JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 96, 283–293 (1983)

Multivariate Padé-Approximants*

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Submitted by R. P. Boas

For an operator $F : \mathbb{R}^n \to \mathbb{R}$, analytic in the origin, the notion of (abstract multivariate Padé-approximant (APA) is introduced, by making use of abstract polynomials. The classical Padé-approximant ($n = 1$) is a special case of the multivariate theory and many interesting properties of classical Padé-approximants remain valid such as covariance properties and the block-structure [Annie A. M. Cuyt, *J. Oper. Theory* **6** (2) (1981), 207–209] of the Padé-table. Also a projection-property for multivariate Padé-approximants is proved.

1. DEFINITION OF MULTIVARIATE PADÉ-APPROXIMANT

Many attempts have been made to generalize the concept of Padé-approximant for multivariate functions. We refer to [1, 4–8].

Another generalisation is the following one. The Banach-space $\mathbb{R}^n$ is normed by one of the Minkowski-norms; we write $0 = (o,..., o)^T$ and $x = (x_1,..., x_n)^T$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be analytic in the origin:

$$\exists r > o : F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(0)\, x^k \qquad \text{for} \quad \|(x_1,..., x_n)\| < r,$$

where $(1/o!)\, F^{(o)}(0)\, x^o = F(0)$ and $F^{(k)}(0)$ denotes the kth Fréchet-derivative of F in 0; $(1/k!)\, F^{(k)}(0)$ is a symmetric k-linear bounded operator: $(\mathbb{R}^n)^k \to \mathbb{R}$ [9, pp. 109–112] and is equal to

$$\sum_{k_1+\cdots+k_n=k} \frac{1}{k_1! \cdots k_n!} \frac{\partial^k F(x)}{\partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n}} \bigg|_{x=0} \qquad x_1^{k_1} \cdots x_n^{k_n}.$$

DEFINITION 1.1. (a) $P : \mathbb{R}^n \to \mathbb{R} : x \to P(x) = A_m x^m + \cdots + A_0$ is an

* This report was written during the fellowship of N.F.W.O. (Belgium).

† Aspirant Belgisch Nationaal Fonds voor Wetenschappelijk Onderzoek.

0022-247X/83 $3.00

abstract polynomial if for $j = 0, \ldots, m$ the A_j are symmetric j-linear bounded operators: $(\mathbb{R}^n)^j \to \mathbb{R}$, in other words, if

$$A_j x^j = \sum_{j_1 + \cdots + j_n = j} a_{j_1 \ldots j_n} x_1^{j_1} \cdots x_n^{j_n} \qquad \text{with} \quad a_{j_1 \ldots j_n} \text{ in } \mathbb{R}.$$

(b) $\partial_0 P = m_1$ is the *order* of the abstract polynomial P if for $0 \leqslant k < m_1 : A_k x^k \equiv 0$ and $A_{m_1} x^{m_1} \not\equiv 0$.

(c) $\partial P = m_2$ is the *exact degree* of the abstract polynomial P if for $m_2 < k \leqslant m : A_k x^k = 0$ and $A_{m_2} x^{m_2} \not\equiv 0$.

We say that $F(x) = O(x^j)$ if

$$\exists J, r \in \mathbb{R}_0^+, o < r < 1 : |F(x)| \leqslant J \cdot \|x\|^j \qquad \text{for} \quad \|x\| < r.$$

DEFINITION 1.2. The couple of abstract polynomials

$$(P(x), Q(x)) = (A_{lm+l} x^{lm+l} + \cdots + A_{lm} x^{lm}, B_{lm+m} x^{lm+m} + \cdots + B_{lm} x^{lm})$$

such that the power series $(F \cdot Q - P)(x) = O(x^{lm+l+m+1})$ (1)

is called a solution of *the Padé-approximation problem of order* (l, m).

The choice of order and degree of P and Q is justified in [2].

