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MULTIVARIATE PADÉ APPROXIMANTS REVISITED

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Abstract.

Several definitions of multivariate Padé approximants have been introduced during the last decade. We will here consider all types of definitions based on the choice that the coefficients in numerator and denominator of the multivariate Padé approximant are defined by means of a linear system of equations. In this case a determinant representation for the multivariate Padé approximant exists. We will show that a general recursive algorithm can be formulated to compute a multivariate Padé approximant given by any definition of this type. Here intermediate results in the recursive computation scheme will also be multivariate Padé approximants. Up to now such a recursive computation of multivariate Padé approximants only seemed possible in some special cases.

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1. General definition.

The framework used to describe the group of definitions based on the use of a linear system of defining equations for the numerator and denominator coefficients, is greatly inspired by [10]. Given a Taylor series expansion

$$f(x,y) = \sum_{(i,j)\in \mathbb{N}^2} c_{ij}x^iy^j$$

with

$$c_{ij} = \frac{1}{i!}\frac{1}{j!}\frac{\partial^{i+j}f}{\partial x^i\partial y^j}\bigg|_{(0,0)}$$

we will compute an approximant $p(x,y)/q(x,y)$ to $f(x,y)$ where $p(x,y)$ and $q(x,y)$ are determined by the accuracy – through – order principle.

The polynomials $p(x,y)$ and $q(x,y)$ are of the form

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}x^iy^j$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}x^iy^j$$

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where N (Numerator) and D (Denominator) are finite subsets of $\mathbb{N}^2$. The sets N and D indicate in fact the "degree" of $p(x, y)$ and $q(x, y)$. Let us denote

$$\# N = n+1$$

$$\# D = m+1.$$

It is now possible to let $p(x, y)$ and $q(x, y)$ satisfy the following condition for the power series $(fq-p)(x, y)$, namely

$$(fq-p)(x, y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus E} d_{ij}x^i y^j \tag{1}$$

if, in analogy with the univariate case, the index set E (Equations) is such that

(2a) $N \subseteq E$

(2b) $\# (E \setminus N) = m = \# D - 1$

(2c) E satisfies the inclusion property

meaning that when a point belongs to the index set E, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in E.

Condition (2a) enables us to split the system of equations

$$d_{ij} = 0, \qquad (i, j) \in E$$

in an inhomogeneous part defining the numerator coefficients

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu, j-\nu} = a_{ij}, \qquad (i, j) \in N$$

and a homogeneous part defining the denominator coefficients

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu, j-\nu} = 0, \qquad (i, j) \in E \setminus N. \tag{3}$$

By convention

$$b_{kl} = 0 \quad \text{if} \quad (k, l) \notin D.$$

Condition (2b) guarantees the existence of a nontrivial denominator $q(x, y)$ because the homogeneous system has one equation less than the number of unknowns and so one unknown coefficient can be chosen freely.

Condition (2c) finally takes care of the Padé approximation property, namely

$$\left(f-\frac{p}{q}\right)(x, y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} e_{ij}x^iy^j.$$

For more information we refer to [5, 6].

Let us now introduce a numbering $r(i, j)$ of the points in $\mathbb{N}^2$, based on the enumeration

$$(0,0),\ \underbrace{(1,0),\ (0,1)}_{\substack{\text{first}\\ \text{diagonal}}},\ \underbrace{(2,0),\ (1,1),\ (0,2)}_{\substack{\text{second}\\ \text{diagonal}}},\ \underbrace{(3,0),\ (2,1),\ (1,2),\ (0,3)}_{\substack{\text{third}\\ \text{diagonal}}},\ \ldots$$

So

$$r(i, j) = \tfrac{1}{2}(i+j)(i+j+1)+j.$$

Since the set N contains $n+1$ points, we can write

$$N = \bigcup_{l=0}^{n} N_l$$

with

$$\emptyset = N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_{n-1} \subset N_n = N$$
$$\# N_l = l+1$$
$$N_l\setminus N_{l-1} = \{(i_l, j_l)\}; \qquad l = 0,\ldots, n$$
$$r(i_l, j_l) > r(i_r, j_r), \qquad l > r$$

In other words, for each $l = 0,\ldots,n$ we add to N_{l-1} one point called (i_l, j_l) which is next in line in $N \cap \mathbb{N}^2$ according to the enumeration given above.

