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ON THE CONVERGENCE OF THE MULTIVARIATE "HOMOGENEOUS" *QD*-ALGORITHM

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Abstract.

The convergence of columns in the univariate *qd*-algorithm to reciprocals of polar singularities of meromorphic functions has often proved to be very useful. A multivariate *qd*-algorithm was discovered in 1982 for the construction of the so-called homogeneous Padé approximants.

In the first section we repeat the univariate convergence results. In the second section we summarize the "homogeneous" multivariate *qd*-algorithm. In the third section a multivariate convergence result is proved by combining results from the previous sections. This convergence result is compared with another theorem for the general order multivariate *qdg*-algorithm. The main difference lies in the fact that the homogeneous form detects the polar singularities "pointwise" while the general form detects them "curvewise".

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1. Convergence of the Univariate *qd*-Algorithm.

Let the function $f(z)$ be known by its formal series expansion

$$f(z) = \sum_{i=0}^{\infty} c_i z^i. \tag{1}$$

The series expansion is taken around the origin only to simplify the notation. We set $c_i = 0$ for $i < 0$. For arbitrary integers n and for integers $m \geq 0$ we define determinants

$$H_m^{(n)} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+m-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+m} \\ \vdots & & & \vdots \\ c_{n+m-1} & c_{n+m} & \cdots & c_{n+2m-2} \end{vmatrix}$$

with $H_0^{(n)} = 1$. The series (1) is termed k-normal if $H_m^{(n)} \neq 0$ for $m = 0, 1, \ldots,$

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k and $n \geq 0$. It is called ultimately k-normal if for every $0 \leq m \leq k$ there exists an $n(m)$ such that $H_m^{(n)} \neq 0$ for $n > n(m)$. With (1) we can also define the qd-scheme where subscripts denote columns and superscripts downward sloping diagonals [5, p. 609]:

(a) the start columns are given by

$$e_0^{(n)} = 0 \qquad n = 1, 2, \ldots$$

$$q_1^{(n)} = \frac{c_{n+1}}{c_n} \qquad n = 0, 1, \ldots$$

(b) and the rhombus rules for continuation of the scheme by

$$e_m^{(n)} = q_m^{(n+1)} - q_m^{(n)} + e_{m-1}^{(n+1)} \qquad m = 1, 2, \ldots \qquad n = 0, 1, \ldots$$

$$q_{m+1}^{(n)} = \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)} \qquad m = 1, 2, \ldots \qquad n = 0, 1, \ldots$$

THEOREM 1. *Let* (1) *be the Taylor series at* $z = 0$ *of a function* f *meromorphic in the disk* $B(0, R) = \{z : |z| < R\}$ *and let the poles* z_i *of* f *in* $B(0, R)$ *be numbered such that*

$$z_0 = 0 < |z_1| \leq |z_2| \leq \ldots < R,$$

each pole occurring as many times in the sequence $\{z_i\}_{i \in N}$ *as indicated by its order. If* f *is ultimately* k*-normal for some integer* $k > 0$, *then the qd-scheme associated with* f *has the following properties (put* $z_{k+1} = \infty$ *if* f *has only* k *poles):*

(a) For each m *with* $0 < m \leq k$ *and* $|z_{m-1}| < |z_m| < |z_{m+1}|$,

$$\lim_{n \to \infty} q_m^{(n)} = z_m^{-1}. \tag{2}$$

(b) For each m *with* $0 < m \leq k$ *and* $|z_m| < |z_{m+1}|$,

$$\lim_{n \to \infty} e_m^{(n)} = 0 \tag{3}$$

PROOF. The proof can be found in [5, pp. 612–613]. ■

Any index m such that the strict inequality

$$|z_m| < |z_{m+1}|$$

holds, is called a critical index. It is clear that the critical indices of a function do not depend on the order in which the poles of equal modulus are numbered. The theorem above states that if m is a critical index and f is ultimately m-normal, then

$$\lim_{n \to \infty} e_m^{(n)} = 0.$$

Thus the *qd*-table of a meromorphic function is divided into subtables by the *e*-columns tending to zero. If a subtable contains j columns of q-values, the presence of j poles of equal modulus is indicated. If $j = 1$ the q-column converges to the reciprocal of the corresponding pole. In [5] it is also indicated how to determine the poles if $j > 1$. We omit summarizing all this as well because it will be clear at the end of the paper how the multivariate results also apply to the "equal modulus" situation.