For every non-negative integers l and m a solution of the problem described in Definition 1.2 exists [2]. We call the quotient of two abstract polynomials $P/Q : \mathbb{R}^n \to \mathbb{R} : x \to P(x)/Q(x)$ reducible if there exist abstract polynomials T, R, S such that $P = T \cdot R$ and $Q = T \cdot S$ and $\partial T \geqslant 1$. If (P, Q) and (R, S) are solutions of (1) (for l and m fixed), then $P(x) \cdot S(x) = Q(x) \cdot R(x)$ for every x in $\mathbb{R}^n$. This "equivalence-property" of solutions of (1) justifies the following definitions.

DEFINITION 1.3. Let (P, Q) be a couple of abstract polynomials satisfying (1), with $Q(x) \not\equiv 0$. Let P_*/Q_* be the irreducible form of P/Q such that $Q_*(0) = 1$. If this form exists, we call it the *normalized (abstract) multivariate Padé-approximant (APA) of order* (l, m) *for* F (normalized (l, m)–APA).

Remark that for the polynomial T such that $(P, Q) = (P_* \cdot T, Q_* \cdot T)$:

$$\partial_0 T = \partial_0 Q - \partial_0 Q_*.$$

For the normalized (l, m)–APA we have

$$l' := \partial P_* \leqslant l,$$

$$m' := \partial Q_* \leqslant m,$$

$$\partial_0 T \geqslant lm.$$

DEFINITION 1.4. Let (P, Q) be a couple of abstract polynomials satisfying (1), with $Q(x) \not\equiv 0$. If the irreducible form P_*/Q_* is such that $\partial_0 Q_* \geqslant 1$, then we call P_*/Q_* the *(abstract) multivariate Padé-approximant of order (l, m) for F* ((l, m)–APA).

The (l, m)–APA is unique up to a multiplicative constant in numerator and denominator.

For the (l, m)–APA

$$\begin{aligned} l' &:= \partial P_* - \partial_0 Q_* \leqslant l, \\ m' &:= \partial Q_* - \partial_0 Q_* \leqslant m, \\ \partial_0 T &\geqslant lm - \partial_0 Q_*. \end{aligned}$$

From now on we shall often consider the normalized (l, m)–APA to be a special case of the (l, m)–APA and not mention the specification normalized.

2. EXISTENCE OF A NONTRIVIAL SOLUTION OF (1)

(a) When $n = 1$, the definition of the abstract Padé-approximant is precisely the classical definition [3].

(b) The problem (1) is equivalent with the solution of two linear systems of equations:

$$\left.\begin{aligned} &C_0 \cdot B_{lm} x^{lm} = A_{lm} x^{lm}, \\ &\vdots \\ &C_l x^l \cdot B_{lm} x^{lm} + \cdots + C_0 \cdot B_{lm+l} x^{lm+l} = A_{lm+l} x^{lm+l}, \end{aligned}\right\} \quad \forall x \in \mathbb{R}^n, \tag{1a}$$

$$\left.\begin{aligned} &C_{l+1} x^{l+1} \cdot B_{lm} x^{lm} + \cdots + C_{l+1-m} x^{l+1-m} \cdot B_{lm+m} x^{lm+m} = 0, \\ &\vdots \\ &C_{l+m} x^{l+m} \cdot B_{lm} x^{lm} + \cdots + C_l x^l \cdot B_{lm+m} x^{lm+m} = 0, \end{aligned}\right\} \quad \forall x \in \mathbb{R}^n, \tag{1b}$$

with $\quad B_{lm+j} x^{lm+j} \equiv 0 \quad$ for $j > m$,

$$C_k x^k = (1/k!)\, F^{(k)}(0)\, x^k \quad \text{for } k \geqslant 0,$$

$$C_k x^k \equiv 0 \quad \text{for } k < 0.$$

The homogeneous system contains $N_e = \binom{n+lm+l+m}{lm+l+m} - \binom{n+lm+l}{lm+l}$ equations in $N_u = \binom{n+lm+m}{lm+m} - \binom{n+lm-1}{lm-1}$ unknown coefficients of the B_{lm+j}. For $n = 2: N_u - N_e = 1$ and so one unknown can be chosen and a nontrivial solution always exists. For $n > 2$: the nontriviality of the solution is proved as follows. Suppose that the matrix

