BIT 28 (1988),98 - 112

A MULTIVARIATE QD-LIKE ALGORITHM

ANNIE CUYT*)
Department of mathematics and computer science, Universiteit Antwerpen (UIA)
Universiteitsplein 1, B–2610 Wilrijk, Belgium

Abstract.

The problem of constructing a univariate rational interpolant or Padé approximant for given data can be solved in various equivalent ways: one can compute the explicit solution of the system of interpolation or approximation conditions, or one can start a recursive algorithm, or one can obtain the rational function as the convergent of an interpolating or corresponding continued fraction.

In case of multivariate functions general order systems of interpolation conditions for a multivariate rational interpolant and general order systems of approximation conditions for a multivariate Padé approximant were respectively solved in [6] and [9]. Equivalent recursive computation schemes were given in [3] for the rational interpolation case and in [5] for the Padé approximation case. At that moment we stated that the next step was to write the general order rational interpolants and Padé approximants as the convergent of a multivariate continued fraction so that the univariate equivalence of the three main defining techniques was also established for the multivariate case: algebraic relations, recurrence relations, continued fractions. In this paper a multivariate qd-like algorithm is developed that serves this purpose.

AMS Subject classifications : 65D05, 41A21.

1. Algebraic relations and Recurrence relations.

Let us restrict everything to the case of two variables for the sake of simplicity. Furthermore we assume that the finite interpolation set $I = \{(i,j) | f_{ij}$ is given at $(x_i, y_j)\}$ is structured so that it satisfies the inclusion property. This means that if a point belongs to the data set, then the rectangular subset of points

Received October 1986. Revised November 1987.

*) Senior Research Assistant NFWO

emanating from the origin with the given point as its furthermost corner also lies in the data set. How this can be achieved in a lot of situations is explained in [6]. If none of the points in $\{(x_i, y_j)\}_{(i,j)\in I}$ coincide then we are dealing with a rational interpolation problem and the values in $\{f_{ij}\}_{(i,j)\in I}$ are function values. If all the interpolation points coincide then the problem is one of Padé approximation and it is well-known that the given data are not function values but Taylor coefficients. If some of the points coincide and some do not then the problem is of a mixed type and it is called a Hermite interpolation problem or a Newton-Padé approximation problem. In [6] is indicated how one should interpret the data f_{ij}: some of them are derivatives and some of them are function values. In the sequel of the text it does not play a role whether one is dealing with coalescent points or not since all the formulas remain valid in both cases and in the mixed case. Nevertheless we shall sometimes indicate how the formulas are to be read if some of the interpolation points coincide.

Consider the following set of basis functions for the real-valued polynomials in two variables

$$B_{ij}(x,y) = \prod_{k=0}^{i-1}(x - x_k)\prod_{\ell=0}^{j-1}(y - y_\ell).$$

Clearly $B_{ij}(x,y)$ is a bivariate polynomial of degree $i+j$. Given the f_{ij}, we can write in a purely formal manner

$$f(x,y) = \sum_{(i,j)\in \mathbb{N}^2} f_{0i,0j}B_{ij}(x,y)$$

where $f_{0i,0j}$ are the bivariate divided differences

$$f_{0i,0j} = f\,[x_0,\ldots,x_i]\,[y_0,\ldots,y_j]$$

given by

$$f\,[x_0,\ldots,x_i]\,[y_0,\ldots,y_j] = \frac{f\,[x_1,\ldots,x_i]\,[y_0,\ldots,y_j] - f\,[x_0,\ldots,x_{i-1}]\,[y_0,\ldots,y_j]}{x_i - x_0}$$

or

$$f\,[x_0,\ldots,x_i]\,[y_0,\ldots,y_j] = \frac{f\,[x_0,\ldots,x_i]\,[y_1,\ldots,y_j] - f\,[x_0,\ldots,x_i]\,[y_0,\ldots,y_{j-1}]}{y_j - y_0}$$

with

$$f\,[x_i]\,[y_j] = f_{ij}.$$

Divided differences with coalescent points $x_i,\ldots,x_{i+k}$ and $y_j,\ldots,y_{j+\ell}$ are given by

$$f\,[x_i,\ldots,x_{i+k}]\,[y_j] = \frac{1}{k!}\frac{\partial^k f}{\partial x^k}_{|(x_i,y_j)}$$

and

$$f[x_i][y_j,\ldots,y_{j+\ell}] = \frac{1}{\ell!}\frac{\partial^\ell f}{\partial y^\ell}_{|(x_i,y_j)}.$$

In order to construct rational interpolants or Padé approximants for the given set I we choose two finite index sets N, a subset of I, and D, a subset of $\mathbb{N}^2$, which determine the "degree" of the numerator and denominator and we put as in [6]

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}B_{ij}(x,y) \qquad N \text{ from "numerator"}$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}B_{ij}(x,y) \qquad D \text{ from "denominator"}$$

(1)
$$(fq-p)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus I} c_{ij}B_{ij}(x,y)$$

I from "interpolation conditions"

