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A Montessus de Ballore Theorem for Multivariate Padé Approximants

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During the last few years several authors have tried to generalize the concept of Padé approximant to multivariate functions and to prove a generalization of Montessus de Ballore's theorem. We refer, e.g., to J. Chisholm and P. Graves-Morris (*Proc. Roy. Soc. London Ser. A* **342** (1975), 341–372), J. Karlsson and H. Wallin ("Padé and Rational Approximations and Applications" (E. B. Saff and R. S. Varga, Eds.), pp. 83–100, Academic Press, 1977), C. H. Lutterodt (*J. Phys. A* **7**, No. 9 (1974), 1027–1037; *J. Math. Anal. Appl.* **53** (1976), 89–98; preprint, Dept. of Mathematics, University of South Florida, Tampa, Florida, 1981). However, it is a very delicate matter to generalize Montessus de Ballore's result from $\mathbb{C}$ to $\mathbb{C}^p$. This problem is discussed in Section 3. A definition of multivariate Padé approximant, which was introduced by A. A. M. Cuyt ("Padé Approximants for Operators: Theory and Applications," Lecture Notes in Mathematics No. 1065, Springer–Verlag, Berlin, 1984; *J. Math. Anal. Appl.* **96** (1983), 283–293) and which is repeated in Section 1, is a generalization that allows one to preserve many of the properties of the univariate Padé approximants: covariance properties, block-structure of the Padé-table, the ε-algorithm, the *qd*-algorithm, and so on. It also allows one to formulate a Montessus de Ballore theorem, which is presented in Section 2; up to now it is probably the most "Montessus de Ballore"-like version existing for the multivariate case. Illustrative numerical results are given in Section 4.

1. MULTIVARIATE PADÉ APPROXIMANTS

Let the multivariate function $f(z_1,..., z_p)$ be holomorphic in the polydisc $B(0, \rho_1,..., \rho_p) = \{(z_1,..., z_p) \in \mathbb{C}^p \mid |z_i| < \rho_i\}$ around the origin,

$$f(z) = \sum_{k=0}^{\infty} C_k z^k \qquad \text{for} \quad z = (z_1,..., z_p) \in B(0, \rho_1,..., \rho_p),$$

where

$$C_k z^k = \sum_{k_1 + \cdots + k_p = k} c_{k_1 \cdots k_p} z_1^{k_1} \cdots z_p^{k_p}$$

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with

$$c_{k_1 \cdots k_p} = \left. \frac{\partial^k f(z_1, \ldots, z_p)}{\partial z_1^{k_1} \cdots \partial z_p^{k_p}} \right|_{(z_1, \ldots, z_p) = (0, \ldots, 0)}.$$

Now choose n and m in $\mathbb{N}$ and find

$$p(z) = \sum_{i=nm}^{nm+n} A_i z^i \qquad \text{with} \quad A_i z^i = \sum_{i_1 + \cdots + i_p = i} a_{i_1 \cdots i_p} z_1^{i_1} \cdots z_p^{i_p}$$

and

$$q(z) = \sum_{j=nm}^{nm+m} B_j z^j \qquad \text{with} \quad B_j z^j = \sum_{j_1 + \cdots + j_p = j} b_{j_1 \cdots j_p} z_1^{j_1} \cdots z_p^{j_p}$$

such that

$$\partial_0 (f \cdot q - p) \geqslant nm + n + m + 1 \tag{1}$$

where ∂_0, the order of the power series, is the degree of the first nonzero term (a term $z_1^{k_1} \cdots z_p^{k_p}$ is said to be of degree $k_1 + \cdots + k_p$). Note the shift of the degrees of p and q over nm. In [3] we proved that this problem always has a nontrivial solution for the $b_{j_1 \cdots j_p}$.

