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The Euler–Minding series for branched continued fractions

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Abstract: In the paper "Branched continued fractions for double power series" [*J. Comput. Appl. Math.* 6 (1980) 121–125] Siemaszko generalizes for branched continued fractions the formula that expresses the difference of two successive convergents of an ordinary continued fraction. However, the generalization is not yet fit to write the branched continued fraction as an Euler–Minding series for the following reason. Indeed a convergent of the branched continued fraction can be written as a partial sum of a series but different convergents are different partial sums of different series. The next convergent cannot be obtained from the previous one by adding some terms. We shall develop here another formula that overcomes this problem.

Let us consider ordinary continued fractions

$$B_i = b_0^{(i)} + \frac{| a_1^{(i)}}{b_1^{(i)} |} + \frac{| a_2^{(i)}}{b_2^{(i)} |} + \cdots, \quad i = 0, 1, 2, \ldots. \tag{1}$$

If $C_n^{(i)}$ denotes the nth convergent of (1) then it is well-known that

$$C_n^{(i)} - C_{n-1}^{(i)} = (-1)^{(n+1)} a_1^{(i)} \cdots a_n^{(i)} / Q_n^{(i)} Q_{n-1}^{(i)}, \quad n = 1, 2, \ldots, \tag{2}$$

$$C_0^{(i)} = b_0^{(i)},$$

where

$$C_n^{(i)} = P_n^{(i)} / Q_n^{(i)},$$

with

$$\begin{cases} P_k^{(i)} = b_k^{(i)} P_{k-1}^{(i)} + a_k^{(i)} P_{k-2}^{(i)}, \\ Q_k^{(i)} = b_k^{(i)} Q_{k-1}^{(i)} + a_k^{(i)} Q_{k-2}^{(i)}, \end{cases} \quad k = 1, \ldots, n, \tag{3}$$

$$P_{-1}^{(i)} = 1 = Q_0^{(i)}, \qquad P_0^{(i)} = b_0^{(i)}, \qquad Q_{-1}^{(i)} = 0.$$

We will now generalize (2) for the branched continued fraction

$$B_0 + \frac{a_1 |}{| B_1} + \frac{a_2 |}{| B_2} + \cdots. \tag{4}$$

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Let us denote by P_n/Q_n the subexpression

$$C_{n,(n,n-1,\ldots,1,0)} = b_0^{(0)} + \sum_{j=1}^{n} \frac{a_j^{(0)}|}{|b_j^{(0)}} + \sum_{i=1}^{n} \frac{a_i \quad |}{\left| b_0^{(i)} + \sum_{j=1}^{n-i} \frac{a_j^{(i)}|}{|b_j^{(i)}} \right.}$$

$$= C_n^{(0)} + \sum_{i=1}^{n} \frac{a_i \,|}{|\, C_{n-i}^{(i)}}. \tag{5}$$

Another subexpression we shall need is

$$\frac{R_k^{(n)}}{S_k^{(n)}} = C_n^{(0)} + \sum_{i=1}^{k} \frac{a_i \,|}{|\, C_{n-1}^{(i)}}, \quad k = 0, \ldots, n,$$

which is in fact the kth convergent of P_n/Q_n. These subconvergents can be ordered in a table

$$\begin{array}{cccc} \frac{R_0^{(0)}}{S_0^{(0)}} & & & \\ \frac{R_0^{(1)}}{S_0^{(1)}} & \frac{R_1^{(1)}}{S_1^{(1)}} & & \\ \frac{R_0^{(2)}}{S_0^{(2)}} & \frac{R_1^{(2)}}{S_1^{(2)}} & \frac{R_2^{(2)}}{S_2^{(2)}} & \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

where we proceed in a certain row from one value to the next one by using (3) for (5):

$$\begin{cases} R_k^{(n)} = C_{n-k}^{(k)} R_{k-1}^{(n)} + a_k R_{k-2}^{(n)}, \\ S_k^{(n)} = C_{n-k}^{(k)} S_{k-1}^{(n)} + a_k S_{k-2}^{(n)}, \end{cases} \quad k = 1, \ldots, n \tag{6}$$

with $R_{-1}^{(n)} = 1 = S_0^{(n)}$, $R_0^{(n)} = C_n^{(0)}$ and $S_{-1}^{(n)} = 0$. If we want to obtain an Euler–Minding series for the branched continued fraction (4) we must compute an expression for the difference

