
	0.25	0.50	0.75
0.25	0.958513709	0.947882237	0.958513709
0.50	0.947882192	0.933717325	0.947882192
0.75	0.958513647	0.947882149	0.958513647

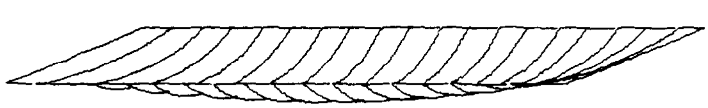

Fig. 6.1.

7. NONLINEAR INTEGRAL EQUATION OF FREDHOLM TYPE

A general nonlinear Fredholm integral equation may be written in the form:

$$F(y) = \int_a^b K(x, t, y(x), y(t))\, dt = 0 \text{ for } a \leq x \leq b.$$

We will treat the equation:

$$F(y) = y(x) - 1 - \frac{\lambda}{2} y(x) \int_0^1 \frac{x}{x+t} y(t)\,dt = 0 \qquad (16)$$

$$\text{for } 0 \le x \le 1 \text{ and } 0 \le \lambda \le 1$$

which was derived by Chandrasekhar [3].

If we write:

$$Ly = \int_0^1 \frac{x}{x+t} y(t)\,dt.$$

then:

$$F'(y_0)y = y - \frac{\lambda}{2}(y \cdot Ly_0 + y_0 \cdot Ly)$$

$$F''(y_0)y^2 = -\lambda y \cdot Ly.$$

For $y_0 = 1$ the Newton-correction is found by solving:

$$\left(1 - \frac{\lambda}{2} x \ln \frac{x+1}{x}\right) a_0(x) - \frac{\lambda}{2} \int_0^1 \frac{x}{x+t} a_0(t)\,dt = \frac{\lambda}{2} x \ln \frac{x+1}{x}$$

which can be converted in a linear integral equation of Fredholm type and second kind:

$$a_0(x) - \frac{\frac{\lambda}{2}}{1 - \frac{\lambda}{2} x \ln \frac{x+1}{x}} \int_0^1 \frac{x}{x+t} a_0(t)\,dt = \frac{\frac{\lambda}{2} x \ln \frac{x+1}{x}}{1 - \frac{\lambda}{2} x \ln \frac{x+1}{x}}.$$

The equation can be written in the form:

$$(I - \mathscr{L}) a_0(x) = \frac{\frac{\lambda}{2} x \ln \frac{x+1}{x}}{1 - \frac{\lambda}{2} x \ln \frac{x+1}{x}}$$

where

$$\mathscr{L}y = \int_0^1 \frac{\frac{\lambda}{2} x}{\left(1 - \frac{\lambda}{2} x \ln \frac{x+1}{x}\right)(x+t)} y(t)\,dt$$

Now

$$\|I - \mathscr{L}\| \le 1 + \frac{(\lambda/2) \ln 2}{1 - (\lambda/2) \ln 2} < 2$$

and so we can try to invert $I - \mathscr{L}$ as we previously did (cfr. boundary value problems).

Take again $K = I$. Then $I - K(I - \mathscr{L}) = \mathscr{L}$ with $\|\mathscr{L}\| \le \frac{\lambda \ln 2}{2 - \lambda \ln 2} < 1$

and

$$(I-\mathscr{L})^{-1}=\sum_{n=0}^{\infty}\mathscr{L}^n$$

The Newton-correction can be computed as follows:

$$a_0^{(0)}(x)=0$$

$$a_0^{(i)}(x)=\frac{\frac{\lambda}{2}x\ln\frac{x+1}{x}}{1-\frac{\lambda}{2}x\ln\frac{x+1}{x}}+\mathscr{L}a_0^{(i-1)}(x) \tag{17}$$

The correction $b_0(x)$ is calculated analogously:

$$b_0^{(0)}(x)=0$$

$$b_0^{(i)}(x)=\frac{-\lambda a_0(x)}{1-\frac{\lambda}{2}x\ln\frac{x+1}{x}}\int_0^1\frac{x}{x+t}a_0(t)\,dt+\mathscr{L}b_0^{(i-1)}(x) \tag{18}$$

where $a_0(x)$ is the last approximation $a_0^{(j)}(x)$ for the Newton-correction