$$\begin{pmatrix} C_{l+1}x^{l+1} & \cdots & C_{l+1-m}x^{l+1-m} \\ \vdots & & \vdots \\ C_{l+m}x^{l+m} & & C_l x^l \end{pmatrix}$$

of the homogeneous system (1b) has rank k, in other words, that a vector x in $\mathbb{R}^n$ exists such that the determinant of a $k \times k$ submatrix is nonzero. In any case $0 \leqslant k \leqslant m$. The homogeneous system (1b) can now be reduced to a homogeneous system of k equations in $k+1$ of the unknown $B_{lm+j}x^{lm+j}$ $(j=0,\ldots,m)$:

$$\begin{aligned} &\sum_{i=0}^{k} C_{l+h_1-j_i}x^{l+h_1-j_i}B_{lm+j_i}x^{lm+j_i}=0,\\ &\quad\vdots\\ &\sum_{i=0}^{k} C_{l+h_k-j_i}x^{l+h_k-j_i}B_{lm+j_i}x^{lm+j_i}=0,\\ &\text{with}\quad 1\leqslant h_i\leqslant m \quad \text{for } i=1,\ldots,k,\\ &\text{and}\quad 0\leqslant j_i\leqslant m \quad \text{for } i=0,\ldots,k,\\ &\qquad j_0<j_1<\cdots<j_k. \end{aligned} \tag{1c}$$

In fact we have removed $(m-k)$ rows and $(m-k)$ columns of the coefficient matrix of system (1b) to obtain the coefficient matrix of system (1c). We will number the rows that we have removed $\bar{h}_1,\ldots,\bar{h}_{m-k}$ and the columns that we have removed $\bar{j}_1+1,\ldots,\bar{j}_{m-k}+1$ (notice that the rows that we have retained are numbered $h_1,\ldots,h_k$ and the columns $j_0+1,\ldots,j_k+1$).

If $k=m$ then a solution of (1b) can be calculated by means of the following determinants.

$$B_{lm}x^{lm}=\begin{vmatrix} C_l x^l & & C_{l+1-m}x^{l+1-m} \\ \vdots & & \vdots \\ C_{l+m-1}x^{l+m-1} & \cdots & C_l x^l \end{vmatrix},$$

$$B_{lm+j}x^{lm+j}=\begin{vmatrix} C_l x^l & \boxed{\begin{matrix}-C_{l+1}x^{l+1}\\ \vdots \\ -C_{l+m}x^{l+m}\end{matrix}} & \cdots & C_{l+1-m}x^{l+1-m} \\ \vdots & & & \vdots \\ C_{l+m-1}x^{l+m-1} & & & C_l x^l \end{vmatrix}.$$

jth column in $B_{lm}x^{lm}$ replaced by this column $(j=1,\ldots,m)$

Let us introduce the following notations:

$$D(x)=\begin{vmatrix} C_{l+\bar{h}_1-\bar{j}_1}x^{l+\bar{h}_1-\bar{j}_1} & \cdots & C_{l+\bar{h}_1-\bar{j}_{m-k}}x^{l+\bar{h}_1-\bar{j}_{m-k}} \\ \vdots & & \vdots \\ C_{l+\bar{h}_{m-k}-\bar{j}_1}x^{l+\bar{h}_{m-k}-\bar{j}_1} & \cdots & C_{l+\bar{h}_{m-k}-\bar{j}_{m-k}}x^{l+\bar{h}_{m-k}-\bar{j}_{m-k}} \end{vmatrix}$$

$$D_{j_0}(x)=\begin{vmatrix} C_{l+h_1-j_1}x^{l+h_1-j_1} & \cdots & C_{l+h_1-j_k}x^{l+h_1-j_k} \\ \vdots & & \vdots \\ C_{l+h_k-j_1}x^{l+h_k-j_1} & \cdot & C_{l+h_k-j_k}x^{l+h_k-j_k} \end{vmatrix},$$

$$D_{j_i}(x)=\begin{vmatrix} C_{l+h_1-j_1}x^{l+h_1-j_1} & \boxed{-C_{l+h_1-j_0}x^{l+h_1-j_0}} & \cdots & C_{l+h_1-j_k}x^{l+h_1-j_k} \\ \vdots & \vdots & & \vdots \\ C_{l+h_k-j_1}x^{l+h_k-j_1} & \boxed{-C_{l+h_k-j_0}x^{l+h_k-j_0}} & & C_{l+h_k-j_k}x^{l+h_k-j_k} \end{vmatrix}$$