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{R_n^{(n)}}{S_n^{(n)}} - \frac{R_{n-1}^{(n-1)}}{S_{n-1}^{(n-1)}} \tag{7}$$

Remark that in comparison with P_{n-1}/Q_{n-1} the expression P_n/Q_n contains an extra term in each of the involved convergents of B_i. Also B_n is not taken into account in P_{n-1}/Q_{n-1}. In order to compute (7) we must be able to proceed from one row in the table of subconvergents to the next row. The following lemma is a means to calculate the differences $R_k^{(n)} - R_k^{(n-1)}$ and $S_k^{(n)} - S_k^{(n-1)}$.

Lemma. *For $n \geqslant 2$ and $k = 1, \ldots, n-1$*

$$R_k^{(n)} - R_k^{(n-1)} = C_{n-k}^{(k)}\left(R_{k-1}^{(n)} - R_{k-1}^{(n-1)}\right) + a_k\left(R_{k-2}^{(n)} - R_{k-2}^{(n-1)}\right)$$

$$+ (-1)^{n-k+1} \frac{a_1^{(k)} \cdots a_{n-k}^{(k)}}{Q_{n-k}^{(k)} Q_{n-k-1}^{(k)}} R_{k-1}^{(n-1)},$$

$$S_k^{(n)} - S_k^{(n-1)} = C_{n-k}^{(k)}\left(S_{k-1}^{(n)} - S_{k-1}^{(n-1)}\right) + a_k\left(S_{k-2}^{(n)} - S_{k-2}^{(n-1)}\right)$$
$$+(-1)^{n-k+1}\frac{a_1^{(k)}\cdots a_{n-k}^{(k)}}{Q_{n-k}^{(k)}Q_{n-k-1}^{(k)}}S_{k-1}^{(n-1)},$$

with

$$R_{-1}^{(n)} - R_{-1}^{(n-1)} = S_{-1}^{(n)} - S_{-1}^{(n-1)} = S_0^{(n)} - S_0^{(n-1)} = 0$$

and

$$R_0^{(n)} - R_0^{(n-1)} = (-1)^{n+1}\frac{a_1^{(0)}\cdots a_n^{(0)}}{Q_n^{(0)}Q_{n-1}^{(0)}}.$$

Proof. We shall perform the proof only for $R_k^{(n)} - R_k^{(n-1)}$ because it is completely analogous for $S_k^{(n)} - S_k^{(n-1)}$. Choose k and n and write down the recurrence relation (6) for row n and row $n-1$ in the table of subconvergents:

$$R_k^{(n)} = C_{n-k}^{(k)}R_{k-1}^{(n)} + a_kR_{k-2}^{(n)},$$
$$R_k^{(n-1)} = C_{n-1-k}^{(k)}R_{k-1}^{(n-1)} + a_kR_{k-2}^{(n-1)}.$$

By subtracting we get

$$R_k^{(n)} - R_k^{(n-1)} = C_{n-k}^{(k)}\left(R_{k-1}^{(n)} - R_{k-1}^{(n-1)}\right) + a_k\left(R_{k-2}^{(n)} - R_{k-2}^{(n-1)}\right)$$
$$+\left(C_{n-k}^{(k)} - C_{n-k-1}^{(k)}\right)R_{k-1}^{(n-1)}$$

where by (2)

$$C_{n-k}^{(k)} - C_{n-k-1}^{(k)} = (-1)^{n-k+1}\frac{a_1^{(k)}\cdots a_{n-k}^{(k)}}{Q_{n-k}^{(k)}Q_{n-k-1}^{(k)}}.$$

The first three starting values are easy to check and for $R_0^{(n)} - R_0^{(n-1)}$ again (2) is used. □

