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THE ε-ALGORITHM AND PADÉ-APPROXIMANTS IN OPERATOR THEORY*

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Abstract. The ε-algorithm of Wynn is closely related to the Padé-table of a univariate function in the following sense: if we apply the ε-algorithm to the partial sums of the power series $f(x)=\sum_{i=0}^{\infty}c_i x^i$ then $\varepsilon_{2m}^{(l-m)}$ is the (l,m) Padé-approximant to $f(x)$ where l is the degree of the numerator and m is the degree of the denominator [C. Brezinski, *Algorithmes d'accélération de la convergence*, Editions Technip, Paris, 1978, pp. 66–68]. In this paper we see that the Padé-approximants for nonlinear operators F: $X\rightarrow Y$, where X is a Banach space and Y a commutative Banach algebra, introduced in [Springer Lect. Notes in Math. 765, 1979, pp. 61–87], satisfy the same property as the univariate Padé-approximants.

1. Padé-approximants in operator theory. We briefly repeat the definition of Padé-approximants in operator theory and a determinental formula for their calculation. More details can be found in [3] and [4].

Let X be a Banach space and Y a commutative Banach algebra (0 denotes the unit for the addition and I the unit for the multiplication). Let F: $X\rightarrow Y$ be analytic in the open ball $B(0,r)$ with centre $0\in X$ and radius $r>0$ [5, pp. 113]:

$$F(x)=\sum_{k=0}^{\infty}\frac{1}{k!}F^{(k)}(0)x^k \quad \text{for } \|x\|<r,$$

where $F^{(k)}(0)$ is the kth Fréchet-derivative of F in 0 and thus a symmetric k-linear bounded operator, and $(1/0!)F^{(0)}(0)x^0=F(0)$.

DEFINITION 1.1. $F(x)=O(x^k)$ $(k\in\mathbb{N})$ if nonnegative constants $r<1$ and K exist such that $\|F(x)\|\leq K\|x\|^k$ for $\|x\|<r$.

Write $D(F)=\{x\in X \mid F(x)$ is regular in Y, i.e. there exists $y\in Y$: $F(x)\cdot y=I=y\cdot F(x)\}$. We shall denote by y^{-1} the inverse element of y in Y for the multiplication in that Banach algebra.

DEFINITION 1.2. An *abstract polynomial* is a nonlinear operator P: $X\rightarrow Y$ with $P(x)=A_n x^n+A_{n-1}x^{n-1}+\cdots+A_0$, where A_i is a symmetric i-linear bounded operator $(i=0,\cdots,n)$ [5, pp. 194].

DEFINITION 1.3. The couple of abstract polynomials

$$(P(x),Q(x))=\left(\sum_{i=0}^{n}A_{nm+i}x^{nm+i},\ \sum_{j=0}^{m}B_{nm+j}x^{nm+j}\right)$$

such that the abstract power series $(F\cdot Q\text{-}P)(x)=O(x^{nm+n+m+1})$ is called a *solution of the Padé-approximation problem of order* (n,m). The choice of $P(x)$ and $Q(x)$, or in other words the translation of degrees in P and Q by $n\cdot m$, can be justified as follows [3]. Write $C_k x^k=(1/k!)F^{(k)}(0)x^k$.

The condition in Definition 1.3 is equivalent with (1a) and (1b):

$$\text{(1a)}\quad \begin{aligned}
&C_0\cdot B_{nm}x^{nm}=A_{nm}x^{nm} && \forall x\in X,\\
&C_1x\cdot B_{nm}x^{nm}+C_0\cdot B_{nm+1}x^{nm+1}=A_{nm+1}x^{nm+1} && \forall x\in X,\\
&\vdots\\
&C_nx^n\cdot B_{nm}x^{nm}+\cdots+C_0\cdot B_{nm+n}x^{nm+n}=A_{nm+n}x^{nm+n} && \forall x\in X,
\end{aligned}$$