If we take $j=1$ in both cases we get e.g.:

$$y_1(x)=1+\frac{\frac{\lambda}{2}x\ln\frac{x+1}{x}}{1-\frac{\lambda}{2}x\ln\frac{x+1}{x}}=1+a_0^{(1)}(x)\ \text{(Newton)}$$

$$y_1(x)=1+a_0^{(1)}(x)+\frac{\frac{\lambda}{2}a_0^{(1)}(x)}{1-\frac{\lambda}{2}x\ln\frac{x+1}{x}}\int_0^1\frac{x}{x+t}a_0^{(1)}(t)\,dt\ \text{(Chebyshev)}$$

$$y_1(x)=1+\frac{a_0^{(1)}(x)}{1-\frac{(\lambda/2)}{1-(\lambda/2)x\ln(x+1/x)}\int_0^1\frac{x}{x+t}a_0^{(1)}(t)\,dt}\ \text{(Halley).}$$

It has been proved [11] that the exact solution of (16) is:

$$y^*(x)=\exp\left(\frac{-x}{\pi}\int_0^{\pi/2}\frac{\ln(1-\lambda\theta\cot g\theta)}{x^2\sin^2\theta+\cos^2\theta}\,d\theta\right)0\le\lambda\le1 \tag{19}$$

Rall mentions the fact that $y_0(x)=1$ is a satisfactory initial approximation for the Newton-iteration only if ([10], p. 77):

$$0\le\lambda\le\frac{\sqrt{2}-1}{\ln 2}=0.59758\ldots.$$

For other λ we need other initial approximations. If we want to know the solutions $y^*(x)$ for $\lambda=(l/10)$ $(l=0,\ldots,10)$ we could use a tactic known as continuation: the solution for $\lambda=(l/10)$ is used as an initial approximation for the calculation of the solution for $\lambda=(l+1)/10$. Now for $\lambda=0$ the exact solution of (16) is $y^*(x)=1$.

For the computation of the integrals in (17) and (18) we have used the nine-point Gaussian integration rule ([1], p. 916):

$$\int_0^1 f(x)\,\mathrm{d}x \cong \sum_{k=1}^{9} w_k f(x_k)$$

where

	x_k
$k = 1$	0.0159198802461869
2	0.0819844463366821
3	0.1933142836497048
4	0.3378732882980955
5	0.5000000000000000
6	0.6621267117019045
7	0.8066857163502952
8	0.9180155536633179
9	0.9840801197538131

and w_k are the solution of the linear system

$$\sum_{k=1}^{9} x_k^{\,l-1} w_k = \frac{1}{l}(l = 1, \dots, 9).$$

This integration rule enables us to calculate $a_0^{(j)}(x_k)$ and $b_o^{(j)}(x_k)$ to the accuracy desired. It also enables us to calculate further iteration steps $y_{i+1}(x_k)$:

$$Ly_i(x_k) = x_k \sum_{l=1}^{9} \frac{w_l}{x_k + x_l}\, y_i(x_l) \quad k = 1, \dots, 9$$

$$F(y_i)(x_k) = -1 + y_i(x_k)\left(1 - \frac{\lambda}{2}\, Ly_i(x_k)\right)$$

$$a_i^{(0)}(x_k) = 0 \text{ and } b_i^{(0)}(x_k) = 0$$

$$a_i^{(j)}(x_k) = \frac{1 - y_i(x_k)\left(1 - \frac{\lambda}{2} Ly_i(x_k)\right)}{1 - \frac{\lambda}{2} Ly_i(x_k)} + \frac{\frac{\lambda}{2} y_i(x_k) x_k}{1 - \frac{\lambda}{2} Ly_i(x_k)} \left(\sum_{l=1}^{9} \frac{w_l}{x_k + x_l}\, a_i^{(j-1)}(x_l)\right)$$

to the desired accuracy and

$$b_i^{(j)}(x_k) = \frac{\frac{\lambda}{2} x_k}{1 - \frac{\lambda}{2} Ly_i(x_k)} \left[\sum_{l=1}^{9} \frac{w_l}{x_k + x_l} (-2a_i(x_k) a_i(x_l) + y_i(x_k) b_i^{(j-1)}(x_l))\right]$$

to the desired accuracy where $a_i(x_k)$ is the last approximation $a_i^{(j)}(x_k)$ to the Newton-correction.

We can continue the iteration until $\max_{k=1,\dots,9} |y_i(x_k) - y_{i-1}(x_k)| \le \epsilon$. We give the solution $y^*(x_k)$ for $\lambda = (l/10)(l = 1, \dots, 10)$ and the number of iteration steps for the different iterative procedures necessary to achieve convergence to 8 decimal digits ($\epsilon = 5.(-9)$).