↑ ith column in $D_{j_0}(x)$ replaced by this column ($i=1,..., k$)

Then clearly, because of the Laplacian expansion of the determinant $B_{lm+j}x^{lm+j}$, for $j=j_0, j_1,..., j_k$,

$$B_{lm+j_i}x^{lm+j_i}=D(x)\cdot D_{j_i}(x)+\cdots$$

where $D(x)$ is a p-linear bounded operator with $0\leqslant p\leqslant lm+j_0$.

For $k<m$ a nontrivial solution of system (1b) is now given by

$$B_{lm+j}x^{lm+j}=0 \qquad \text{for } j=\bar{j}_i\ (i=1,..., m-k),$$

$$B_{lm+j}x^{lm+j}=E_px^p\cdot D_{j_i}(x) \qquad \text{for } j=j_i\ (i=0,..., k),$$

with E_px^p a nontrivial symmetric p-linear bounded operator: $(\mathbb{R}^n)^p\to\mathbb{R}$, because one of the $D_{j_i}(x)$ is nontrivial. We also prove the following important theorem.

THEOREM 2.1. *Let* P_*/Q_* be the (l, m)–APA *for* F. *Then there exists* s, $lm-\partial_0 Q_*\leqslant s\leqslant lm-\partial_0 Q_*+\min(l-l', m-m')$, *and a nontrivial symmetric s-linear bounded operator* $D_s:(\mathbb{R}^n)^s\to\mathbb{R}$, *such that* $(P_*(x)\cdot D_sx^s, Q_*(x)\cdot D_sx^s)$ *satisfies* (1).

Proof. Because (1) is solvable for every l, $m\in\mathbb{N}$ we may consider abstract polynomials P and Q that satisfy (1) and supply P_* and Q_* with $Q(x)\not\equiv 0$. Because of Definition 1.4 or 1.3, there exists an abstract polynomial T such that $P=P_*\cdot T$ and $Q=Q_*\cdot T$. Now $\partial_0 T=\partial_0 Q-\partial_0 Q_*\geqslant lm-\partial_0 Q_*$. We write

$$s = \partial_0 T = lm - \partial_0 Q_* + r \leqslant \partial T \quad \begin{array}{l} \nearrow \; \partial P - \partial P_* \leqslant lm + l - l' - \partial_0 Q_* \\ \searrow \; \partial Q - \partial Q_* \leqslant lm + m - m' - \partial_0 Q_* \end{array}$$

with $r \geqslant 0$. So $lm - \partial_0 Q_* \leqslant s \leqslant lm - \partial_0 Q_* + \min(l - l', m - m')$. For $T(x) = D_s x^2 + D_{s+1} x^{s+1} + \cdots : \; |(F \cdot Q_* - P_*) \cdot D_s|(x) = 0(x^{lm+l+m+1})$ because of the equivalence of (1) with (1a) and (1b).

We illustrate this theorem with an example. Let $F : \mathbb{R}^2 \to \mathbb{R} : \binom{x_1}{x_2} \to 1 + \sin(x_1 + x_1 x_2)$. The (1, 2)–APA is

$$\frac{x_1 - x_2 + \frac{5}{6}x_1^2 - 2x_1x_2}{x_1 - x_2 - x_1x_2 - \frac{1}{6}x_1^2 + x_1x_2^2 + \frac{1}{6}x_1^3}.$$

Theorem 2.1 holds with $s = 1$, $\partial_0 Q_* = 1$ and $D_1\binom{x_1}{x_2} = x_1$.