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with $B_{nm+j}x^{nm+j} \equiv 0$ if $j>m$;

$$
\text{(1b)}\qquad
\begin{array}{ll}
C_{n+1}x^{n+1} \cdot B_{nm}x^{nm} + \cdots + C_{n+1-m}x^{n+1-m} \cdot B_{nm+m}x^{nm+m} = 0 & \forall x \in X, \\
\vdots & \\
C_{n+m}x^{n+m} \cdot B_{nm}x^{nm} + \cdots + C_{n}x^{n} \cdot B_{nm+m}x^{nm+m} = 0 & \forall x \in X,
\end{array}
$$

with $C_k x^k \equiv 0$ if $k<0$. A solution of (1b) can be computed by means of the following determinants in Y; these formulas are direct generalizations of the classical formulas for the solution of a homogeneous system of m equations in $m+1$ unknowns $B_{nm+j}x^{nm+j}$ $(j=0,\cdots,m)$:

The nm-linear bounded operator

$$
\begin{vmatrix}
C_n x^n & \cdots & C_{n+1-m}x^{n+1-m} \\
C_{n+1}x^{n+1} & \cdots & C_{n+2-m}x^{n+2-m} \\
\vdots & & \vdots \\
C_{n-1+m}x^{n-1+m} & \cdots & C_n x^n
\end{vmatrix} = B_{nm}x^{nm},
$$

the $(nm+j)$-linear bounded operator

$$
\begin{vmatrix}
C_n x^n & \cdots & -C_{n+1}x^{n+1} & \cdots & C_{n+1-m}x^{n+1-m} \\
\vdots & & \vdots & & \vdots \\
C_{n-1+m}x^{n-1+m} & & -C_{n+m}x^{n+m} & \cdots & C_n x^n
\end{vmatrix} = B_{nm+j}x^{nm+j},
$$

$\uparrow$

jth column in $B_{nm}x^{nm}$
replaced by this column $\qquad 1 \le j \le m.$

For every solution of (1b) a solution of (1a) can be calculated by substitution of the $B_{nm+j}x^{nm+j}$ $(j=0,\cdots,m)$ in the left hand side of (1a). So using the classical formulas for the solution of a homogeneous system of equations we get immediately the translation of degrees by $n \cdot m$ in $P(x)$ and $Q(x)$. As a result of these formulas we can also write down the following determinental formulas for $P(x)$ and $Q(x)$:

$$
Q(x) = \begin{vmatrix}
I & & \cdots & I \\
C_{n+1}x^{n+1} & C_n x^n & \cdots & C_{n+1-m}x^{n+1-m} \\
\vdots & \vdots & & \vdots \\
C_{n+m}x^{n+m} & C_{n+m-1}x^{n+m-1} & \cdots & C_n x^n
\end{vmatrix},
$$

$$
P(x) = \begin{vmatrix}
F_n(x) & F_{n-1}(x) & \cdots & F_{n-m}(x) \\
C_{n+1}x^{n+1} & C_n x^n & \cdots & C_{n+1-m}x^{n+1-m} \\
\vdots & \vdots & & \vdots \\
C_{n+m}x^{n+m} & C_{n+m-1}x^{n+m-1} & \cdots & C_n x^n
\end{vmatrix}
$$

where $F_i(x) = \Sigma_{k=0}^{i} C_k x^k$ and $F_i(x) \equiv 0$ for $i<0$. We shall now see how the determinant representations of $P(x)$ and $Q(x)$ link this solution of the Padé-approximation problem of order (n,m) to the ε-algorithm.

2. The ε-algorithm. The ε-algorithm is a nonlinear algorithm due to Wynn [2, pp. 42]; input are the elements of a sequence $\{S_i | i=0,1,\cdots\}$. The following computations are performed:

$$\begin{aligned}
&\varepsilon_{-1}^{(i)}=0, \qquad i=0,1,\cdots,\\
&\varepsilon_{0}^{(i)}=S_i, \qquad i=0,1,\cdots,\\
&\varepsilon_{2j}^{(-j-1)}=0, \qquad j=0,1,\cdots,\\
&\varepsilon_{j+1}^{(i)}=\varepsilon_{j-1}^{(i+1)}+\left[\varepsilon_{j}^{(i+1)}-\varepsilon_{j}^{(i)}\right]^{-1}, \qquad j=0,1,\cdots, \quad i=-j,-j+1,\cdots.
\end{aligned}$$