From the above theorem the *qd*-scheme seems to be an ingenious tool for determining, under certain conditions, the poles of a meromorphic function f directly from its Taylor series at the origin. Any q-column corresponding to a simple pole of isolated modulus would be flanked by e-columns that tend to zero. If f is rational, the last e-column is even theoretically equal to zero, as can be seen from the following theorem. The proof hereof is based on the next lemma [5, pp. 610–613].

LEMMA 1. *Let f be given its formal Taylor series expansion* (1). *If there exists a positive integer k such that f is k-normal, then the values $q_m^{(n)}$ and $e_m^{(n)}$ exist for $m = 1, \ldots, k$ and $n \geq 0$ and they are given by*

$$q_m^{(n)} = \frac{H_m^{(n+1)} H_{m-1}^{(n)}}{H_m^{(n)} H_{m-1}^{(n+1)}},$$

$$e_m^{(n)} = \frac{H_{m+1}^{(n)} H_{m-1}^{(n+1)}}{H_m^{(n)} H_m^{(n+1)}}.$$

THEOREM 2. *Let* (1) *be the Taylor series at $z = 0$ of a rational function of degree n in the numerator and $m \leq n$ in the denominator. Then if the series f is m-normal,*

$$e_m^{(n-m+h)} = 0, \qquad h > 0.$$

These convergence results are closely linked to the convergence theorem of de Montessus de Ballore for Padé approximants of meromorphic functions. The reason is that the q- and e-values appear in the partial numerators and denominators of the corresponding continued fraction

$$f(z) = c_0 + \sum_{i=1}^{\infty} \left(\frac{-q_i^{(1)} z|}{|1} + \frac{-e_i^{(1)} z|}{|1} \right).$$

In order to prepare for a multivariate version we rewrite this continued fraction as

$$f(z) = c_0 + \sum_{i=1}^{\infty} \left(\frac{-Q_i^{(1)}(z)|}{|1} + \frac{-E_i^{(1)}(z)|}{|1} \right)$$

introducing

(4a) $$Q_i^{(n)}(z) = q_i^{(n)} \cdot z, \qquad E_i^{(n)}(z) = e_i^{(n)} \cdot z,$$

with starting values

$$E_0^{(n)}(z) = 0 \qquad n = 1, 2, \ldots, \tag{4b}$$

$$Q_1^{(n)}(z) = \frac{c_{n+1}z^{n+1}}{c_n z^n}, \qquad n = 0, 1, \ldots, \tag{4c}$$

and the same continuation rules as above.

2. The Homogeneous Multivariate *qd*-Algorithm.

We restrict ourselves to the case of two variables because the generalization to functions of more variables is only notationally more difficult. Let $f(x, y)$ be given by its formal Taylor series expansion

$$f(x, y) = \sum_{(i,j)\in \mathbf{N}^2} c_{ij}x^i y^j \tag{5}$$

with $c_{ij} = 0$ if < 0 or $j < 0$. This sum over index pairs in $\mathbf{N}^2$ is rewritten as a sum over indices in $\mathbf{N}$ by grouping terms into homogeneous expressions:

$$f(x, y) = \sum_{l\in\mathbf{N}} \left(\sum_{i+j=l} c_{ij}x^i y^j \right).$$

The homogeneous multivariate *qd*-algorithm is then defined by:

$$E_0^{(n)}(x, y) = 0 \qquad n = 1, 2, \ldots, \tag{6a}$$

$$Q_1^{(n)}(x, y) = \frac{\sum_{i+j=n+1} c_{ij}x^i y^j}{\sum_{i+j=n} c_{ij}x^i y^j}, \tag{6b}$$

$$E_m^{(n)}(x, y) = Q_m^{(n+1)}(x, y) - Q_m^{(n)}(x, y) + E_{m-1}^{(n+1)}(x, y),$$

$$m = 1, 2, \ldots, \qquad n = 0, 1, \ldots, \tag{6c}$$

$$Q_{m+1}^{(n)}(x, y) = \frac{E_m^{(n+1)}(x, y)Q_m^{(n+1)}(x, y)}{E_m^{(n)}(x, y)},$$

$$m = 1, 2, \ldots, \qquad n = 0, 1, \ldots.$$

If we arrange the values $Q_m^{(n)}(x, y)$ and $E_m^{(n)}(x, y)$ as in the univariate case, where subscripts indicate columns and superscripts indicate downward sloping diagonals, then the entire construction is very similar to the univariate reformulation (4). Diagonal homogeneous Padé approximants are now obtained from the continued fraction [2]

$$f(x, y) = c_{00} + \sum_{i=1}^{\infty} \left(\frac{-Q_i^{(1)}(x, y)|}{|1} + \frac{-E_i^{(1)}(x, y)|}{|1} \right).$$