When we compare this theorem with the similar one for the classical Padé-approximant we remark that the term lm in s is due to the choice of the order of the couple of polynomials (P, Q) in Definition 1.2 and that the term $(-\partial_0 Q_*)$ in s is due to the fact that sometimes the abstract Padé-approximant cannot be normalized as in Definition 1.3.

3. Covariance Properties

Several covariance properties can already be found in [3, pp. 204–206], where they are formulated for operator Padé-approximants (the multivariate Padé-approximants are a special case); normalized (l, m)–APA are transformed into normalized (l, m)–APA and (l, m)–APA are transformed into (l, m)–APA.

Another property, especially for multivariate Padé-approximants, is added and proved here.

Theorem 3.1. *Let* $y_i = a_i x_i / (1 + b_1 x_1 + \cdots + b_n x_n)$ *for* $i = 1, \ldots, n$ *and* $y = (y_1, \ldots, y_n)^T$. *Let the* (l, l)–APA *for* $F(x)$ *be* $P_*(x)/Q_*(x)$ *and let*

$$G(x) := F(y),$$

$$R_*(x) := P_*(y) \cdot (1 + b_1 x_1 + \cdots + b_n x_n)^k$$

and

$$S_*(x) := Q_*(y) \cdot (1 + b_1 x_1 + \cdots + b_n x_n)^k$$

with $k = \max(\partial P_*, \partial Q_*)$. *Then the* (l, l)–APA *for* $G(x)$ *is* $R_*(x)/S_*(x)$.

Proof. Because of Theorem 2.1 there exists a positive integer s, $l^2 - \partial_0 Q_* \leqslant s \leqslant l^2 - \partial_0 Q_* + \min(l - \partial P_* + \partial_0 Q_*, l - \partial Q_* + \partial_0 Q_*)$, and a nontrivial symmetric s-linear bounded operator $D_s : (\mathbb{R}^n)^s \to \mathbb{R}$ such that $[(F \cdot Q_* - P_*) \cdot D_s](x) = o(x^{l^2+2l+1})$. We write

$$D_s(y) = \frac{D_s(a_1 x_1, ..., a_n x_n)}{(1 + b_1 x_1 + \cdots + b_n x_n)^s} = \frac{\bar{D}_s(x)}{(1 + b_1 x_1 + \cdots + b_n x_n)^s}.$$

For $k = \max(\partial P_*, \partial Q_*)$:

$$\partial_0(R_* \cdot \bar{D}_s) \geqslant \partial_0(P_* \cdot D_s) \geqslant l^2,$$
$$\partial_0(S_* \cdot \bar{D}_s) \geqslant \partial_0(Q_* \cdot D_s) \geqslant l^2,$$
$$\max[\partial(R_* \cdot \bar{D}_s), \partial(S_* \cdot \bar{D}_s)] \leqslant k + s \leqslant l^2 + l.$$

So

$$\begin{aligned}
&[(G \cdot S_* - R_*) \cdot \bar{D}_s](x) \\
&\quad = [(F \cdot Q_* - P_*) \cdot D_s](y) \cdot (1 + b_1 x_1 + \cdots + b_n x_n)^{k+s} \\
&\quad = O(y^{l^2+2l+1}) \cdot (1 + b_1 x_1 + \cdots + b_n x_n)^{k+s} \\
&\quad = O(x^{l^2+2l+1}).
\end{aligned}$$

And thus $[(G \cdot S_* - R_*) \cdot \bar{D}_s](x) = O(x^{l^2+2l+1})$.