The rational interpolant $(p/q)(x,y)$ will then be denoted by

$$[N/D]_I.$$

Let us introduce a numbering $r(i,j)$ of the points in $\mathbb{N}^2$ based on the enumeration

$$(0,0),\ \underbrace{(1,0),(0,1),}_{\text{first diagonal}}\ \underbrace{(2,0),(1,1),(0,2),}_{\text{second diagonal}}\underbrace{(3,0),(2,1),(1,2),(0,3)}_{\text{third diagonal}},\ldots$$

so that

$$r(i,j) = \frac{(i+j)(i+j+1)}{2} + j - i.$$

If we denote $\#N = n+1$ then we can write

$$N = \bigcup_{\ell=0}^{n} N_\ell$$

with

$$\emptyset = N_{-1} \subset N_0 \subset N_1 \subset \ldots \subset N_{n-1} \subset N_n = N$$

$$\#N_\ell = \ell + 1$$

$$N_\ell \setminus N_{\ell-1} = \{(i_\ell, j_\ell)\}; \qquad \ell = 0,\ldots,n$$

$$r(i_\ell, j_\ell) > r(i_r, j_r); \qquad \ell > r.$$

In other words, for each $\ell = 0,\ldots,n$ we add to $N_{\ell-1}$ the point (i_ℓ, j_ℓ) which is the next in line in $N \cap \mathbb{N}^2$ according to the enumeration given above. Denote $\#D = m+1$ and proceed in the same way. Then

$$D = \bigcup_{\ell=0}^{m} D_\ell$$

with

$$D_{-1} = \emptyset, \qquad \#D_\ell = \ell+1, \qquad D_\ell \setminus D_{\ell-1} = \{(d_\ell, e_\ell)\}\,; \qquad \ell = 0,\ldots,m.$$

Since (1) can be rewritten as

$$\text{(2)} \qquad \begin{aligned} (fq)_{0i,0j} = p_{0i,0j} = a_{ij}, &\qquad (i,j) \in N \\ (fq)_{0i,0j} = 0, &\qquad (i,j) \in I \setminus N \end{aligned}$$

we will assume that the interpolation set I is such that exactly m of the homogeneous equations (2) are linearly independent. Degenerate cases can be avoided by adding interpolation data to the set I until the rank of (2) is equal to m. It is obvious that this condition guarantees the existence of a nontrivial solution of (2) given by the following determinant expressions, because the number of unknowns in the homogeneous system is now one more than its rank. We group the respective m elements of $I \setminus N$ that supply the linearly independent equations in the set H and number them also following the enumeration given above,

$$H = \bigcup_{\ell=1}^{m} H_\ell \subseteq I \setminus N$$

with

$$H_0 = \emptyset, \qquad \#H_\ell = \ell, \qquad H_\ell \setminus H_{\ell-1} = \{(h_\ell, k_\ell)\}\,; \qquad \ell = 1,\ldots,m.$$

The polynomials $p(x,y)$ and $q(x,y)$ satisfying (1) are then given by [6]

$$\text{(3a)} \quad p(x,y) = \begin{vmatrix} \sum_{(i,j)\in N} f_{d_0 i, e_0 j} B_{ij}(x,y) & \cdots & \sum_{(i,j)\in N} f_{d_m i, e_m j} B_{ij}(x,y) \\ f_{d_0 h_1, e_0 k_1} & \cdots & f_{d_m h_1, e_m k_1} \\ \vdots & & \vdots \\ f_{d_0 h_m, e_0 k_m} & \cdots & f_{d_m h_m, e_m k_m} \end{vmatrix}$$

$$\text{(3b)} \qquad q(x,y) = \begin{vmatrix} B_{d_0 e_0}(x,y) & \cdots & B_{d_m e_m}(x,y) \\ f_{d_0 h_1, e_0 k_1} & \cdots & f_{d_m h_1, e_m k_1} \\ \vdots & & \vdots \\ f_{d_0 h_m, e_0 k_m} & \cdots & f_{d_m h_m, e_m k_m} \end{vmatrix}$$

where

$$f_{d_i h_j, e_i k_j} = f\left[x_{d_i}, \ldots, x_{h_j}\right]\left[y_{e_i}, \ldots, y_{k_j}\right]$$

with

$$f_{d_i h_j, e_i k_j} = 0 \quad \text{if} \quad d_i > h_j \quad \text{or} \quad e_i > k_j.$$

In [5] these determinant formulas are given when all the interpolation points coincide and a lot of specific choices for N, D and I are described. In [4] is illustrated that the covariance properties satisfied by these multivariate Padé approximants are determined by the structure of the index sets N, D and I.