Once we have calculated a pair of polynomials (p, q) that satisfies (1), we are going to look for the irreducible form $(p_{(n,m)}/q_{(n,m)})(z)$ of $(p/q)(z)$. Different solutions (p_1, q_1) and (p_2, q_2) of (1) have the same irreducible form since we can prove the equivalency of the solutions; i.e., [4]

$$(p_1 q_2)(z) = (p_2 q_1)(z) \qquad \forall z \in \mathbb{C}^p.$$

By computing $(p_{(n,m)}/q_{(n,m)})(z)$, possibly a polynomial $t(z)$ has been cancelled in the numerator and denominator of $(p/q)(z)$. Thus the degrees of $p_{(n,m)}$ and $q_{(n,m)}$ may be shifted back a bit.

We can easily show that [4]

$$\partial_0 p_{(n,m)} \geqslant \partial_0 q_{(n,m)}$$

and this justifies the following definition.

Let ∂ denote the exact degree of a polynomial.

DEFINITION 1.1. We call $\partial_1 p_{(n,m)} = \partial p_{(n,m)} - \partial_0 q_{(n,m)}$ the *pseudo-degree* of $p_{(n,m)}$ and $\partial_1 q_{(n,m)} = \partial q_{(n,m)} - \partial_0 q_{(n,m)}$ the pseudo-degree of $q_{(n,m)}$.

For these pseudo-degrees we can write the inequalities

$$\partial_1 p_{(n,m)} \leqslant n$$

$$\partial_1 q_{(n,m)} \leqslant m.$$

Now we can formulate the definition of multivariate Padé approximant.

DEFINITION 1.2. The (n, m) multivariate Padé approximant ((n, m) MPA) is the irreducible form $(p_{(n,m)}/q_{(n,m)})(z)$ of $(p/q)(z)$ where p and q satisfy (1).

Because we cancelled $t(z)$ in the numerator and denominator of $(p/q)(z)$, the pair of polynomials $(p_{(n,m)}(z), q_{(n,m)}(z))$ no longer necessarily satisfies (1). However, the following results hold.

Analogously to the univariate case, we can show that [4]

$$\partial_0(f \cdot q_{(n,m)} - p_{(n,m)}) \geqslant \partial_0 q_{(n,m)} + \partial_1 p_{(n,m)} + \partial_1 q_{(n,m)} + t + 1$$

with $t \geqslant \max(n - \partial_1 p_{(n,m)}, m - \partial_1 q_{(n,m)})$. If we define the defect

$$d_{n,m} = \min(n - \partial_1 p_{(n,m)}, m - \partial_1 q_{(n,m)})$$

then we can also write

$$\partial_0(f \cdot q_{(n,m)} - p_{(n,m)}) \geqslant \partial_0 q_{(n,m)} + n + m + 1 - d_{n,m}.$$

The term $\partial_0 q_{(n,m)}$ is a consequence of what is still left of the shift of the degrees and the term $-d_{n,m}$ is a consequence of dividing out the polynomial $t(z)$ in the solution $(p(z), q(z))$.

Let us illustrate some of the preceding remarks by a simple example. Consider

$$f(z_1, z_2) = 1 + \frac{z_1}{0.1 - z_2} + \sin(z_1 z_2).$$

Take $n = 1 = m$. Then $p(z)$ and $q(z)$ are of the form

$$p(z) = a_{10} z_1 + a_{01} z_2 + a_{20} z_1^2 + a_{11} z_1 z_2 + a_{02} z_2^2$$
$$q(z) = b_{10} z_1 + b_{01} z_2 + b_{20} z_1^2 + b_{11} z_1 z_2 + b_{02} z_2^2.$$

Note that the degrees are shifted over $nm = 1$. A solution of (1) is given by

$$\frac{p(z)}{q(z)} = \frac{10z_1 + 100z_1^2 - 101z_1 z_2}{10z_1 - 101z_1 z_2}$$

while the irreducible form is

$$\frac{p_{(1,1)}(z)}{q_{(1,1)}(z)} = \frac{1 + 10z_1 - 10.1z_2}{1 - 10.1z_2}.$$

Here $\partial_0 q_{(1,1)} = 0$ because we cancelled $t(z) = 10z_1$ in the numerator and denominator; thus $\partial_1 p_{(1,1)} = \partial p_{(1,1)} \leqslant 1$ and $\partial_1 q_{(1,1)} = \partial q_{(1,1)} \leqslant 1$.