The same can be done for

$$D = \bigcup_{l=0}^{m} D_l$$

with

$$D_{-1} = \emptyset, \qquad D_l\setminus D_{l-1} = \{(d_l, e_l)\}, \qquad l = 0,\ldots, m.$$

For the sake of simplicity we assume that the homogeneous system of equations (3) has maximal rank. From numerical experiments we know that this is most often the case. However, what follows can be extended to the case when this is not true, by adding points to the set $E\setminus N$ until the rank deficiency has disappeared, but at this moment this would only complicate the notation. As a consequence of (2a), and (2b), we know

$$\# E = n+m+1.$$

Hence we can write

$$E = \bigcup_{l=0}^{n+m} E_l$$

with

$$\begin{aligned} E_l &= N_l, & l &= 0,\ldots,n \\ E_{n+l}\backslash E_{n+l-1} &= \{(i_{n+l}, j_{n+l})\}, & l &= 1,\ldots,m \\ r(i_{n+l}, j_{n+l}) &> r(i_r, j_r), & n+l &> r \geqq n+1. \end{aligned}$$

It was shown in [10] that a determinantal representation for

$$p_h(x, y) = \sum_{(i,j)\in N_h} a_{ij}x^iy^j, \qquad 0 \leqq h \leqq n$$

and

$$q_k(x, y) = \sum_{(i,j)\in D_k} b_{ij}x^iy^j, \qquad 0 \leqq k \leqq m$$

satisfying

$$(fq_k - p_h)(x, y) = \sum_{(i,j)\in \mathbb{N}^2\backslash E_{h+k}} d_{ij}x^iy^j$$

is given by

$$\text{(4a)} \qquad p_h(x, y) = \begin{vmatrix} \sum_{(i,j)\in N_h} c_{i-d_0, j-e_0}x^iy^j & \cdots & \sum_{(i,j)\in N_h} c_{i-d_k, j-e_k}x^iy^j \\ c_{i_{h+1}-d_0, j_{h+1}-e_0} & \cdots & c_{i_{h+1}-d_k, j_{h+1}-e_k} \\ \vdots & & \vdots \\ c_{i_{h+k}-d_0, j_{h+k}-e_0} & \cdots & c_{i_{h+k}-d_k, j_{h+k}-e_k} \end{vmatrix}$$

$$\text{(4b)} \qquad q_k(x, y) = \begin{vmatrix} x^{d_0}y^{e_0} & \cdots & x^{d_k}y^{e_k} \\ c_{i_{h+1}-d_0, j_{h+1}-e_0} & \cdots & c_{i_{h+1}-d_k, j_{h+1}-e_k} \\ \vdots & & \vdots \\ c_{i_{h+k}-d_0, j_{h+k}-e_0} & \cdots & c_{i_{h+k}-d_k, j_{h+k}-e_k} \end{vmatrix}$$

where

$$c_{ij} = 0 \quad \text{if} \quad i < 0 \quad \text{or} \quad j < 0.$$

A solution of the original problem (1) is then given by $p_n(x, y)/q_m(x, y)$ because

$$N_n = N, \quad D_m = D \qquad \text{and} \qquad E_{n+m} = E.$$

In order to show that this general setting can handle quite a number of previously given definitions of multivariate Padé approximations, we shall now give the sets N, D and E for several of these definitions. When we are dealing with Karlsson-Wallin [9] Padé approximants; we must choose ν and μ in $\mathbb{N}$ and construct

$$N = \{(i, j) \in \mathbb{N}^2 | 0 \leqq i+j \leqq \nu\}$$
$$D = \{(d, e) \in \mathbb{N}^2 | 0 \leqq d+e \leqq \mu\}$$

which are triangular sets. In this way

$$\# N = n+1 = \tfrac{1}{2}(\nu+1)(\nu+2)$$
$$\# D = m+1 = \tfrac{1}{2}(\mu+1)(\mu+2).$$