The formulas (3) can be rewritten so that they can be computed recursively. Multiplying the $(\ell+1)$th row in $p(x,y)$ and $q(x,y)$ by $B_{h_\ell k_\ell}(x,y)$ $(\ell = 1,\ldots,m)$, and dividing the $(\ell+1)$th column by $B_{d_\ell e_\ell}(x,y)$ $(\ell = 0,\ldots,m)$ results in

$$p(x,y) = \begin{vmatrix} \sum_{(i,j)\in N} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y) & \cdots & \sum_{(i,j)\in N} f_{d_m i, e_m j} B_{d_m i, e_m j}(x,y) \\ f_{d_0 h_1, e_0 k_1} B_{d_0 h_1, e_0 k_1}(x,y) & \cdots & f_{d_m h_1, e_m k_1} B_{d_m h_1, e_m k_1}(x,y) \\ \vdots & & \vdots \\ f_{d_0 h_m, e_0 k_m} B_{d_0 h_m, e_0 k_m}(x,y) & \cdots & f_{d_m h_m, e_m k_m} B_{d_m h_m, e_m k_m}(x,y) \end{vmatrix}$$

(4a)

$$q(x,y) = \begin{vmatrix} 1 & \cdots & 1 \\ f_{d_0 h_1, e_0 k_1} B_{d_0 h_1, e_0 k_1}(x,y) & \cdots & f_{d_m h_1, e_m k_1} B_{d_m h_1, e_m k_1}(x,y) \\ \vdots & & \vdots \\ f_{d_0 h_m, e_0 k_m} B_{d_0 h_m, e_0 k_m}(x,y) & \cdots & f_{d_m h_m, e_m k_m} B_{d_m h_m, e_m k_m}(x,y) \end{vmatrix}$$

(4b)

where for $k \le i$ and $\ell \le j$

$$B_{ki,\ell j}(x,y) = \frac{B_{ij}(x,y)}{B_{k\ell}(x,y)} = (x-x_k)\ldots(x-x_{i-1})(y-y_\ell)\ldots(y-y_{j-1})$$

and for $k > i$ or $\ell > j$

$$f_{ki,lj} = 0.$$

For such a quotient of determinants the E-algorithm is particularly suitable [3]:

$$E_0^{(\ell)} = \sum_{(i,j)\in N_\ell} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y); \qquad \ell = 0,\ldots,n+m$$

$$g_{0,r}^{(\ell)} = \sum_{(i,j)\in N_\ell} f_{d_r i, e_r j} B_{d_r i, e_r j}(x,y) - \sum_{(i,j)\in N_\ell} f_{d_{r-1} i, e_{r-1} j} B_{d_{r-1} i, e_{r-1} j}(x,y)$$
$$r = 1,\ldots,m; \qquad \ell = 0,\ldots,n+m$$

$$\text{(5a)} \quad E_r^{(\ell)} = \frac{E_{r-1}^{(\ell)} g_{r-1,r}^{(\ell+1)} - E_{r-1}^{(\ell+1)} g_{r-1,r}^{(\ell)}}{g_{r-1,r}^{(\ell+1)} - g_{r-1,r}^{(\ell)}}; \qquad \ell = 0,1,\ldots,n; \qquad r = 1,2,\ldots,m$$

$$\text{(5b)} \quad g_{r,s}^{(\ell)} = \frac{g_{r-1,s}^{(\ell)} g_{r-1,r}^{(\ell+1)} - g_{r-1,s}^{(\ell+1)} g_{r-1,r}^{(\ell)}}{g_{r-1,r}^{(\ell+1)} - g_{r-1,r}^{(\ell)}}; \qquad s = r+1, r+2, \ldots$$

The values $E_r^{(\ell)}$ and $g_{r,s}^{(\ell)}$ are stored as indicated below.

Table 1.

$$\begin{array}{ccccc}
E_0^{(0)} & & & & \\
 & E_1^{(0)} & & & \\
E_0^{(1)} & & \ddots & & \\
 & E_1^{(1)} & E_m^{(0)} & & \\
E_0^{(2)} & \vdots & \vdots & \ddots & \\
\vdots & & & & E_{n+m}^{(0)} \\
 & & E_m^{(n)} & & \\
 & E_1^{(n+m-1)} & & & \\
E_0^{(n+m)} & & & &
\end{array}$$

Table 2.

$$\begin{array}{c|cc|c|ccc|c|cc}
g_{0,1}^{(0)} & g_{0,2}^{(0)} & & & g_{0,r}^{(0)} & & & & g_{0,m}^{(0)} & \\
 & & g_{1,2}^{(0)} & & & g_{1,r}^{(0)} & & & & \ddots \\
g_{0,1}^{(1)} & g_{0,2}^{(1)} & & & g_{0,r}^{(1)} & & \ddots & & g_{0,m}^{(1)} & \\
 & & g_{1,2}^{(1)} & & & g_{1,r}^{(1)} & g_{r-1,r}^{(0)} & & \vdots & \\
g_{0,1}^{(2)} & g_{0,2}^{(2)} & \vdots & \cdots & g_{0,r}^{(2)} & \vdots & \vdots & \cdots & & \\
\vdots & \vdots & & & \vdots & & g_{r-1,r}^{(n+m-r+1)} & & & \\
 & & & & & \iddots & & & & \\
 & & g_{1,2}^{(n+m-1)} & & & g_{1,r}^{(n+m-1)} & & & & \iddots \\
g_{0,1}^{(n+m)} & g_{0,2}^{(n+m)} & & & g_{0,r}^{(n+m)} & & & & g_{0,m}^{(n+m)} &
\end{array}$$