Take $n = 1$ and $m = 2$. The (1, 2) MPA is given by

$$\frac{p_{(1,2)}(z)}{q_{(1,2)}(z)} = \frac{z_1 - 1.01z_2 + 10z_1^2 + 10z_2^2 - 20.2z_1z_2}{z_1 - 1.01z_2 + 10z_2^2 - 10.1z_1z_2 + 2.01z_1z_2^2}.$$

The shift nm was equal to 2, but we could only divide out a polynomial $t(z)$ with $\partial_0 t = 1$. So $\partial_0 q_{(1,2)} = 1$ and this leftover of the shift of the degrees has an influence on $\partial_0(f \cdot q_{(1,2)} - p_{(1,2)})$. The pseudo-degrees are $\partial_1 p_{(1,2)} = 2 - \partial_0 q_{(1,2)} \leqslant 1$ and $\partial_1 q_{(1,2)} = 3 - \partial_0 q_{(1,2)} \leqslant 2$.

We will restrict ourselves now mainly to those multivariate Padé approximants where $\partial_0 q_{(n,m)} = 0$ and thus where the denominator starts with a constant term. The shift over nm has disappeared in this case.

2. Montessus de Ballore Theorem

The ring $H(B(0, \rho_1, ..., \rho_p))$ of holomorphic complex-valued functions in $B(0, \rho_1, ..., \rho_p)$ inherits its topology from the ring $C(B(0, \rho_1, ..., \rho_p))$ of continuous complex-valued functions in $B(0, \rho_1, ..., \rho_p)$ and the topology on $C(B(0, \rho_1, ..., \rho_p))$ is given by the following metric. Let $(K_j)_j$ be a sequence of compact subsets of $B(0, \rho_1, ..., \rho_p)$ such that

$$K_j \subset K_{j+1} \qquad \text{and} \qquad \bigcup_{j=1}^{\infty} K_j = B(0, \rho_1, ..., \rho_p)$$

and for elements $f, g \in C(B(0, \rho_1, ..., \rho_p))$ define

$$d(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|f - g\|_{K_j}}{1 + \|f - g\|_{K_j}}$$

where $\|f - g\|_{K_j} = \sup_{z \in K_j} |(f - g)(z)|$ (this value is a well-defined finite real number since f is continuous and K_j is compact). So the topology of $H(B(0, \rho_1, ..., \rho_p))$ is that of uniform convergence on compact subsets.

As a consequence we shall mean by

$$(f_i)_i \to f \qquad \text{uniformly on compact } K$$

where f and f_i $(i \in \mathbb{N})$ are holomorphic functions on $B(0, \rho_1, ..., \rho_p)$, that

$$\lim_{i \to \infty} \|f_i - f\|_K = 0.$$

Before going on to the question of convergence of multivariate Padé approximants, we want to repeat a univariate theorem that will serve as a starting point for our generalization. For the proof we refer to [6, p. 90].

THEOREM 2.1. *Let f be a meromorphic function of one complex variable in* $\{z \in \mathbb{C} \mid |z| < \rho\}$ *with poles* $g_1, ..., g_\mu$ *(counted with their multiplicities). Then for m fixed,* $m \geq \mu$, *there exist* $m - \mu$ *points* $g_{\mu+1}, ..., g_m$ *and a subsequence of* $((p_{(n,m)}/q_{(n,m)})(z))_n$ *converging uniformly to f on compact subsets of* $\{z \in C \mid |z| < \rho\} \setminus \{g_j \mid 1 \leq j \leq m\}$.

Montessus de Ballore's well-known univariate convergence theorem is obtained as a corollary. We shall now try to formulate the multivariate analogon of this theorem.