From the above lemma we see that up to an additional correction term the values $R_k^{(n)} - R_k^{(n-1)}$ and $S_k^{(n)} - S_k^{(n-1)}$ also satisfy a three-term recurrence relation. By means of this result we can write for the numerator of (7):

$$R_n^{(n)}S_{n-1}^{(n-1)} - S_n^{(n)}R_{n-1}^{(n-1)}$$
$$= \left(C_0^{(n)}R_{n-1}^{(n)} + a_nR_{n-2}^{(n)}\right)S_{n-1}^{(n-1)} - \left(C_0^{(n)}S_{n-1}^{(n)} + a_nS_{n-2}^{(n)}\right)R_{n-1}^{(n-1)}$$
$$= C_0^{(n)}S_{n-1}^{(n-1)}\Big[R_{n-1}^{(n-1)} + C_1^{(n-1)}\left(R_{n-2}^{(n)} - R_{n-2}^{(n-1)}\right)$$
$$+ a_{n-1}\left(R_{n-3}^{(n)} - R_{n-3}^{(n-1)}\right) + \left(a_1^{(n-1)}/b_1^{(n-1)}\right)R_{n-2}^{(n-1)}\Big]$$
$$- C_0^{(n)}R_{n-1}^{(n-1)}\Big[S_{n-1}^{(n-1)} + C_1^{(n-1)}\left(S_{n-2}^{(n)} - S_{n-2}^{(n-1)}\right)$$
$$+ a_{n-1}\left(S_{n-3}^{(n)} - S_{n-3}^{(n-1)}\right) + \left(a_1^{(n-1)}/b_1^{(n-1)}\right)S_{n-2}^{(n-1)}\Big]$$
$$+ a_n\left(R_{n-2}^{(n)}S_{n-1}^{(n-1)} - S_{n-2}^{(n)}R_{n-1}^{(n-1)}\right)$$

$$= \left(C_0^{(n)}C_1^{(n-1)} + a_n\right)\left[S_{n-1}^{(n-1)}\left(R_{n-2}^{(n)} - R_{n-2}^{(n-1)}\right) - R_{n-1}^{(n-1)}\left(S_{n-2}^{(n)} - S_{n-2}^{(n-1)}\right)\right]$$
$$+ C_0^{(n)}a_{n-1}\left[S_{n-1}^{(n-1)}\left(R_{n-3}^{(n)} - R_{n-3}^{(n-1)}\right) - R_{n-1}^{(n-1)}\left(S_{n-3}^{(n)} - S_{n-3}^{(n-1)}\right)\right]$$
$$+ \left(C_0^{(n)}\left(a_1^{(n-1)}/b_1^{(n-1)}\right) + a_n\right)\left[R_{n-2}^{(n-1)}S_{n-1}^{(n-1)} - S_{n-2}^{(n-1)}R_{n-1}^{(n-1)}\right]$$

where

$$R_{n-2}^{(n-1)}S_{n-1}^{(n-1)} - S_{n-2}^{(n-1)}R_{n-1}^{(n-1)} = (-1)^{n-1}a_1 \cdots a_{n-1}$$

because $R_{n-2}^{(n-1)}/S_{n-2}^{(n-1)}$ and $R_{n-1}^{(n-1)}/S_{n-1}^{(n-1)}$ are consecutive convergents of the finite continued fraction

$$C_{n-1}^{(0)} + \sum_{i=1}^{n-1} \frac{a_i}{C_{n-1-i}^{(i)}}.$$

In this way

$$P_n/Q_n - P_{n-1}/Q_{n-1}$$
$$= \frac{\left(C_0^{(n)}C_1^{(n-1)} + a_n\right)\left(R_{n-2}^{(n)} - R_{n-2}^{(n-1)}\right) + C_0^{(n)}a_{n-1}\left(R_{n-3}^{(n)} - R_{n-3}^{(n-1)}\right)}{Q_n}$$
$$- \frac{P_{n-1}}{Q_nQ_{n-1}}\left[\left(C_0^{(n)}C_1^{(n-1)} + a_n\right)\left(S_{n-2}^{(n)} - S_{n-2}^{(n-1)}\right) + C_0^{(n)}a_{n-1}\left(S_{n-3}^{(n)} - S_{n-3}^{(n-1)}\right)\right]$$
$$+ (-1)^{n-1}\frac{a_1 \cdots a_{n-1}}{Q_nQ_{n-1}}\left(C_0^{(n)}\frac{a_1^{(n-1)}}{b_1^{(n-1)}} + a_n\right). \tag{8}$$