As a result of these computations

$$[N/D]_I = E_m^{(n)}.$$

Since the solution $p(x,y)/q(x,y)$ of (2) is unique due to fact that the rank of (2) is m, the value $E_m^{(n)}$ itself does not depend upon the numbering of the points within the sets N, D and H. But this numbering affects the interpolation conditions satisfied by the intermediate E-values. For $\ell = 0, \ldots, n$ and $r = 0, \ldots, m$ [3]

$$E_r^{(\ell)} = [N_\ell/D_r]_{N_\ell \cup \underbrace{\{(i_{\ell+1}, j_{\ell+1}), \ldots, (i_n, j_n), (h_1, k_1), \ldots, (h_{r-n+\ell}, k_{r-n+\ell})\}}_{r \text{ points}}}.$$

2. Continued fraction representation and the qdg-algorithm.

Let us now suppose for the sake of simplicity that the homogeneous system of equations (2) has maximal rank, in other words $H = I\backslash N$. As a consequence we have

$$\#I = n + m + 1.$$

Hence we can write

$$I = \bigcup_{\ell=0}^{n+m} I_\ell$$

with

$$\begin{aligned} I_\ell &= N_\ell; && \ell = 0, \dots, n \\ I_{n+\ell}\backslash I_{n+\ell-1} &= \{(i_{n+\ell}, j_{n+\ell})\}; && \ell = 1, \dots, m \\ r(i_{n+\ell}, j_{n+\ell}) &> r(i_r, j_r); && n + \ell > r \geq n + 1 \end{aligned}$$

With the subsets N_ℓ, D_r and $I_{\ell+r}$ rational interpolants

$$[N_\ell / D_r]_{I_{\ell+r}}$$

can be constructed which satisfy only part of the interpolation conditions and which are of lower "degree". To this end we assume that the numbering $r(i_r, j_r)$ of the points in $\mathbb{N}^2$ is such that the inclusion property of the set I is carried over to the subsets I_ℓ. We can now fill a table with rational interpolants or Padé approximants.

Table 3.

$$\begin{array}{cccc} [N_0/D_0]_{I_0} & [N_0/D_1]_{I_1} & [N_0/D_2]_{I_2} & \dots \\ [N_1/D_0]_{I_1} & [N_1/D_1]_{I_2} & [N_1/D_2]_{I_3} & \dots \\ [N_2/D_0]_{I_2} & [N_2/D_1]_{I_3} & [N_2/D_2]_{I_4} & \dots \\ \vdots & \vdots & \vdots & \end{array}$$

where

$$[N/D]_I = [N_n/D_m]_{I_{n+m}}.$$

Our aim is to consider descending staircases in this table of multivariate rational functions:

$$(6) \qquad \begin{array}{cccc} [N_s/D_0]_{I_s} & & & \\ [N_{s+1}/D_0]_{I_{s+1}} & [N_{s+1}/D_1]_{I_{s+2}} & & \\ & [N_{s+2}/D_1]_{I_{s+3}} & [N_{s+2}/D_2]_{I_{s+4}} & \\ & & \vdots & \dots \end{array}$$

and construct continued fractions of which the ℓth convergent equals the ℓth interpolant on the staircase. We restrict ourselves to the case where every three successive elements in (6) are different. It is well-known that a continued fraction of which the ℓth convergent is the ℓth element of a given sequence $\{C_\ell\}_{\ell\in N}$ with every three successive elements different from each other, is given by

$$C_0 + \frac{C_1 - C_0|}{|1} + \sum_{\ell=2}^{\infty} \frac{\left|\dfrac{C_{\ell-1}-C_\ell}{C_{\ell-1}-C_{\ell-2}}\right.}{\left|\dfrac{C_\ell - C_{\ell-2}}{C_{\ell-1}-C_{\ell-2}}\right.}.$$

Let us compute the partial numerators and denominators of this continued fraction for the elements

$$C_{\ell+r} = [N_{\ell+s}/D_r]_{I_{\ell+r+s}}, \qquad s \geq 0; \qquad \ell + r = 0, 1, 2, \ldots$$

on the descending staircase (6). In the notation of the previous section we already have

$$C_0 = \sum_{(i,j)\in N_s} f_{d_{0i},e_{0j}} B_{d_{0i},e_{0j}}(x,y)$$

$$C_1 - C_0 = \sum_{(i,j)\in N_{s+1}} f_{d_{0i},e_{0j}} B_{d_{0i},e_{0j}}(x,y) - \sum_{(i,j)\in N_s} f_{d_{0i},e_{0j}} B_{d_{0i},e_{0j}}(x,y) \cdot$$

$$= f_{d_{0i_{s+1}},e_{0j_{s+1}}} B_{d_{0i_{s+1}},e_{0j_{s+1}}}(x,y)$$

We shall now distinguish between even and odd numerators and denominators. For this purpose we introduce the notations

$$-q_\ell^{(s+1)} = \frac{C_{2\ell-1} - C_{2\ell}}{C_{2\ell-1} - C_{2\ell-2}}$$

$$-e_\ell^{(s+1)} = \frac{C_{2\ell} - C_{2\ell+1}}{C_{2\ell} - C_{2\ell-1}}$$

for the partial numerators. Consequently we can write for the partial denominators

$$1 + q_\ell^{(s+1)} = \frac{C_{2\ell} - C_{2\ell-2}}{C_{2\ell-1} - C_{2\ell-2}}$$

$$1 + e_\ell^{(s+1)} = \frac{C_{2\ell+1} - C_{2\ell-1}}{C_{2\ell} - C_{2\ell-1}}.$$