The $\varepsilon_j^{(i)}$ can be ordered in a table where (i) indicates a diagonal and j a column:

$$\begin{array}{ccccc}
 & \varepsilon_0^{(-1)}=0 & & \varepsilon_2^{(-2)}=0 & \cdots\\
\varepsilon_{-1}^{(0)}=0 & & \varepsilon_1^{(-1)} & & \\
 & \varepsilon_0^{(0)}=S_0 & & \varepsilon_2^{(-1)} & \cdots\\
\varepsilon_{-1}^{(1)}=0 & & \varepsilon_1^{(0)} & & \\
 & \varepsilon_0^{(1)}=S_1 & & \varepsilon_2^{(0)} & \cdots\\
\varepsilon_{-1}^{(2)}=0 & & \varepsilon_1^{(1)} & & \\
 & \varepsilon_0^{(2)}=S_2 & & \varepsilon_2^{(1)} & \cdots\\
\varepsilon_{-1}^{(3)}=0 & \vdots & \varepsilon_1^{(2)} & \vdots & \\
\vdots & & \vdots & &
\end{array}$$

Let us now take $\{S_i | i=0,1,\cdots\}\subseteq Y$ and denote by $\Delta S_i = S_{i+1}-S_i$ and $\Delta^2 S_i=\Delta S_{i+1}-\Delta S_i$. Write

$$H_j(S_i)=\begin{vmatrix} S_i & \cdots & S_{i+j-1}\\ \vdots & & \vdots \\ S_{i+j-1} & \cdots & S_{i+2j-2}\end{vmatrix}.$$

We can prove the following property for the $\varepsilon_j^{(i)}$. The proof is very technical and similar to the proof in [2, pp. 44–46].

THEOREM 2.1. *If $H_{j-1}(\Delta^2 S_{i+1})$ and $H_j(\Delta^2 S_i)$ are regular in Y, then*

$$\varepsilon_{2j}^{(i)}=\frac{\begin{vmatrix} S_{i+j} & \cdots & & S_i\\ \Delta S_{i+j} & \cdots & \Delta S_{i+1} & \Delta S_i\\ \vdots & \ddots & \vdots & \vdots\\ \Delta S_{i+2j-1} & \cdots & \Delta S_{i+j} & \Delta S_{i+j-1}\end{vmatrix}}{\begin{vmatrix} I & \cdots & I\\ \Delta S_{i+j} & \cdots & \Delta S_i\\ \vdots & & \vdots\\ \Delta S_{i+2j-1} & \cdots & \Delta S_{i+j-1}\end{vmatrix}},$$

and if $H_j(\Delta S_{i+1})$ *and* $H_{j+1}(\Delta S_i)$ *are regular in* Y, *then*

$$\varepsilon_{2j+1}^{(i)}=\frac{\begin{vmatrix} I & \cdots & I \\ \Delta^2 S_{i+j} & \cdots & \Delta^2 S_i \\ \vdots & & \vdots \\ \Delta^2 S_{i+2j-1} & \cdots & \Delta^2 S_{i+j-1} \end{vmatrix}}{\begin{vmatrix} \Delta S_{i+j} & \cdots & \Delta S_i \\ \Delta^2 S_{i+j} & \cdots & \Delta^2 S_i \\ \vdots & & \vdots \\ \Delta^2 S_{i+2j-1} & \cdots & \Delta^2 S_{i+j-1} \end{vmatrix}}$$

with $S_i=0$ *for* $i<0$.