In ([10], p. 78) Rall has approximated the integral equation (16) by:

$$\xi_k - 1 - \frac{\lambda}{2}\,\xi_k x_k \sum_{l=1}^{9} \frac{w_l \xi_l}{x_k + x_l} \quad \text{where} \quad \xi_k = y(x_k) \quad \text{for} \quad k = 1, \ldots, 9.$$

This fixed point problem can be solved by repeated substitution and the method of continuation. The number of iterations required now to obtain convergence to eight decimal places is also shown in the following table. We notice a significant difference. All the computations are performed in double precision accuracy (about 16 decimal digits).

Table 7.1.

$y^*(x_k)$

x_k	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$	$\lambda = 0.6$	$\lambda = 0.7$	$\lambda = 0.8$	$\lambda = 0.9$	$\lambda = 1.0$
k = 1	1.00336988	1.00687491	1.01053648	1.01438330	1.01845565	1.02281372	1.02755593	1.03286827	1.03921408	1.05118792
2	1.01089751	1.02250589	1.03495048	1.04840010	1.06309273	1.07938509	1.09786206	1.11963041	1.14744810	1.20857560
3	1.01829895	1.03817912	1.05995679	1.08406415	1.11111907	1.14206127	1.17845395	1.22329757	1.28417328	1.43721371
4	1.02435468	1.05121660	1.08114154	1.11490155	1.15362003	1.19903811	1.25411164	1.32464119	1.42564702	1.71373669
5	1.02892234	1.06117663	1.09755911	1.13919206	1.18773513	1.24580716	1.31794506	1.41326257	1.55603380	2.01277877
6	1.03220523	1.06840231	1.10959650	1.15722111	1.21342320	1.28164104	1.36793401	1.48472695	1.66599369	2.30601170
7	1.03445865	1.07339474	1.11797620	1.16988217	1.23165182	1.30739606	1.40445447	1.53811996	1.75110922	2.56462107
8	1.03589121	1.07658239	1.12335356	1.17805495	1.24350201	1.32428633	1.42867803	1.57409995	1.80995660	2.76255967
9	1.03664375	1.07826123	1.12619412	1.18238740	1.24981062	1.33332581	1.44173217	1.59367978	1.84249994	2.87963509
Newton	3	3	3	4	4	4	4	4	5	17
Chebyshev	3	3	3	3	3	3	3	3	4	13
Halley	3	3	3	3	3	3	3	3	3	12
Fixed point	7	9	10	12	15	18	22	29	45	14293

For the calculation of $y^*(x_k)$ for a chosen λ, (19) has been rewritten as follows[11] to remove singularities in the integrand for small x and great λ:

$$y^*(x) = \exp(z^*(x))$$

with

$$z^*(x) = \frac{1}{\pi}\int_0^{\pi/2} [f(\theta) - g(\theta) + h(\theta)]\, d\theta + z_2(x) - z_3(x)$$

where

$$f(\theta) = \lambda \text{ Arctg}\,(xtg\,\theta)\,\frac{\theta \text{ cose } c^2\theta - \text{cotg}\,\theta}{1 - \lambda\theta\,\text{cotg}\,\theta}$$

$$g(\theta) = \frac{\pi}{2}\lambda \text{ Arctg}\,(x \text{ tg}\,\theta)$$

$$h(\theta) = \frac{2x(1-\lambda)}{1 - \lambda + \frac{1}{3}\lambda\theta^2}$$

$$z_2(x) = \begin{cases} \frac{1}{2}\lambda\left\{\frac{\pi^2}{8} - \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(\frac{1-x}{1+x}\right)^{2n+1}\right\} & \text{for } 1 \geq x > \sqrt{2} - 1 \\ x\frac{1}{2}\lambda\left\{\frac{1}{2}\ln \times \ln\frac{1-x}{1+x} + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^2}\right\} & \text{for } 0 \leq x \leq \sqrt{2} - 1 \end{cases}$$

$$z_3(x) = \frac{2x}{\pi}\sqrt{\left(\frac{3(1-\lambda)}{\lambda}\right)}\text{Arctg}\left(\frac{\pi}{2}\sqrt{\left(\frac{\lambda}{3(1-\lambda)}\right)}\right).$$

The convergence to 8 decimal places of the different methods of approximation does not imply that those 8 digits are significant digits for $y^*(x_k)$.

For small x_k and great λ the iterative methods do not converge to $y^*(x_k)$ but to a function in the neighbourhood of $y^*(x_k)$.

Let us denote by $y_I(x_k)$ the solution obtained by performing one of the iterative procedures Newton, Chebyshev or Halley (for each iterative procedure after a different number of iteration steps) and let us denote by $y_F(x_k)$ the solution obtained after rewriting (16) as a fixed point problem.