In $q_\ell^{(s+1)}$ the convergents

$$\begin{matrix} \ldots & C_{2\ell-2} & \\ & C_{2\ell-1} & C_{2\ell} \\ & & \vdots \end{matrix}$$

of (6) are involved, in other words the rational interpolants

$$\begin{array}{ccc} \cdots & [N_{\ell+s-1}/D_{\ell-1}]_{I_{2\ell+s-2}} & \\ & [N_{\ell+s}/D_{\ell-1}]_{I_{2\ell+s-1}} & [N_{\ell+s}/D_{\ell}]_{I_{2\ell+s}} \\ & & \vdots \end{array}$$

or, in the notation of the previous section,

$$\begin{array}{ccc} \cdots & E_{\ell-1}^{(s+\ell-1)} & \\ & E_{\ell-1}^{(s+\ell)} & E_{\ell}^{(s+\ell)} \\ & & \vdots \end{array}$$

Hence, by using (5a)

$$\begin{aligned} q_\ell^{(s+1)} = \frac{C_{2\ell}-C_{2\ell-1}}{C_{2\ell-1}-C_{2\ell-2}} &= \frac{E_\ell^{(s+\ell)}-E_{\ell-1}^{(s+\ell)}}{E_{\ell-1}^{(s+\ell)}-E_{\ell-1}^{(s+\ell-1)}} \\ &= \frac{(E_{\ell-1}^{(s+\ell)}-E_{\ell-1}^{(s+\ell+1)})}{(E_{\ell-1}^{(s+\ell)}-E_{\ell-1}^{(s+\ell-1)})} \; \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell+1)}-g_{\ell-1,\ell}^{(s+\ell)}}. \end{aligned} \tag{7}$$

In $e_\ell^{(s+1)}$ the convergents

$$\begin{array}{ccc} \vdots & & \\ C_{2\ell-1} & C_{2\ell} & \\ & C_{2\ell+1} & \cdots \end{array}$$

of (6) are involved, in other words the rational interpolants

$$\begin{array}{ccc} \vdots & & \\ [N_{\ell+s}/D_{\ell-1}]_{I_{2\ell+s-1}} & [N_{\ell+s}/D_\ell]_{I_{2\ell+s}} & \\ & [N_{\ell+1+s}/D_\ell]_{I_{2\ell+1+s}} & \cdots \end{array}$$

or the values

$$\begin{array}{ccc} \vdots & & \\ E_{\ell-1}^{(s+\ell)} & E_\ell^{(s+\ell)} & \\ & E_\ell^{(s+\ell+1)} & \cdots \end{array}$$

In this way we get

$$e_\ell^{(s+1)} = \frac{C_{2\ell+1}-C_{2\ell}}{C_{2\ell}-C_{2\ell-1}} = \frac{E_\ell^{(s+\ell+1)}-E_\ell^{(s+\ell)}}{E_\ell^{(s+\ell)}-E_{\ell-1}^{(s+\ell)}} \tag{8}$$

Combining (7) and (8) we find for $\ell \geq 2$

$$
\begin{aligned}
q_\ell^{(s+1)} &= e_{\ell-1}^{(s+2)} \; \frac{(E_{\ell-2}^{(s+\ell)} - E_{\ell-1}^{(s+\ell)})}{(E_{\ell-1}^{(s+\ell)} - E_{\ell-1}^{(s+\ell-1)})} \; \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell+1)} - g_{\ell-1,\ell}^{(s+\ell)}} \\
&= -e_{\ell-1}^{(s+2)} q_{\ell-1}^{(s+2)} \; \frac{(E_{\ell-2}^{(s+\ell)} - E_{\ell-2}^{(s+\ell-1)})}{(E_{\ell-1}^{(s+\ell)} - E_{\ell-1}^{(s+\ell-1)})} \; \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell+1)} - g_{\ell-1,\ell}^{(s+\ell)}} \\
&= \frac{-e_{\ell-1}^{(s+2)} q_{\ell-1}^{(s+2)}}{e_{\ell-1}^{(s+1)}} \; \frac{(E_{\ell-2}^{(s+\ell)} - E_{\ell-2}^{(s+\ell-1)})}{(E_{\ell-1}^{(s+\ell-1)} - E_{\ell-2}^{(s+\ell-1)})} \; \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell+1)} - g_{\ell-1,\ell}^{(s+\ell)}} \\
&= \frac{e_{\ell-1}^{(s+2)} q_{\ell-1}^{(s+2)}}{e_{\ell-1}^{(s+1)}} \; \frac{g_{\ell-2,\ell-1}^{(s+\ell)} - g_{\ell-2,\ell-1}^{(s+\ell-1)}}{g_{\ell-2,\ell-1}^{(s+\ell-1)}} \; \frac{g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell+1)} - g_{\ell-1,\ell}^{(s+\ell)}} \qquad (9)
\end{aligned}
$$