3. Convergence Results for Multivariate Meromorphic Functions.

It was shown in [1, pp. 22–28] that problem (6) reduces to problem (4) on rays $y = \lambda x$.

THEOREM 3. *Let $f(x, y)$ be given by its formal Taylor series expansion*

$$f(x, y) = \sum_{(i,j)\in\mathbf{N}} c_{ij} x^i y^j,$$

generating $Q_m^{(n)}(x, y)$ and $E_m^{(n)}(x, y)$ through (6 a–c), and let $f_\lambda(x)$ be defined by

$$f_\lambda(x) = \sum_{l\in\mathbf{N}} \left(\sum_{i+j=l} c_{ij}\lambda^j \right) x^l,$$

generating $\tilde{Q}_m^{(n)}(x)$ and $\tilde{E}_m^{(n)}(x)$ through (4 a–c) when used with $c_l = \Sigma_{i+j=l}\, c_{ij}\lambda^j$. Then for $y = \lambda x$,

$$c_{00} + \sum_{i=1}^{2m} \left(\frac{-Q_i^{(n-m+1)}(x, y)|}{|\ 1} + \frac{-E_i^{(n-m+1)}(x, y)|}{|\ 1} \right)$$

$$= c_{00} + \sum_{i=1}^{2m} \left(\frac{-\tilde{Q}_i^{(n-m+1)}(x)|}{|\ 1} + \frac{-\tilde{E}_i^{(n-m+1)}(x)|}{|\ 1} \right),$$

$$n \geq m, \qquad m = 0, 1, \ldots.$$

This result implies that for $n, m \geq 1$

$$Q_m^{(n)}(x, \lambda x) = \tilde{Q}_m^{(n)}(x), \qquad E_m^{(n)}(x, \lambda x) = \tilde{E}_m^{(n)}(x),$$

with as in (4a)

$$\tilde{Q}_m^{(n)}(x) = \tilde{q}_m^{(n)} \cdot x, \qquad \tilde{E}_m^{(n)}(x) = \tilde{e}_m^{(n)} \cdot x.$$

This could also be proved directly, but we want to indicate the link with the result by [1]. When we now want to use the homogeneous *qd*-algorithm to detect the polar singularities of $f(x, y)$, we proceed as follows.

THEOREM 4. *Let (5) be the Taylor series at the origin of a function $f(x, y)$ meromorphic in the polydisc $B(0, R) = \{(x, y): |x| < R, |y| < R\}$, meaning that there exists a polynomial $q(x, y)$ such that $(fq)(x, y)$ is holomorphic in $B(0, R)$. Let for $\lambda \in \boldsymbol{R}$ the function $f_\lambda(x)$ be defined by*

$$f_\lambda(x) = f(x, \lambda x),$$

and let the poles z_i of f_λ in $B(0, R)$ be numbered such that

$$z_0 = 0 < |z_1| \leq |z_2| \leq \ldots < R,$$

each pole occurring as many times in the sequence $\{z_i\}_{i\in N}$ as indicated by its order. If f_λ is ultimately k-normal for some integer $k > 0$, then the homogeneous QD-scheme associated with f has the following properties (put $z_{k+1} = \infty$ if f_λ has only k poles):

(*a*) *For each* m *with* $0 < m \leq k$ *and* $|z_{m-1}| < |z_m| < |z_{m+1}|$,

$$\lim_{n\to\infty} Q_m^{(n)}(x, \lambda x) = z_m^{-1} \cdot x.$$

(*b*) *For each* m *with* $0 < m \leq k$ *and* $|z_m| < |z_{m+1}|$,

$$\lim_{n\to\infty} E_m^{(n)}(x, \lambda x) = 0.$$

PROOF. For the proof we use Theorem 3 and apply the univariate *qd*-algorithm to the function $f_\lambda(x) = f(x, \lambda x)$. This univariate algorithm coincides with the homogeneous one on the ray $y = \lambda x$. By Theorem 1 it detects the poles of f_λ, in other words the poles of $f(x, y)$ on $y = \lambda x$. Doing this for λ varying from $-\infty$ to $+\infty$, one can detect all the poles of $f(x, y)$ point by point. ■

How the parameter λ affects the order in which the poles of $f(x, y)$ are detected pointwise (and not curvewise) can be learned from the numerical example given in the next section.