Let us consider a multivariate function f where the finite singularities of f within $B(0, \rho_1, ..., \rho_p)$ are given by the zeros of the polynomial

$$g_\mu(z) = \sum_{i_1 + \cdots + i_p = 0}^{\mu} g_{i_1 \cdots i_p} z_1^{i_1} \cdots z_p^{i_p}$$

and let $g_\mu(z)$ be such that

$$\max_{z \in \bar{B}(0, \rho_1, ..., \rho_p)} |g_\mu(z)| = 1$$

where $\bar{B}(0, \rho_1, ..., \rho_p) = \{z \in \mathbb{C}^p \mid |z_i| \leq \rho_i\}$. We shall denote the zero set of $g_\mu(z)$ by G_μ:

$$G_\mu = \{z \in \mathbb{C}^p \mid g_\mu(z) = 0\}.$$

From now on, for m fixed we shall always denote by

$$S_m = \left\{ \frac{p_{(n(k),m)}}{q_{(n(k),m)}}(z) \,\middle|\, \partial_0 q_{(n(k),m)} = 0; k = 0, 1, 2, ... \right\}$$

the subsequence of $((p_{(n,m)}/q_{(n,m)})(z))_n$ for which $\partial_0 q_{(n(k),m)} = 0$. So the denominator of every element in S_m starts with a constant term different from zero; i.e., $q_{(n(k),m)}(0) \neq 0$.

THEOREM 2.2. *Suppose* $f(z)$ *is analytic in the origin and meromorphic in* $B(0, \rho_1, ..., \rho_p)$ *with a pole set given by* G_μ. *Let m be fixed and* $m \geq \mu$ *and let* S_m *not be a finite set. Then there exists a polynomial* $q(z)$ *of degree m with zero set* $Q = \{z \in \mathbb{C}^p \mid q(z) = 0\}$ *such that* $Q \cap B(0, \rho_1, ..., \rho_p) \supset G_\mu \cap B(0, \rho_1, ..., \rho_p)$ *and there exists a subsequence of* $((p_{(n,m)}/q_{(n,m)})(z))_n$ *that converges to f uniformly on compact subsets of* $B(0, \rho_1, ..., \rho_p) \setminus Q$.

Proof. Since $g_\mu(z)\cdot f(z)$ is holomorphic on $B(0, \rho_1,..., \rho_p)$, we also have that

$$R_{n,m}(z)=g_\mu(z)[f(z)\,q_{(n,m)}(z)-p_{(n,m)}(z)]$$

is holomorphic on $B(0, \rho_1,..., \rho_p)$. So we can write the following Cauchy integral representation [5, p. 3]:

$$R_{n,m}(z)=\sum_{j=0}^{\infty}\sum_{j_1+\cdots+j_p=j} r_{n,m,j_1,...,j_p} z_1^{j_1}\cdots z_p^{j_p}$$

with

$$r_{n,m,j_1,...,j_p}=\left(\frac{1}{2\pi i}\right)^p\int_{\substack{|t_i|=\rho_i\\ i=1,...,p}}\frac{R_{n,m}(t)\,dt_1\cdots dt_p}{t_1^{j_1+1}\cdots t_p^{j_p+1}}. \tag{2}$$

Since $\partial_0(f\cdot q_{(n,m)}-p_{(n,m)})\geqslant\partial_0 q_{(n,m)}+n+m+1-d_{n,m}$, we know that $\partial_0 R_{n,m}\geqslant\partial_0 q_{(n,m)}+n+m+1-d_{n,m}$. Now $\partial(g_\mu\cdot p_{(n,m)})\leqslant\mu+\partial p_{(n,m)}$ where $\partial p_{(n,m)}=\partial_1 p_{(n,m)}+\partial_0 q_{(n,m)}\leqslant n-d_{n,m}+\partial_0 q_{(n,m)}$ and so $\partial(g_\mu\cdot p_{(n,m)})\leqslant\partial_0 q_{(n,m)}+n+m-d_{n,m}$. Consequently

$$r_{n,m,j_1,...,j_p}=\left(\frac{1}{2\pi i}\right)^p\int_{\substack{|t_i|=\rho_i\\ i=1,...,p}}\frac{f(t)\,q_{(n,m)}(t)\,g_\mu(t)}{t_1^{j_1+1}\cdots t_p^{j_p+1}}\,dt_1\cdots dt_p. \tag{3}$$