and for $\ell \geq 1$

$$
\begin{aligned}
e_\ell^{(s+1)} + 1 &= \frac{E_\ell^{(s+\ell+1)} - E_{\ell-1}^{(s+\ell)}}{E_\ell^{(s+\ell)} - E_{\ell-1}^{(s+\ell)}} \\
&= -\frac{g_{\ell-1,\ell}^{(s+\ell+1)} - g_{\ell-1,\ell}^{(s+\ell)}}{g_{\ell-1,\ell}^{(s+\ell)}} \left(q_\ell^{(s+2)} + 1\right) \qquad (10)
\end{aligned}
$$

If we arrange the values $q_\ell^{(s+1)}$ and $e_\ell^{(s+1)}$ in a table as follows

Table 4.

$$
\begin{array}{ccccccc}
q_1^{(1)} & & & & & & \\
 & e_1^{(1)} & & & & & \\
q_1^{(2)} & & q_2^{(1)} & & & & \\
 & e_1^{(2)} & & e_2^{(1)} & & & \\
q_1^{(3)} & & q_2^{(2)} & & \ddots & & \\
 & e_1^{(3)} & & e_2^{(2)} & & & \\
q_1^{(4)} & & q_2^{(3)} & & \ddots & & \\
\vdots & e_1^{(4)} & \vdots & e_2^{(3)} & & & \\
 & \vdots & & \vdots & & &
\end{array}
$$

where subscripts indicate columns and superscripts indicate downward sloping diagonals, then (9) links the elements in the rhombus

$$
\begin{array}{ccc}
 & e_{\ell-1}^{(s+1)} & \\
q_{\ell-1}^{(s+2)} & & q_\ell^{(s+1)} \\
 & e_{\ell-1}^{(s+2)} &
\end{array}
$$

and (10) links two elements on an upward sloping diagonal

$$\begin{matrix} & e_\ell^{(s+1)} \\ q_\ell^{(s+2)} & \end{matrix}$$

If starting values for $q_\ell^{(s+1)}$ were known, all the values in the multivariate *qd*-table could be computed. These starting values are given by

(11)

$$q_1^{(s+1)} = \frac{E_1^{(s+1)} - E_0^{(s+1)}}{E_0^{(s+1)} - E_0^{(s)}} = \frac{-f_{d_0 i_{s+2}, e_0 j_{s+2}} B_{d_0 i_{s+2}, e_0 j_{s+2}}(x,y)}{f_{d_0 i_{s+1}, e_0 j_{s+1}} B_{d_0 i_{s+1}, e_0 j_{s+1}}(x,y)} \frac{g_{0,1}^{(s+1)}}{g_{0,1}^{(s+2)} - g_{0,1}^{(s+1)}}$$

Finally, we can say that, given a descending staircase (6) of different elements, it is possible to construct a continued fraction of the form

$$\begin{aligned} &[N_s/D_0]_{I_s} + \left|\frac{[N_{s+1}/D_0]_{I_{s+1}} - [N_s/D_0]_{I_s}}{1}\right| + \left|\frac{-q_1^{(s+1)}}{1+q_1^{(s+1)}}\right| + \left|\frac{-e_1^{(s+1)}}{1+e_1^{(s+1)}}\right| + \\ (12)\qquad & \left|\frac{-q_2^{(s+1)}}{1+q_2^{(s+1)}}\right| + \left|\frac{-e_2^{(s+1)}}{1+e_2^{(s+1)}}\right| + \dots \end{aligned}$$

of which the successive convergents equal the successive elements on the descending staircase (6). Here

$$[N_s/D_0]_{I_s} = \sum_{(i,j)\in N_s} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y)$$

$$[N_{s+1}/D_0]_{I_{s+1}} = \sum_{(i,j)\in N_{s+1}} f_{d_0 i, e_0 j} B_{d_0 i, e_0 j}(x,y)$$

and the coefficients $q_\ell^{(s+1)}$ and $e_\ell^{(s+1)}$ can be computed using (9–11). Since the qd-table given in table 4. needs the help-entries $g_{r,s}^{(\ell)}$ from table 2. we have baptised the rules (9–11) the qdg-algorithm. This new algorithm coincides with Rutishauser's qd-algorithm for the computation of univariate Padé approximants and with Claessens' generalized qd-algorithm for the computation of univariate Newton-Padé approximants.