In analogy with the univariate Padé approximation case [5, p. 610] it is also possible to give explicit determinant formulas for the multivariate Q- and E-values. We introduce the notation

$$C_l(x, y) = \sum_{i+j=l} c_{ij} x^i y^j, \qquad l = 0, 1, \dots,$$

and the determinants

$$H_m^{(n)}(x, y) = \begin{vmatrix} C_n(x, y) & C_{n+1}(x, y) & \cdots & C_{n+m-1}(x, y) \\ C_{n+1}(x, y) & C_{n+2}(x, y) & \cdots & C_{n+m}(x, y) \\ \vdots & & & \vdots \\ C_{n+m-1}(x, y) & C_{n+m}(x, y) & \cdots & C_{n+2m-2}(x, y) \end{vmatrix}.$$

The series (5) is termed k-normal if $H_m^{(n)}(x, y) \not\equiv 0$ for $m = 0, 1, \dots, k$ and $n \geq 0$. It is called ultimately k-normal if for every $0 \leq m \leq k$ there exists an $n(m)$ such that $H_m^{(n)}(x, y) \not\equiv 0$ for $n > n(m)$. By means of the determinant identities of Sylvester and Schweins and using the continued fraction representation for homogeneous Padé approximants, we can prove the following lemma for k-normal multivariate series [2].

LEMMA 2. *Let* $f(x, y)$ *be given by its formal Taylor series expansion* (5). *If there exists a positive integer* k *such that* $f(x, y)$ *is* k*-normal then the functions* $Q_m^{(n)}(x, y)$ *and* $E_m^{(n)}(x, y)$ *exist for* $m = 1, \dots, k$ *and* $n \geq 0$ *and they are given by*

$$Q_m^{(n)}(x, y) = \frac{H_m^{(n+1)} H_{m-1}^{(n)}}{H_m^{(n)} H_{m-1}^{(n+1)}}(x, y),$$

$$E_m^{(n)}(x, y) = \frac{H_{m+1}^{(n)} H_{m-1}^{(n+1)}}{H_m^{(n)} H_m^{(n+1)}}(x, y).$$

Before proceeding to the next section we can complete the list of results with the following multivariate analogue of Theorem 2.

THEOREM 5. *Let* (5) *be the Taylor series at the origin of a multivariate rational function of homogeneous degree n in the numerator and* $m \leq n$ *in the denominator. Then if the series* $f(x, y)$ *is m-normal,*

$$E_m^{(n-m+h)}(x, y) \equiv 0, \qquad h > 0.$$

PROOF. The proof is based on the consistency property of homogeneous Padé approximants which can be found in [3, p. 65] and which says that if one computes the homogeneous multivariate Padé approximant $r_{st}(x, y)$ of degree $s \geq n$ in the numerator and $t \geq m$ in the denominator for the rational function $f(x, y)$, then one finds $r_{st} = f$. The approximation method is consistent: it does not return another rational function than the one to be approximated. This implies that the homogeneous expression of degree t in the denominator of $r_{st}(x, y)$ is zero for every $s \geq n$ and $t > m$. This expression is a factor of the homogeneous expression $H_t^{(s-t+2)}(x, y)$ of degree $s(t + 1)$. For $t = m + 1$ and $s = n + h$ with $h \geq 0$ this gives $H_{m+1}^{(n-m+h+1)}(x, y) \equiv 0$. From Lemma 2 we know that this determinant appears in the numerator of $E_m^{(n-m+h)}(x, y)$ with $h > 0$ and hence the proof is completed. ∎