For Padé approximants introduced by Lutterodt [13], we have with ν_1, ν_2, μ_1, μ_2 fixed

$$N = \{(i, j) \in \mathcal{N}^2 | 0 \leqq i \leqq \nu_1, \qquad 0 \leqq j \leqq \nu_2\}$$
$$D = \{d, e) \in \mathcal{N}^2 | 0 \leqq d \leqq \mu_1, \qquad 0 \leqq e \leqq \mu_2\}$$

which are rectangular sets. Now

$$\# N = n+1 = (\nu_1+1)(\nu_2+1)$$
$$\# D = m+1 = (\mu_1+1)(\mu_2+1).$$

For these two types of multivariate Padé approximants, the only demands for the set E are the conditions (2a, b, c).

Multivariate Padé approximants of order (ν, μ) introduced by Cuyt [3] appear to have numerator and denominator index sets given by

$$N = \{(i, j) \in \mathbb{N}^2 | \nu \cdot \mu \leqq i+j \leqq \nu \cdot \mu+\nu\}$$
$$D = \{(d, e) \in \mathbb{N}^2 | \nu \cdot \mu \leqq d+e \leqq \nu \cdot \mu+\mu\}$$

which resemble triangular sets. Here

$$E = \{(i, j) \in \mathbb{N}^2 | 0 \leqq i+j \leqq \nu \cdot \mu+\nu+\mu\}$$

The approximants introduced by the group working in Canterbury [2, 8] were constructed from

$$N = \{(i,j) \in \mathbb{N}^2 | 0 \leqq i \leqq \nu_1, \quad 0 \leqq j \leqq \nu_2\}$$
$$D = \{(d,e) \in \mathbb{N}^2 | 0 \leqq d \leqq \mu_1, \quad 0 \leqq e \leqq \mu_2\}$$

$$\begin{aligned} E = N \bigcup D \bigcup \{(i,j) \in \mathbb{N}^2 | 0 \leqq j \leqq \min(\nu_2, \mu_2), \quad & \max(\nu_1, \mu_1) < i \leqq \nu_1 + \mu_1, \\ & i + j \leqq \nu_1 + \mu_1\} \\ \bigcup \{(i,j) \in \mathbb{N}^2 | 0 \leqq i \leqq \min(\nu_1, \mu_1), \quad & \max(\nu_2, \mu_2) < j \leqq \nu_2 + \mu_2, \\ & i + j \leqq \nu_2 + \mu_2\} \end{aligned}$$

with the additional requirements

$$d_{\nu_1+\mu_1+1-l,\,l} + d_{l,\,\nu_2+\mu_2+1-l} = 0, \qquad l = 1, \ldots, \min(\nu_1, \mu_1, \nu_2, \mu_2).$$

These additional requirements alter the determinantal representations (4a) and (4b) but the structure of the determinants remains the same. For more details we refer to [10]. We will show that in each of these cases the rational function $p_n(x,y)/q_m(x,y)$ can be computed recursively, starting from $p_0(x, y)/q_0(x,y)$ and building intermediate values $p_h(x,y)/q_k(x,y)$.

2. Recursive algorithm.

The algorithm which we shall give is a generalization of the ε-algorithm, but a special case of the more general E-algorithm [1]. The formulas (4) can be rewritten as follows. Multiply the $(l+1)^{th}$ row in $p_h(x,y)$ and $q_k(x,y)$ by $x^{i_{h+l}}y^{j_{h+1}}$ $(l = 1, \ldots, k)$ and afterwards divide the $(l+1)^{th}$ column by $x^{d_l}y^{e_l}$ $(l = 0, \ldots, k)$.