We will now show that the irreducible form of $[R_* \cdot \bar{D}_s](x)/[S_* \cdot \bar{D}_s](x)$ is $R_*(x)/S_*(x)$. Suppose

$$\begin{aligned} R_*(x) &= \bar{U}(x) \cdot \bar{V}(x) \\ S_*(x) &= \bar{U}(x) \cdot \bar{W}(x) \end{aligned} \qquad \text{with} \quad \partial \bar{U} \geqslant 1.$$

Since

$$\frac{a_1 x_1}{y_1} = \frac{a_2 x_2}{y_2} = \cdots = \frac{a_n x_n}{y_n} = 1 + \sum_{i=1}^{n} b_i x_i$$

we know that

$$x_i = \frac{a_n y_i}{a_i y_n} x_n \qquad \text{for} \quad i = 1, ..., n.$$

Consequently

$$1 + \sum_{i=1}^{n} b_i x_i = 1 + x_n \sum_{i=1}^{n} b_i \frac{a_n y_i}{a_i y_n} = \frac{a_n x_n}{y_n}$$

or

$$x_n = 1\Bigg/\left(\frac{a_n}{y_n} - b_n - \sum_{i=1}^{n-1} b_i \frac{a_n y_i}{y_n a_i}\right).$$

So we can write

$$x_i = y_i\Bigg/\left(a_i\left(1 - \sum_{i=1}^{n} b_i \frac{y_i}{a_i}\right)\right).$$

Thus

$$R_*(x) = P_*(y)\left(1 + \sum_{i=1}^{n} b_i x_i\right)^k,$$

$$S_*(x) = Q_*(y)\left(1 + \sum_{i=1}^{n} b_i x_i\right)^k,$$

implies that

$$P_*(y) = \bar{U}(x)\cdot\bar{V}(x)\left(1 - \sum_{i=1}^{n} \frac{b_i}{a_i} y_i\right)^k,$$

$$Q_*(y) = \bar{U}(x)\cdot\bar{W}(x)\left(1 - \sum_{i=1}^{n} \frac{b_i}{a_i} y_i\right)^k,$$

and thus

$$P_*(y) = U(y)\cdot V(y),$$

$$Q_*(y) = U(y)\cdot W(y),$$

with

$$U(y) = \bar{U}\left(y_1\Bigg/\left(a_1\left(1 - \sum_{i=1}^{n} \frac{b_i}{a_i} y_i\right)\right),\dots, y_n\Bigg/\left(a_n\left(1 - \sum_{i=1}^{n} \frac{b_i}{a_i} y_i\right)\right)\right)$$
$$\cdot\left(1 - \sum_{i=1}^{n} \frac{b_i}{a_i} y_i\right)^{k'},$$

$$V(y) = \bar{V}(x_1,\dots, x_n)\cdot\left(1 - \sum_{i=1}^{n} \frac{b_i}{a_i} y_i\right)^{k-k'},$$

$$W(y) = \bar{W}(x_1,\dots, x_n)\cdot\left(1 - \sum_{i=1}^{n} \frac{b_i}{a_i} y_i\right)^{k-k'},$$

$$k' = \partial\bar{U}\ (\partial\bar{U} + \partial\bar{V} \leqslant k \text{ and } \partial\bar{U} + \partial\bar{W} \leqslant k).$$

This contradicts the fact that P_*/Q_* is irreducible.

Since $\partial_0 S_* = \partial_0 Q_*$ and $S_*(0) = Q_*(0)$, the normalized (l, l)–APA for F is transformed into the normalized (l, l)–APA for G and the (l, l)–APA for F is transformed into the (l, l)–APA for G.

4. Projection-Property

We introduce the following notations:

$$
{}^j\tilde{x} = (x_1, \ldots, x_{j-1}, o, x_{j+1}, \ldots, x_n),
$$
$$
x_{'j'} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).
$$

THEOREM 5.1. *If the* (l, m)–APA *for* $F : \mathbb{R}^n \to \mathbb{R}$ *is* P_*/Q_* *and*

$$
j \in \{1, \ldots, n\},
$$
$$
S(x_{'j'}) := Q_*({}^j\tilde{x}) \not\equiv 0,
$$
$$
R(x_{'j'}) := P_*({}^j\tilde{x}),
$$
$$
G_j(x_{'j'}) := F({}^j\tilde{x}),
$$

then the (l, m)–APA *for* $G_j : \mathbb{R}^{n-1} \to \mathbb{R}$ *is the irreducible form* R_*/S_* *of* R/S.