In analogy with the univariate Padé approximation case [8 p. 610] and the univariate rational Hermite interpolation case [2] it is also possible to give explicit determinant formulas for the partial numerators in (12). Let us introduce the notations

$$\triangle t_r(\ell) = f_{d_r i_{\ell+1}, e_r j_{\ell+1}} B_{d_r i_{\ell+1}, e_r j_{\ell+1}}(x,y); \qquad r = 0,1,\dots \qquad \ell = 0,1,\dots$$

$$t_r(0) = f_{d_r i_0, e_r j_0} B_{d_r i_0, e_r j_0}(x,y)$$

$$t_r(\ell) = t_r(0) + \sum_{i=0}^{\ell-1} \triangle t_r(i)$$

Remember that $\triangle t_r(\ell) = 0$ for $i_{\ell+1} < d_r$ or $j_{\ell+1} < e_r$. We also introduce the notations

$$H_0(h,k) = \begin{vmatrix} \triangle\, t_0(h) & \dots & \triangle\, t_{k-1}(h) \\ \vdots & & \vdots \\ \triangle\, t_0(h+k-1) & \dots & \triangle\, t_{k-1}(h+k-1) \end{vmatrix}; \qquad H_0(h,0) = 0$$

$$H_1(h,k) = \begin{vmatrix} 1 & \dots & 1 \\ \triangle\, t_0(h) & \dots & \triangle\, t_k(h) \\ \vdots & & \vdots \\ \triangle\, t_0(h+k-1) & \dots & \triangle\, t_k(h+k-1) \end{vmatrix}; \qquad H_1(h,-1) = 0$$

$$H_2(h,k) = \begin{vmatrix} t_0(h) & \dots & t_k(h) \\ \triangle\, t_0(h) & \dots & \triangle\, t_k(h) \\ \vdots & & \vdots \\ \triangle\, t_0(h+k-1) & \dots & \triangle\, t_k(h+k-1) \end{vmatrix}; \qquad H_2(h,-1) = 0$$

$$H_3(h,k) = \begin{vmatrix} 1 & \dots & 1 \\ t_0(h) & \dots & t_k(h) \\ \triangle\, t_0(h) & \dots & \triangle\, t_k(h) \\ \vdots & & \vdots \\ \triangle\, t_0(h+k-2) & \dots & \triangle\, t_k(h+k-2) \end{vmatrix}; \qquad H_3(h,-1) = 0$$

We know from (4) that

$$\frac{H_2(h,k)}{H_1(h,k)} = [N_h/D_k]_{I_{h+k}}.$$

Besides the differences $\triangle t_r(\ell)$ we can also consider

$$\delta t_r(\ell) = t_{r+1}(\ell) - t_r(\ell)$$

and introduce the notations

$$G_0(h,k) = \begin{vmatrix} \delta t_0(h) & \dots & \delta t_0(h+k-1) \\ \vdots & & \vdots \\ \delta t_{k-1}(h) & \dots & \delta t_{k-1}(h+k-1) \end{vmatrix}; \qquad G_0(h,0) = 0$$

$$G_1(h,k) = \begin{vmatrix} 1 & \dots & 1 \\ \delta t_0(h) & \dots & \delta t_0(h+k) \\ \vdots & & \vdots \\ \delta t_{k-1}(h) & \dots & \delta t_{k-1}(h+k) \end{vmatrix}; \qquad G_1(h,-1) = 0$$

$$G_2(h,k) = \begin{vmatrix} t_0(h) & \dots & t_0(h+k) \\ \delta t_0(h) & \dots & \delta t_0(h+k) \\ \vdots & & \vdots \\ \delta t_{k-1}(h) & \dots & \delta t_{k-1}(h+k) \end{vmatrix}; \qquad G_2(h,-1) = 0$$

$$G_3(h,k) = \begin{vmatrix} 1 & \dots & 1 \\ t_0(h) & \dots & t_0(h+k) \\ \delta t_0(h) & \dots & \delta t_0(h+k) \\ \vdots & & \vdots \\ \delta t_{k-2}(h) & \dots & \delta t_{k-2}(h+k) \end{vmatrix}; \qquad G_3(h,-1) = 0$$

For the H-values it is well-known by the Schweins expansion [1 p. 43] that

(13) $$H_1(h,k)H_2(h,k-1) - H_1(h,k-1)H_2(h,k) = H_3(h,k)H_0(h,k).$$

For the G-values one can prove using the Sylvester-identity [7] that

(14) $$G_1(h-1,k)G_2(h,k) - G_1(h,k)G_2(h-1,k) = G_3(h-1,k+1)G_0(h,k).$$

(15) $$G_1(h-1,k)G_0(h,k+1) - G_1(h,k)G_0(h-1,k+1) = G_1(h-1,k+1)G_0(h,k).$$

Some easy computations show that the G-values are very related to the H-values. For $k \geq 1$ we have

$$H_0(h,k) = G_3(h,k)$$
$$H_3(h,k) = G_0(h,k)$$

and for $k \geq 0$

$$H_1(h,k) = G_1(h,k)$$
$$H_2(h,k) = G_2(h,k)$$

Hence we know from (13) and (14) that

(16) $$G_1(h,k)G_2(h,k-1) - G_1(h,k-1)G_2(h,k) = G_0(h,k)G_3(h,k).$$

and that for $k \geq 1$ also

(17) $$H_1(h-1,k)H_2(h,k) - H_1(h,k)H_2(h-1,k) = H_0(h-1,k+1)H_3(h,k).$$

By means of these formulas we can prove the following theorem.