Suppose that $q_{(n,m)}(z)$ has been normalized such that

$$\max_{z\in\bar{B}(0,\rho_1,...,\rho_p)}|q_{(n,m)}(z)|=1.$$

We can also bound $(g_\mu\cdot f)(z)$ by

$$M_{g_\mu\cdot f}=\max_{z\in\bar{B}(0,\rho_1,...,\rho_p)}|(g_\mu\cdot f)(z)|<\infty.$$

Thus

$$|R_{n,m}(z)|\leqslant\sum_{j\geqslant\partial_0 q_{(n,m)}+n+m+1-d_{n,m}}\left(\sum_{j_1+\cdots+j_p=j}|r_{n,m,j_1,...,j_p}|\,|z_1|^{j_1}\cdots|z_p|^{j_p}\right)$$

with

$$|r_{n,m,j_1,...,j_p}|\leqslant\left|\frac{1}{2\pi i}\right|^p\frac{M_{g_\mu\cdot f}(2\pi)^p\,\rho_1\cdots\rho_p}{\rho_1^{j_1+1}\cdots\rho_p^{j_p+1}}.$$

So

$$|R_{n,m}(z)| \leqslant \sum_{j_1+\cdots+j_p \geqslant \partial_0 q_{(n,m)}+n+m+1-d_{n,m}} M_{g_\mu \cdot f} \left(\frac{|z_1|}{\rho_1}\right)^{j_1} \cdots \left(\frac{|z_p|}{\rho_p}\right)^{j_p} \tag{4}$$

which goes to zero if $n \to \infty$ and $z \in B(0, \rho_1, ..., \rho_p)$.

The sequence of denominators of the elements of S_m is uniformly bounded by 1 on compact subsets of $\bar{B}(0, \rho_1, ..., \rho_p)$ because of the normalization we introduced. Hence, by Vitali's theorem [5, p. 11], it contains a convergent subsequence. We shall denote this by $(q_{(n_i(k),m)}(z))_i \to q(z)$ on compact subsets of $B(0, \rho_1, ..., \rho_p)$ where $q(z)$ is also a polynomial of degree m.

Let us take a look at the sequence $(p_{(n_i(k),m)})_i$. Since $g_\mu(z) f(z) q_{(n_i(k),m)} - g_\mu(z) p_{(n_i(k),m)}(z)$ goes to zero for z in $B(0, \rho_1, ..., \rho_p)$ and since $q_{(n_i(k),m)}(z)$ converges to $q(z)$ for $i \to \infty$ and z in $B(0, \rho_1, ..., \rho_p)$ we can also write $(p_{(n_i(k),m)}(z))_i \to p(z)$ on compact subsets of $B(0, \rho_1, ..., \rho_p)$ where $p(z)$ is a holomorphic function on $B(0, \rho_1, ..., \rho_p)$. Then in the limit $(g_\mu \cdot f \cdot q - g_\mu \cdot p)(z) = 0$ for z in $B(0, \rho_1, ..., \rho_p)$. If $z \in G_\mu \cap B(0, \rho_1, ..., \rho_p)$, then $g_\mu(z) = 0$; since $(f \cdot g_\mu)(z) \neq 0$ in a dense set of $G_\mu \cap B(0, \rho_1, ..., \rho_p)$ we have $q(z) = 0$. Consequently $G_\mu \cap B(0, \rho_1, ..., \rho_p)$ is a subset of $Q \cap B(0, \rho_1, ..., \rho_p)$. Let K be a compact subset of $B(0, \rho_1, ..., \rho_p) \backslash Q$. Then for i large enough we know that $q_{(n_i(k),m)}(z) \neq 0$ for z in K. Let ρ_j' be chosen such that $K \subset \bar{B}(0, \rho_1', ..., \rho_p') \subset B(0, \rho_1, ..., \rho_p)$. Then, because of (4), we can write

$$\|R_{n_i(k),m}\|_K \leqslant M_{g_\mu \cdot f} \sum_{j_1+\cdots+j_p \geqslant n_i(k)+m+1-d_{n_i(k),m}} \left(\frac{\rho_1'}{\rho_1}\right)^{j_1} \cdots \left(\frac{\rho_p'}{\rho_p}\right)^{j_p}.$$