4. Numerical Illustration and Comparison.

In what follows we discuss functions $f(x, y)$ which are meromorphic in a polydisc $B(0, 0; R_1, R_2) = \{(x, y) : |x| < R_1, |y| < R_2\}$, meaning that there exists a polynomial

$$D(x, y) = \sum_{l=0}^{m} \sum_{i+j-l} d_{ij} x^i y^j$$

of homogeneous degree m such that $(fD)(x, y)$ is analytic in the polydisc above. Consider the meromorphic finction

$$f(x, y) = \frac{\exp(x + 2y)}{-5x^3 - x^2y - 5xy^2 - y^3 - 5x^2 + 8xy + 7y^2 + 40x - 4y - 30}$$
$$= \frac{\exp(x + 2y)}{(5 - 5x - y)(x^2 + 2x + y^2 - 2y - 6)}.$$

Its polar singularities consist of a "straight line" through (0, 5) and (1, 0) and a "circle" with center $(-1, 1)$ and radius $2\sqrt{2}$, see Figure 1.

Let us explore several rays $y = \lambda x$ and apply Theorem 3. For $\lambda = 0$ the poles are found in the following order:

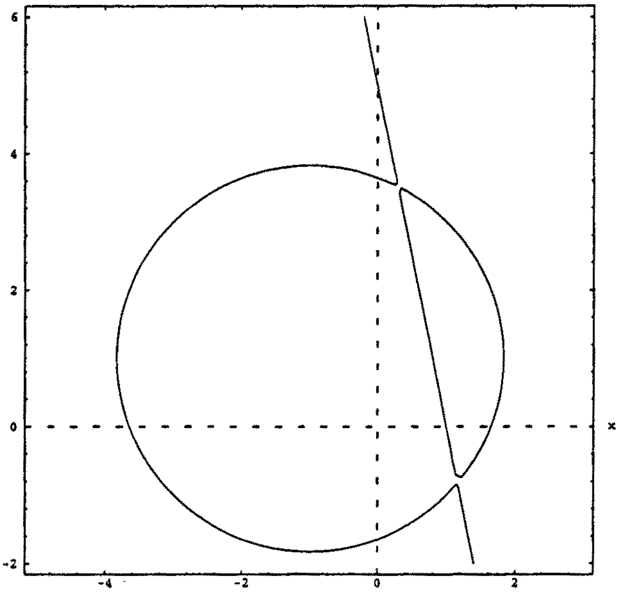

Figure 1. Polar singularities of $f(x, y)$.

$$\lim_{n\to\infty} Q_1^{(n)}(x, \lambda x) = \lim_{n\to\infty} \tilde{Q}_1^{(n)}(x) = 1x,$$

$$\lim_{n\to\infty} Q_2^{(n)}(x, \lambda x) = \lim_{n\to\infty} \tilde{Q}_2^{(n)}(x) = (-1 + \sqrt{7})^{-1}x,$$

$$\lim_{n\to\infty} Q_3^{(n)}(x, \lambda x) = \lim_{n\to\infty} \tilde{Q}_3^{(n)}(x) = (-1 - \sqrt{7})^{-1}x.$$

An approximation for the poles on $y = 0$ is then given by the three points

$$(1, 0), \qquad (-1 + \sqrt{7}, 0), \qquad (-1 - \sqrt{7}, 0),$$

the first one coming from the line, the second one being the intersection point of the x-axis with the circle closest to the origin and the third one being the other intersection point of the circle with the x-axis. For $\lambda = -1$ the application of Theorem 3 and Theorem 1 gives:

$$\lim_{n\to\infty} Q_1^{(n)}(x, \lambda x) = \lim_{n\to\infty} \tilde{Q}_1^{(n)}(x) = 1x,$$

$$\lim_{n\to\infty} Q_2^{(n)}(x, \lambda x) = \lim_{n\to\infty} \tilde{Q}_2^{(n)}(x) = (5/4)^{-1}x,$$

$$\lim_{n\to\infty} Q_3^{(n)}(x, \lambda x) = \lim_{n\to\infty} \tilde{Q}_3^{(n)}(x) = -(3)^{-1}x.$$