We remark that (8) reduces to (2) if the continued fraction (4) is not branched because then $R_k^{(k)} = R_k^{(n)}$ and $S_k^{(k)} = S_k^{(n)}$ for all $n \geq k$. Consequently the classical Euler–Minding series will turn out to be a special case of the Euler–Minding series for branched continued fractions.

Theorem. *For $n \geq 2$ the convergent $C_{n,(n,n-1,\ldots,1,0)}$ of the branched continued fraction* (4) *can be written as*

$$\frac{P_n}{Q_n} = C_1^{(0)} + \frac{a_1}{C_0^{(1)}} + \sum_{i=2}^{n}(-1)^{i+1}\frac{a_1 \cdots a_{i-1}}{Q_iQ_{i-1}}\left(a_i + C_0^{(i)}\frac{a_1^{(i-1)}}{b_1^{(i-1)}}\right)$$
$$+ \sum_{i=2}^{n}\frac{\left(a_i + C_0^{(i)}C_1^{(i-1)}\right)\left(R_{i-2}^{(i)} - R_{i-2}^{(i-1)}\right) + C_0^{(i)}a_{i-1}\left(R_{i-3}^{(i)} - R_{i-3}^{(i-1)}\right)}{Q_i}$$
$$- \sum_{i=2}^{n}\frac{P_{i-1}}{Q_{i-1}}\frac{\left(a_i + C_0^{(i)}C_1^{(i-1)}\right)\left(S_{i-2}^{(i)} - S_{i-2}^{(i-1)}\right) + C_0^{(i)}a_{i-1}\left(S_{i-3}^{(i)} - S_{i-3}^{(i-1)}\right)}{Q_i}.$$

Proof. The result is obvious if we write

$$\frac{P_n}{Q_n} = \frac{P_1}{Q_1} + \sum_{i=2}^{n}\left(\frac{P_i}{Q_i} - \frac{P_{i-1}}{Q_{i-1}}\right)$$

and insert (8) for $P_i/Q_i - P_{i-1}/Q_{i-1}$. □

As a result of the previous theorem we can associate with the branched continued fraction (4) the series

$$C_1^{(0)} + \frac{a_1}{C_0^{(1)}} + \sum_{i=2}^{\infty} \left\{ (-1)^{i+1} \frac{a_1 \cdots a_{i-1}}{Q_i Q_{i-1}} \left(a_i + C_0^{(i)} \frac{a_1^{(i-1)}}{b_1^{(i-1)}} \right) \right.$$
$$+ \frac{\left(a_i + C_0^{(i)} C_1^{(i-1)}\right)\left(R_{i-2}^{(i)} - R_{i-2}^{(i-1)}\right) + C_0^{(i)} a_{i-1}\left(R_{i-3}^{(i)} - R_{i-3}^{(i-1)}\right)}{Q_i}$$
$$\left. - \frac{P_{i-1}}{Q_{i-1}} \frac{\left(a_i + C_0^{(i)} C_1^{(i-1)}\right)\left(S_{i-2}^{(i)} - S_{i-2}^{(i-1)}\right) + C_0^{(i)} a_{i-1}\left(S_{i-3}^{(i)} - S_{i-3}^{(i-1)}\right)}{Q_i} \right\}$$

of which the successive partial sums equal the successive convergents $C_{n,(n,n-1,\ldots,1,0)}$ of (4).

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References

[1] W. Siemaszko, Branched continued fraction for double power series, *J. Comput. Appl. Math.* **6** (1980) 121–125.