This results in

(5a)
$$p_h(x,y) = \begin{vmatrix} \sum_{(i,j)\in N_h} c_{i-d_0,j-e_0} x^{i-d_0} y^{j-e_0} & \cdots & \sum_{(i,j)\in N_h} c_{i-d_k,j-e_k} x^{i-d_k} y^{j-e_k} \\ c_{i_{h+1}-d_0,j_{h+1}-e_0} x^{i_{h+1}-d_0} y^{j_{h+1}-e_0} & \cdots & c_{i_{h+1}-d_k,j_{h+1}-e_k} x^{i_{h+1}-d_k} y^{j_{h+1}-e_k} \\ \vdots & & \vdots \\ c_{i_{h+k}-d_0,j_{h+k}-e_0} x^{i_{h+k}-d_0} y^{j_{h+k}-e_0} & \cdots & c_{i_{h+k}-d_k,j_{h+k}-e_k} x^{i_{h+k}-d_k} y^{j_{h+k}-e_k} \end{vmatrix}$$

(5b)
$$q_k(x,y) = \begin{vmatrix} 1 & \cdots & 1 \\ c_{i_{h+1}-d_0,j_{h+1}-e_0} x^{i_{h+1}-d_0} y^{j_{h+1}-e_0} & \cdots & c_{i_{h+1}-d_k,j_{h+1}-e_k} x^{i_{h+1}-d_k} y^{j_{h+1}-e_k} \\ \vdots & & \vdots \\ c_{i_{h+k}-d_0,j_{h+k}-e_0} x^{i_{h+k}-d_0} y^{j_{h+k}-e_0} & \cdots & c_{i_{h+k}-d_k,j_{h+k}-e_k} x^{i_{h+k}-d_k} y^{j_{h+k}-e_k} \end{vmatrix}$$

We can easily construct $(k+1)$ series of which the successive partial sums can be found in the columns of $p_h(x, y)$. Take

$$t_0(0) = c_{i_0-d_0, j_0-e_0} x^{i_0-d_0} y^{j_0-e_0}$$
$$\Delta t_0(l-1) = t_0(l)-t_0(l-1) = c_{i_l-d_0, j_l-e_0} x^{i_l-d_0} y^{j_l-e_0}, \qquad l = 1,\dots,h+k$$

In this way

$$t_0(h) = \sum_{(i,j)\in N_h} c_{i-d_0, j-e_0} x^{i-d_0} y^{j-e_0}.$$

We remark that $\Delta t_0(l-1) = 0$ as long as $i_l < d_0$ or $j_l < e_0 (l = 1,\dots,h+k)$. In this way we obtain the first column of $p_h(x, y)$. We can proceed in the same way for the other columns. Define for $r = 1,\dots,k$

$$t_r(0) = c_{i_0-d_r, j_0-e_r} x^{i_0-d_r} y^{j_0-e_r}$$
$$\Delta t_r(l-1) = t_r(l)-t_r(l-1) = c_{i_l-d_r, j_l-e_r} x^{i_l-d_r} y^{j_l-e_r}, \qquad l = 1,\dots,h+k.$$

Hence

$$t_r(h) = \sum_{(i,j)\in N_h} c_{i-d_r, j-e_r} x^{i-d_r} y^{j-e_r}$$

and the $(r+1)$th column of $p_h(x, y)$ is obtained. Again $\Delta t_r(l-1) = 0$ for $i_l < d_r$ or $j_l < e_r$.