Proof. Since the (l, m)–APA is $P_*/Q_* : \partial_0(F \cdot Q_* - P_*) = \partial_0 Q_* + l' + m' + t + 1$ with $t \geqslant 0$ [3, p. 208]. Using a Minkowski-norm in $\mathbb{R}^n : \|{}^j\tilde{x}\|$ in $\mathbb{R}^n$ equals $\|x_{'j'}\|$ in $\mathbb{R}^{n-1}$. Thus $(F \cdot Q_* - P_*)({}^j\tilde{x}) = (G_j \cdot S - R)(x_{'j'}) = O(x_{'j'}^{\partial_0 Q_* + l' + m' + t + 1})$. Now $\partial P_* = \partial_0 Q_* + l' \leqslant \partial P \leqslant lm + l$ and $\partial Q_* = \partial_0 Q_* + m' \leqslant \partial Q \leqslant lm + m$ imply $lm - \partial_0 Q_* + \min(l - l', m - m') \geqslant 0$. Take $s = lm - \partial_0 Q_* + \min(l - l', m - m')$ and a symmetric s-linear bounded operator $D_s : (\mathbb{R}^{n-1})^s \to \mathbb{R}$ with $D_s x_{'j'}^s \not\equiv 0$. The couple of polynomials $(R \cdot D_s, S \cdot D_s)$ satisfies (1) for G_j since

$$
\partial_0(R \cdot D_s) = \partial_0 R + s \geqslant \partial_0 P_* + s \geqslant lm,
$$
$$
\partial_0(S \cdot D_s) = \partial_0 S + s \geqslant \partial_0 Q_* + s \geqslant lm,
$$
$$
\partial(R \cdot D_s) = \partial R + s \leqslant \partial P_* + s = lm + \min(l, m - m' + l')
$$
$$
\leqslant lm + l,
$$
$$
\partial(S \cdot D_s) = \partial S + s \leqslant \partial Q_* + s = lm + \min(l - l' + m', m)
$$
$$
\leqslant lm + m,
$$

$$\begin{aligned}\partial_0[(G_j \cdot S - R) \cdot D_s] &= \partial_0 Q_* + l' + m' + t + 1 + s \\ &= lm + \min(l + m' + t, m + l' + t) + 1 \\ &\geqslant lm + l + m + 1 \qquad \text{since} \\ & m \leqslant m' + t, \qquad l \leqslant l' + t.\end{aligned}$$

Also $(S \cdot D_s)(x_{'j'}) \not\equiv 0$ and if $Q_*(0) = 1$ then $S_*(0) = 1$.

We give some examples and illustrate that it is very well possible that if P_*/Q_* is the (l, m)–APA for $F(x)$, then R_*/S_* is the normalized (l, m)–APA for $G_j(x_{'j'})$. Take $F : \mathbb{R}^2 \to \mathbb{R} : \binom{x}{x_2} \to \frac{1}{2}(1 + e^{x_1 + x_2})$. The normalized (1, 1)–APA for F is $1/(1 - \frac{1}{2}(x_1 + x_2))$. For $j = 1$:

$$x_1 = o,$$

$$G_1 : \mathbb{R} \to \mathbb{R} : x_2 \to \tfrac{1}{2}(1 + e^{x_2}),$$

normalized (1, 1)–APA for G_1 is $1/(1 - \frac{1}{2}x_2)$.

For $j = 2$:

$$x_2 = o,$$

$$G_2 : \mathbb{R} \to \mathbb{R} : x_1 \to \tfrac{1}{2}(1 + e^{x_1}),$$

normalized (1, 1)–APA for G_2 is $1/(1 - \frac{1}{2}x_1)$.

Take

$$F : \mathbb{R}^2 \to \mathbb{R} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \frac{x_1 e^{x_1} - x_2 e^{x_2}}{x_1 - x_2}.$$

The (1, 1)–APA for F is

$$\frac{x_1 + x_2 + 0.5(x_1^2 + 3x_1 x_2 + x_2^2)}{x_1 + x_2 - 0.5(x_1^2 + x_1 x_2 + x_2^2)}.$$

For $j = 2$:

$$x_2 = o,$$

$$G_2 : \mathbb{R} \to \mathbb{R} : x_1 \to e^{x_1},$$

normalized (1, 1)–APA for G_2 is $(1 + 0.5x_1)/(1 - 0.5x_1)$.