We write $r_i = \lfloor (1/p)(n_i(k) + m - d_{n_i(k),m} + 1) \rfloor$ where $\lfloor \quad \rfloor$ denotes the integer part and p is the number of variables. So

$$\begin{aligned} \|R_{n_i(k),m}\|_K &\leqslant M_{g_\mu \cdot f} \sum_{\substack{j_l \geqslant r_i \\ l=1,...,p}} \left(\frac{\rho_1'}{\rho_1}\right)^{j_1} \cdots \left(\frac{\rho_p'}{\rho_p}\right)^{j_p} \\ &\leqslant M_{g_\mu \cdot f} \sum_{l=1}^{p} \left[\left(\frac{\rho_l'}{\rho_l}\right)^{r_i} \prod_{\substack{j=1 \\ j \neq l}}^{p} \frac{1}{1-(\rho_j'/\rho_j)}\right]. \end{aligned}$$

If $i \to \infty$, $r_i \to \infty$ also and thus $\|f - p_{(n_i(k),m)}/q_{(n_i(k),m)}\|_K \to 0$ on compact subsets K of $B(0, \rho_1, ..., \rho_p) \backslash Q$.

From the multivariate analogon of Theorem 2.1 we now get the following multivariate corollary.

COROLLARY 2.1. *Suppose $f(z)$ is analytic in the origin and meromorphic in $B(0, \rho_1, ..., \rho_p)$ with a pole set given by G_μ. Let S_μ not be a finite set. Then the sequence $((p_{(n(k),\mu)}/q_{(n(k),\mu)})(z))_k$, i.e., the elements in S_μ, converges to f*

uniformly on compact subsets of $B(0, \rho_1, \ldots, \rho_p) \backslash G_\mu$ *and the sequence* $(q_{(n(k),\mu)}(z))_k$ *converges to* $g_\mu(z)$ *uniformly on compact subsets of* $B(0, \rho_1, \ldots, \rho_p)$*; i.e., the poles of* $(p_{(n(k),\mu)}/q_{(n(k),\mu)})(z)$ *converge to the poles of* f.

Proof. In the proof of Theorem 2.1 we obtained that S_μ contains a convergent subsequence $q_{(n_i(k),\mu)}(z) \to q(z)$ uniformly on compact subsets of $B(0, \rho_1, \ldots, \rho_p)$ and that $G_\mu \cap B(0, \rho_1, \ldots, \rho_p)$ is a subset of $Q \cap B(0, \rho_1, \ldots, \rho_p)$. Here $\partial q \leqslant \mu = \partial g_\mu$.

Since each of the irreducible factors in $g_\mu(z)$ (counted with its multiplicity) has points inside $B(0, \rho_1, \ldots, \rho_p)$ where it vanishes, we know that each of the irreducible factors in $g_\mu(z)$ is also a factor of $q(z)$ [1, p. 232]. Hence $q(z) = g_\mu(z)$. Consequently the whole sequence $(q_{(n(k),\mu)}(z))$ must converge to $g_\mu(z)$ uniformly on compact subsets of $B(0, \rho_1, \ldots, \rho_p)$ since every subsequence contains a uniformly convergent subsequence to $g_\mu(z)$ on compact subsets of $B(0, \rho_1, \ldots, \rho_p)$, and the whole sequence $(p_{(n(k),\mu)}(z))$ converges to $p(z)$ uniformly on compact subsets of $B(0, \rho_1, \ldots, \rho_p)$. So we can finish the proof as in Theorem 2.2.

3. Discussion

We shall now discuss the differences between these theorems and the results obtained by other authors.