Consequently

$$p_h(x, y) \equiv \begin{vmatrix} t_0(h) & \cdots & t_k(h) \\ \Delta t_0(h) & \cdots & \Delta t_k(h) \\ \vdots & & \vdots \\ \Delta t_0(h+k-1) & \cdots & \Delta t_k(h+k-1) \end{vmatrix} \tag{6a}$$

$$q_k(x, y) = \begin{vmatrix} 1 & \cdots & 1 \\ \Delta t_0(h) & \cdots & \Delta t_k(h) \\ \vdots & & \vdots \\ \Delta t_0(h+k-1) & \cdots & \Delta t_k(h+k-1) \end{vmatrix} \tag{6b}$$

This quotient of determinants can easily be computed using the E-algorithm [1]:

$$\varepsilon_0^{(l)} = t_0(l), \qquad l = 0,\dots,h+k$$
$$g_{0,r}^{(l)} = t_r(l)-t_{r-1}(l), \qquad r = 1,\dots,k; \quad l = 0,\dots,h+k$$

(7a) $$\varepsilon_r^{(l)} = \frac{\varepsilon_{r-1}^{(l)} g_{r-1,r}^{(l+1)} - \varepsilon_{r-1}^{(l+1)} g_{r-1,r}^{(l)}}{g_{r-1,r}^{(l+1)} - g_{r-1,r}^{(l)}}, \qquad l = 0, \ldots, h; \quad r = 1, \ldots, k$$

(7b) $$g_{r,s}^{(l)} = \frac{g_{r-1,s}^{(l)} g_{r-1,r}^{(l+1)} - g_{r-1,s}^{(l+1)} g_{r-1,r}^{(l)}}{g_{r-1,r}^{(l+1)} - g_{r-1,r}^{(l)}}, \qquad s = r+1, r+2, \ldots$$

$$\begin{array}{cccc}
\varepsilon_0^{(0)} & & & \\
 & \varepsilon_1^{(0)} & & \\
\varepsilon_0^{(1)} & & \ddots & \\
 & \varepsilon_1^{(1)} & & \varepsilon_k^{(0)} \\
\varepsilon_0^{(2)} & \vdots & & \vdots \\
\vdots & & & \varepsilon_k^{(h)} \\
 & \varepsilon_1^{(h+k-1)} & \cdots & \\
\varepsilon_0^{(h+k)} & & &
\end{array}$$

$$\begin{array}{c|cc|c|ccc}
g_{0,1}^{(0)} & g_{0,2}^{(0)} & & & g_{0,k}^{(0)} & & \\
 & & g_{1,2}^{(0)} & & & g_{1,k}^{(0)} & \\
g_{0,1}^{(1)} & g_{0,2}^{(1)} & & & g_{0,k}^{(1)} & & \ddots \\
 & & g_{1,2}^{(1)} & \cdots & & g_{1,k}^{(1)} & g_{k-1,k}^{(0)} \\
g_{0,1}^{(2)} & g_{0,2}^{(2)} & \vdots & & g_{0,k}^{(2)} & \vdots & \vdots \\
\vdots & \vdots & & & \vdots & & g_{k-1,k}^{(h+1)} \\
 & & & & & & \\
 & & g_{1,2}^{(h+k-1)} & & & g_{1,k}^{(h+k-1)} & \\
g_{0,1}^{(h+k)} & g_{0,2}^{(h+k)} & & & g_{0,k}^{(h+k)} & &
\end{array}$$

Finally with $h = n$ and $k = m$, i.e. with $N_h = N$, $D_k = D$ and $E_{h+k} = E$, we get

$$\frac{p_n(x, y)}{q_m(x, y)} = \varepsilon_m^{(n)}$$

while intermediate values in the computation scheme are also multivariate Padé approximants since

$$\varepsilon_k^{(h)} = \frac{p_h(x, y)}{q_k(x, y)}$$

and thus

$$f - \varepsilon_k^{(h)} = \sum_{(i,j) \in \mathbb{N}^2 \setminus E_{h+k}} e_{ij} x^i y^j.$$

So we see that Padé approximants originally only introduced via defining equations, can now also be given via a recursive scheme. The next step is to write them as the convergent of a multivariate continued fraction. This will be the subject of further research. Then the univariate equivalence of the three main defining techniques for Padé approximants is also established for the multivariate case: algebraic relations, recurrence relations, continued fractions. Readers interested in a comparison of numerical results for the definitions of multivariate Padé approximants treated here are referred to [4, 7, 11].

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