We also searched for a product property of the following kind. Let $(P_1/Q_1)(x_1, \ldots, x_k)$ be the (l, m)–APA for $F_1 : \mathbb{R}^k \to \mathbb{R}$ and let $(P_2/Q_2)(x_{k+1}, \ldots, x_n)$ be the (l, m)–APA for $F_2 : \mathbb{R}^{n-k} \to \mathbb{R}$. Then is

$$\frac{P}{Q}(x) = \frac{P_1(x_1, \ldots, x_k) \cdot P_2(x_{k+1}, \ldots, x_n)}{Q_1(x_1, \ldots, x_k) \cdot Q_2(x_{k+1}, \ldots, x_n)}$$

the (l, m)-APA for $F: \mathbb{R}^n \to \mathbb{R}: x \to F(x) = F_1(x_1, ..., x_k) \cdot F_2(x_{k+1}, ..., x_n)$? In fact it is not obvious that the multivariate approximants should have this property. The following counterexample proves it.

Let $F_1: \mathbb{R} \to \mathbb{R}: x_1 \to e^{x_1}$ and $F_2: \mathbb{R} \to \mathbb{R}: x_2 \to e^{x_2}$. Then $F: \mathbb{R}^2 \to \mathbb{R}: \binom{x_1}{x_2} \to e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$. Take $l = 1$ and $m = 2$.

The (1, 2)-APA for F_1 is

$$\frac{P_1}{Q_1}(x_1) = \frac{1 + \frac{1}{3}x_1}{1 - \frac{2}{3}x_1 + \frac{1}{6}x_1^2}$$

and for F_2 is

$$\frac{P_2}{Q_2}(x_2) = \frac{1 + \frac{1}{3}x_2}{1 - \frac{2}{3}x_2 + \frac{1}{6}x_2^2}.$$

The (1, 2)-APA for F is

$$\frac{1 + \frac{1}{3}(x_1 + x_2)}{1 - \frac{2}{3}(x_1 + x_2) + \frac{1}{6}(x_1 + x_2)^2} \neq \frac{P_1(x_1) \cdot P_2(x_2)}{Q_1(x_1) \cdot Q_2(x_2)}.$$

Another kind of product-property, however, has been proved in [3].

References

1. J. S. R. Chisholm, *N*-variable rational approximants, *in* "Padé and Rational Approximation: Theory and Applications" (E. B. Saff and R. S. Varga, Eds.), pp. 23–42, Academic Press, London/New York, 1977.
2. A. A. M. Cuyt, "Abstract Padé Approximants in Operator Theory," pp. 61–87, Lecture Notes in Mathematics No. 765, Springer-Verlag, Berlin/New York, 1979.
3. A. A. M. Cuyt, On the properties of abstract rational (1-point) approximants, *J. Oper. Theory* **6** (2) (1981), 195–216.
4. R. Hughes Jones, General rational approximants in *N*-variables, *J. Approx. Theory* **16** (1976), 201–233.
5. J. Karlsson and H. Wallin, Rational approximation by an interpolation procedure in several variables, *in* "Padé and Rational Approximation: Theory and Applications" (E. B. Saff and R. S. Varga, Eds.), pp. 83–100, Academic Press, London/New York, 1977.
6. D. Levin, General order Padé-type rational approximants defined from double power series, *J. Inst. Math. Appl.* **18** (1976), 1–8.
7. C. H. Lutterodt, Rational approximants to holomorphic functions in *n*-dimensions, *J. Math. Anal. Appl.* **53** (1976), 89–98.
8. C. H. Lutterodt, A two-dimensional analogue of Padé-approximant theory, *J. Phys. A* **7** (9) (1974), 1027–1037.
9. L. B. Rall, "Computational Solution of Nonlinear Operator Equations," Wiley, New York, 1969.

Printed in Belgium