Of course we restrict ourselves to the case that the $\varepsilon_j^{(i)}$ are finite; since the ε-algorithm is a nonlinear algorithm, it can always happen that $\varepsilon_{j+1}^{(i)}$ does not exist (when $\varepsilon_j^{(i+1)}-\varepsilon_j^{(i)}$ is singular in Y). It is easy to see now that for $S_i=F_i(x)$, i.e., the partial sums of $F(x)=\sum_{k=0}^{\infty}C_k x^k$, we get

$$\varepsilon_{2m}^{(n-m)}=\frac{\begin{vmatrix} F_n(x) & \cdots & F_{n-m}(x) \\ C_{n+1}x^{n+1} & \cdots & C_{n-m+1}x^{n-m+1} \\ \vdots & & \vdots \\ C_{n+m}x^{n+m} & \cdots & C_n x^n \end{vmatrix}}{\begin{vmatrix} I & \cdots & I \\ C_{n+1}x^{n+1} & \cdots & C_{n+1-m}x^{n+1-m} \\ \vdots & & \vdots \\ C_{n+m}x^{n+m} & \cdots & C_n x^n \end{vmatrix}},$$

The numerator and denominator of $\varepsilon_{2m}^{(n-m)}$ are the determinental formulas for $P(x)$ and $Q(x)$, the solution of the Padé-approximation problem of order (n,m). Let us illustrate this by calculating part of the ε-table for the following nonlinear operator:

$$F\colon C'([1,T])\to C([1,T])$$

$$: x(t)\to e^{x(t)}\frac{dx}{dt}-(1+a),$$

with a a small nonnegative number. The Taylor series expansion is

$$F(x)=\frac{dx}{dt}\sum_{k=0}^{\infty}\frac{1}{k!}[x(t)]^k-(1+a).$$

For the ε-table, we get

$$
\begin{array}{ccccc}
 & 0 & & 0 & \cdots \\
0 & & \dfrac{-1}{1+a} & & \\
 & -(1+a) & & \dfrac{-(1+a)^2}{1+a+\dfrac{dx}{dt}} & \cdots \\
0 & & \dfrac{1}{\dfrac{dx}{dt}} & & \\
 & \dfrac{dx}{dt}-(1+a) & & \dfrac{\dfrac{dx}{dt}-(1+a)(1-x(t))}{1-x(t)} & \cdots \\
0 & & \dfrac{1}{x(t)\dfrac{dx}{dt}} & \vdots & \\
 & \dfrac{dx}{dt}(1+x(t))-(1+a) & \vdots & & \\
 & \vdots & & & \\
0 & & & & \\
\vdots & & & &
\end{array}
$$

3. Applications in operator theory. Several types of nonlinear operator equations

$$F(x)=0$$

can be solved by means of Padé-approximants in operator theory; we mention for instance systems of nonlinear equations, initial value problems, boundary value problems, partial differential equations and nonlinear integral equations. The well-known Newton and Chebyshev iteration [7, pp.205] result respectively from the use of the solution of the Padé-approximation problem of order (1,0) and (2,0) [5], [6]. An interesting new iterative procedure of third order,

$$x_{i+1}=x_i+\frac{\left(-F_i'^{-1}F_i\right)\cdot\left(-F_i'^{-1}F_i\right)}{-F_i'^{-1}F_i+\frac{1}{2}F_i'^{-1}F_i''\left(-F_i'^{-1}F_i\right)^2}$$

where

$F_i=F(x_i)$,

$F_i'=F'(x_i)$ a linear operator (1st Fréchet-derivative at x_i),

$F_i''=F''(x_i)$ a bilinear operator (2nd Fréchet-derivative at x_i),
the division is a multiplication by the inverse element of the denominator,

which we called the Halley iteration [5], [6], proves to be especially interesting in the neighbourhood of singularities because it is derived from the solution of the Padé-approximation problem of order (1, 1). If we use the ε-algorithm for the calculation of the next iteration step in Halley's method, we have

$$\varepsilon_0^{(0)} = x_i,$$

$$\varepsilon_1^{(0)} = \left[-F_i'^{-1}F_i\right]^{-1},$$

$$\varepsilon_0^{(1)} = x_i - F_i'^{-1}F_i, \qquad \varepsilon_2^{(0)} = x_{i+1}.$$

$$\varepsilon_1^{(1)} = -2\left[F_i'^{-1}F_i''\left(-F_i'^{-1}F_i\right)^2\right]^{-1},$$

$$\varepsilon_0^{(2)} = x_i - F_i'^{-1}F_i - \frac{1}{2}F_i'^{-1}F_i''\left(-F_i'^{-1}F_i\right)^2,$$

For numerical examples and results we refer to [5], [6].

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