THEOREM. *For the partial numerators $q_\ell^{(s+1)}$ and $e_\ell^{(s+1)}$ in the continued fraction (12) of which the successive convergents equal the successive elements on the descending staircase (6), the following determinant formulas hold:*

(18) $$q_\ell^{(s+1)} = -\frac{H_0(s+\ell,\ell)H_1(s+\ell-1,\ell-1)H_3(s+\ell,\ell)}{H_0(s+\ell-1,\ell)H_1(s+\ell,\ell)H_3(s+\ell,\ell-1)}$$

$$e_\ell^{(s+1)} = -\frac{H_0(s+\ell,\ell+1)H_1(s+\ell,\ell-1)H_3(s+\ell+1,\ell)}{H_0(s+\ell,\ell)H_1(s+\ell+1,\ell)H_3(s+\ell,\ell)}. \tag{19}$$

PROOF. We know from (7) and (4) that

$$\begin{aligned} q_\ell^{(s+1)} &= \frac{E_\ell^{(s+\ell)} - E_{\ell-1}^{(s+\ell)}}{E_{\ell-1}^{(s+\ell)} - E_{\ell-1}^{(s+\ell-1)}} \\ &= \frac{\dfrac{H_2(s+\ell,\ell)}{H_1(s+\ell,\ell)} - \dfrac{H_2(s+\ell,\ell-1)}{H_1(s+\ell,\ell-1)}}{\dfrac{H_2(s+\ell,\ell-1)}{H_1(s+\ell,\ell-1)} - \dfrac{H_2(s+\ell-1,\ell-1)}{H_1(s+\ell-1,\ell-1)}} \end{aligned}$$

Using (13) and (17) we get

$$\begin{aligned} q_\ell^{(s+1)} &= -\frac{H_3(s+\ell,\ell)H_0(s+\ell,\ell)}{H_1(s+\ell,\ell)H_1(s+\ell,\ell-1)} \Big/ \frac{H_0(s+\ell-1,\ell)H_3(s+\ell,\ell-1)}{H_1(s+\ell,\ell-1)H_1(s+\ell-1,\ell-1)} \\ &= -\frac{H_0(s+\ell,\ell)H_1(s+\ell-1,\ell-1)H_3(s+\ell,\ell)}{H_0(s+\ell-1,\ell)H_1(s+\ell,\ell)H_3(s+\ell,\ell-1)} \end{aligned}$$

The formula for $e_\ell^{(s+1)}$ is proved in a completely analogous way. ∎

Note that one can prove, using (14) and (15) that

$$\frac{H_2(h,k)}{H_1(h,k)} = E_k^{(h)} = \frac{G_2(h,k)}{G_1(h,k)}$$

$$\begin{aligned} &= \frac{\dfrac{G_2(h,k-1)}{G_1(h,k-1)}\dfrac{G_0(h+1,k)}{G_1(h+1,k-1)} - \dfrac{G_2(h+1,k-1)}{G_1(h+1,k-1)}\dfrac{G_0(h,k)}{G_1(h,k-1)}}{\dfrac{G_0(h+1,k)}{G_1(h+1,k-1)} - \dfrac{G_0(h,k)}{G_1(h,k-1)}} \\ &= \frac{E_{k-1}^{(h)}\dfrac{G_0(h+1,k)}{G_1(h+1,k-1)} - E_{k-1}^{(h+1)}\dfrac{G_0(h,k)}{G_1(h,k-1)}}{\dfrac{G_0(h+1,k)}{G_1(h+1,k-1)} - \dfrac{G_0(h,k)}{G_1(h,k-1)}} \end{aligned}$$

Referring to (4a) we see that

$$\frac{G_0(h+1,k)}{G_1(h+1,k-1)} = g_{k-1,k}^{(h+1)}$$

$$\frac{G_0(h,k)}{G_1(h,k-1)} = g_{k-1,k}^{(h)}$$

Obviously the formulas from the recursive computation scheme and those of the qdg-algorithm are closely linked. This is to be expected if we want to develop a multivariate theory with the properties of the univariate theory.

References

1. Brezinski C., *Accélération de la convergence en Analyse Numérique*, LNM **584**, Springer Verlag, Berlin.
2. Claessens G., *A generalization of the qd-algorithm*, J. Comput. Appl. Math. **7** (1981), 237–247.
3. Cuyt A., *A recursive computation scheme for multivariate rational interpolants*, SIAM J. Numer. Anal. (to appear).
4. Cuyt A., *A review of multivariate Padé approximation theory*, J. Comput. Appl. Math. **12 & 13** (1985), 221–232.
5. Cuyt A., *Multivariate Padé approximants revisited*, BIT **26** (1986), 71–79.
6. Cuyt A. and Verdonk B., *General order Newton-Padé approximants for multivariate functions*, Numer. Math. **43** (1984), 293–307.
7. Gantmacher F. R., *The Theory of Matrices*, Chelsea Publ. Co., New York, 1960.
8. Henrici P., *Applied and Computational Complex Analysis, vol. I*, J. Wiley, New York, 1974.
9. Levin D., *General order Padé type rational approximants defined from double power series*, J. Inst. Math. Appl. **18** (1976), 1–8.