Many papers have been published that define a generalization of the Padé approximant for multivariate functions. Each definition of a multivariate Padé approximant $(p/q)(z_1, \ldots, z_p)$ is based on

$$(f \cdot q - p)(z_1, \ldots, z_p) = \sum_{k_1, \ldots, k_p = 0}^{\infty} d_{k_1 \cdots k_p} z_1^{k_1} \cdots z_p^{k_p}$$

with

$$d_{k_1 \cdots k_p} = 0 \qquad \text{for} \quad (k_1, \ldots, k_p) \in E \subseteq \mathbb{N}^p.$$

The set E is called the interpolation set; the choice of E, $p(z_1, \ldots, z_p)$, and $q(z_1, \ldots, z_p)$ determines the type of approximant. In [4] the choices for p, q, and E are given for Levin's general order Padé-type rational approximants [7], Chisholm's diagonal approximants, Hughes Jones' off-diagonal approximants, Lutterodt's approximants, Karlsson–Wallin approximants, and the multivariate Padé approximants repeated here in Section 2. For the approximants introduced by the Canterbury group, by Karlsson and Wallin, and by Lutterodt a convergence result as given in Theorem 2.1 here is not possible because the transition from (2) to (3) is not valid. For their definition terms of $g_\mu \cdot p(z_1, \ldots, z_p)$ can influence $r_{n,m,j_1,\ldots,j_p}$ with $(j_1, \ldots, j_p) \in \mathbb{N}^p \backslash E$. Thus $R_{n,m}(z)$ cannot be bounded by (4) as is done here.

We refer to [6] where this kind of remark is made for the rational approximants introduced by the Canterbury group and those introduced by Karlsson and Wallin. We refer the reader to [10] where he or she can establish a serious gap in the convergence proofs for Lutterodt's approximants because this remark is not taken into account.

4. Numerical Example

Again consider

$$f(z_1, z_2) = 1 + \frac{z_1}{0.1 - z_2} + \sin(z_1 \cdot z_2).$$

Take $m = 1$ and

$$\begin{aligned} n(k) &= k && \text{for} \quad k = 0, \ldots, 4 \\ &= k + 2j && \text{for} \quad k = 2j+3,\, 2j+4, \text{ and } j = 1, 2, 3, \ldots. \end{aligned}$$

The $(n(k), 1)$ MPA equals

$$\begin{aligned} &\frac{1}{1 - 10z_1} && \text{for} \quad k = 0 \\ &\frac{1 + 10z_1 - 10.1z_2}{1 - 10.1z_2} && \text{for} \quad k = 1 \\ &\frac{1 + 10z_1 - (1000/101)\, z_2 + (201/101)\, z_1 z_2}{1 - (1000/101)\, z_2} && \text{for} \quad k = 2 \\ &\frac{\sum_{i=0}^{n(k)} C_i z^i - 10z_2 \sum_{i=0}^{n(k)-1} C_i z^i}{1 - 10z_2} && \text{for} \quad k > 2. \end{aligned}$$

Clearly the $q_{(n(k),1)}(z)$ converge to $1 - 10z_2$ and

$$\|f - (n(k), 1)\,\text{MPA}\|_K = \left\| \frac{\sum_{i=n(k)+1}^{\infty} C_i z^i - 10z_2 \sum_{i=n(k)}^{\infty} C_i z^i}{1 - 10z_2} \right\|_K \to 0$$

for $k \to \infty$ and K a compact subset of $B(0, \rho_1, \rho_2) \backslash \{z \in \mathbb{C}^2 \mid z_2 = 0.1\}$. In Table 4.1 one can find the function values of the $(n, 1)$ MPA ($n = 0, \ldots, 18$) for $z_1 = 0.5$ and $z_2 = 0.2$, which is outside the region of convergence of the Taylor series development. One can compare these values with

$$f(0.5, 0.2) = -3.9001665833531.$$

All the computations were performed in double precision arithmetic.

TABLE 4.1

-0.2500000000000
-3.9019607843137
-4.3040404040404
-3.9000000000000
-3.9000000000000
-3.9001666670139
-3.8995000006944
-3.9001666666667
-3.9001666666666
-3.9001665833333
-3.9001669166668
-3.9001665833334
-3.9001665833334
-3.9001665833530
-3.9001665832729
-3.9001665833566
-3.9001665833421
-3.9001665833639
-3.9001665833639

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