In the following table one can find $|y^*(x_k) - y_I(x_k)|$ and $|y^*(x_k) - y_F(x_k)|$ for $k = 1, \ldots, 9$ and $\lambda = 0.1, \ldots, 1.0$. For small $x_k (k = 1, 2)$ generally $y_F(x_k) \leq y^*(x_k) \leq y_I(x_k)$.

Table 7.2

$|y^*(x_k) - y_I(x_k)|$ $(\varepsilon = 5.(-9))$

k	λ=0.1	λ=0.2	λ=0.3	λ=0.4	λ=0.5	λ=0.6	λ=0.7	λ=0.8	λ=0.9	λ=1.0
1	3.7(-5)	7.5(-5)	1.1(-4)	1.5(-4)	1.9(-4)	2.3(-4)	2.7(-4)	3.0(-4)	3.5(-4)	3.9(-4)
2	5.0(-7)	9.4(-7)	1.3(-6)	1.6(-6)	1.9(-6)	2.0(-6)	2.1(-6)	2.1(-6)	2.1(-6)	7.1(-6)
3	8.9(-9)	4.7(-8)	1.2(-7)	2.2(-7)	3.6(-7)	5.4(-7)	7.7(-7)	1.0(-6)	1.4(-6)	1.2(-5)
4	7.8(-9)	3.2(-8)	7.6(-8)	1.4(-7)	2.2(-7)	3.3(-7)	4.8(-7)	6.6(-7)	9.0(-7)	2.8(-5)
5	≤ε	2.2(-8)	5.1(-8)	9.5(-8)	1.5(-7)	2.3(-7)	3.4(-7)	4.7(-7)	6.5(-7)	5.1(-5)
6	≤ε	1.7(-8)	3.9(-8)	7.3(-8)	1.2(-7)	1.8(-7)	2.6(-7)	3.7(-7)	5.3(-7)	7.8(-5)
7	≤ε	1.4(-8)	3.2(-8)	6.0(-8)	9.9(-8)	1.5(-7)	2.2(-7)	3.2(-7)	4.5(-7)	1.1(-4)
8	≤ε	1.2(-8)	2.9(-8)	5.3(-8)	8.8(-8)	1.3(-7)	2.0(-7)	2.8(-7)	4.1(-7)	1.3(-4)
9	≤ε	1.1(-8)	2.7(-8)	5.0(-8)	8.2(-8)	1.3(-7)	1.9(-7)	2.7(-7)	3.9(-7)	1.5(-4)

$|y^*(x_k) - y_F(x_k)|$ $(\varepsilon=5.(-9))$

k	λ=0.1	λ=0.2	λ=0.3	λ=0.4	λ=0.5	λ=0.6	λ=0.7	λ=0.8	λ=0.9	λ=1.0
1	3.7(-5)	7.5(-5)	1.1(-4)	1.5(-4)	1.9(-4)	2.3(-4)	2.7(-4)	3.1(-4)	3.5(-4)	3.9(-4)
2	5.0(-7)	9.4(-7)	1.3(-6)	1.6(-6)	1.9(-6)	2.0(-6)	2.1(-6)	2.1(-6)	2.1(-6)	9.7(-6)
3	8.8(-9)	4.7(-8)	1.2(-7)	2.2(-7)	3.6(-7)	5.4(-7)	7.7(-7)	1.0(-6)	1.4(-6)	2.0(-5)
4	7.8(-9)	3.2(-8)	7.5(-8)	1.4(-7)	2.2(-7)	3.3(-7)	4.8(-7)	6.6(-7)	8.9(-7)	4.4(-5)
5	5.3(-9)	2.2(-8)	5.1(-8)	9.4(-8)	1.5(-7)	2.3(-7)	3.3(-7)	4.7(-7)	6.5(-7)	7.8(-5)
6	≤ε	1.7(-8)	3.8(-8)	7.2(-8)	1.2(-7)	1.8(-7)	2.6(-7)	3.7(-7)	5.2(-7)	1.2(-4)
7	≤ε	1.4(-8)	3.2(-8)	5.9(-8)	9.8(-8)	1.5(-7)	2.2(-7)	3.1(-7)	4.5(-7)	1.6(-4)
8	≤ε	1.2(-8)	2.8(-8)	5.2(-8)	8.7(-8)	1.3(-7)	1.9(-7)	2.8(-7)	4.0(-7)	2.0(-4)
9	≤ε	1.1(-8)	2.6(-8)	4.8(-8)	8.1(-8)	1.2(-7)	1.8(-7)	2.6(-7)	3.8(-7)	2.2(-4)

Only for $\lambda = 1.0$ one notices slight differences.

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