An approximation of the poles on $y = -x$ is then given by the three points

$$(1, -1), \qquad (5/4, -5/4), \qquad (-3, 3),$$

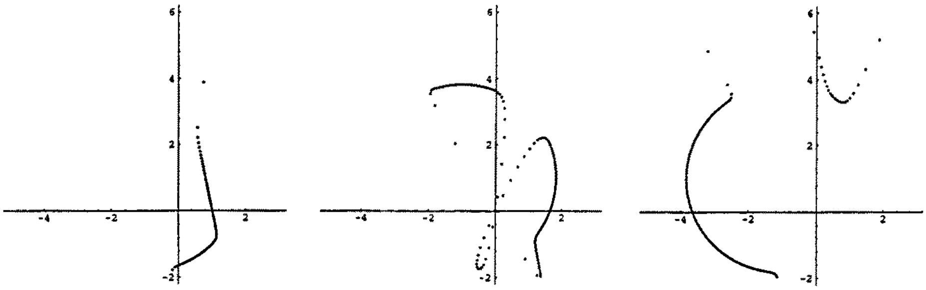

Figure 2a, b, c. Poles found from the first, second and third Q-columns.

the first pole coming from the circle, the second one from the line and the third one being the other point on the circle further away from the origin.

Figures 2a, 2b, and 2c depict respectively all the poles found from the first, second and third Q-columns, "all" meaning that we scanned the rays $y = \lambda x$ for $\lambda = -\infty \rightarrow +\infty$. One can see that after each column, parts of the line and the circle are discovered. However the "straight line" and the "circle" are not recaptured as separate entities. We also print the $Q_1^{(24)}(x, y)$-function that has been used to construct Figure 2a. The functions $Q_2^{(22)}(x, y)$ and $Q_3^{(20)}(x, y)$ are similar but longer expressions and are therefore omitted. All computations were performed in exact rational arithmetic using Mathematica. Note that the Q-functions projected on the rays have the form $\alpha(\lambda)x$. If we denote the projections of $Q_1^{(24)}$, $Q_2^{(22)}$ and $Q_3^{(20)}$ on $y = \lambda x$ respectively by $q_1^{(24)}(\lambda)x$, $q_2^{(22)}(\lambda)x$ and $q_3^{(20)}(\lambda)x$, then the location of the poles is indicated by

$$([q_i^{(26-2i)}(\lambda)]^{-1}, \lambda[q_i^{(26-2i)}(\lambda)]^{-1}) \qquad i = 1, 2, 3$$

$$Q_1^{(24)}(x, y) = \frac{\begin{gathered} 1.x^{25} + 5.867x^{24}y + 17.32x^{23}y^2 + 33.49x^{22}y^3 + 47.49x^{21}y^4 \\ + 52.3x^{20}y^5 + 46.5x^{19}y^6 + 34.11x^{18}y^7 + 21.1x^{17}y^8 \\ + 11.06x^{16}y^9 + 5.04x^{15}y^{10} + 1.943x^{14}y^{11} + 0.6869x^{13}y^{12} \\ + 0.1865x^{12}y^{13} + 0.06081x^{11}y^{14} + 0.006405x^{10}y^{15} \\ + 0.005508x^9y^{16} - 0.001171x^8y^{17} + 0.0007941x^7y^{18} \\ - 0.0002851x^6y^{19} + 0.0001043x^5y^{20} - 0.00003069x^4y^{21} \\ + 7.565\cdot 10^{-6}x^3y^{22} - 1.439\cdot 10^{-6}x^2y^{23} + 1.923\cdot 10^{-7}xy^{24} \\ - 1.382\cdot 10^{-8}y^{25} \end{gathered}}{\begin{gathered} 1.x^{24} + 5.667x^{23}y + 16.18x^{22}y^2 + 30.25x^{21}y^3 + 41.44x^{20}y^4 \\ + 44.01x^{19}y^5 + 37.69x^{18}y^6 + 26.57x^{17}y^7 + 15.79x^{16}y^8 \\ + 7.911x^{15}y^9 + 3.452x^{14}y^{10} + 1.259^{13}y^{11} + 0.429x^{12}y^{12} \\ + 0.106x^{11}y^{13} + 0.03554x^{10}y^{14} + 0.002156x^9y^{15} \\ + 0.00327x^8y^{16} - 0.0008024x^7y^{17} + 0.0004399x^6y^{18} \\ - 0.0001453x^5y^{19} + 0.00004598x^4y^{20} - 0.00001144x^3y^{21} \\ + 2.249\cdot 10^{-6}x^2y^{22} - 3.076\cdot 10^{-7}xy^{23} + 2.277\cdot 10^{-8}y^{24} \end{gathered}}$$

$$1. + 5.867\lambda + 17.32\lambda^2 + 33.49\lambda^3 + 47.49\lambda^4 + 52.3\lambda^5$$

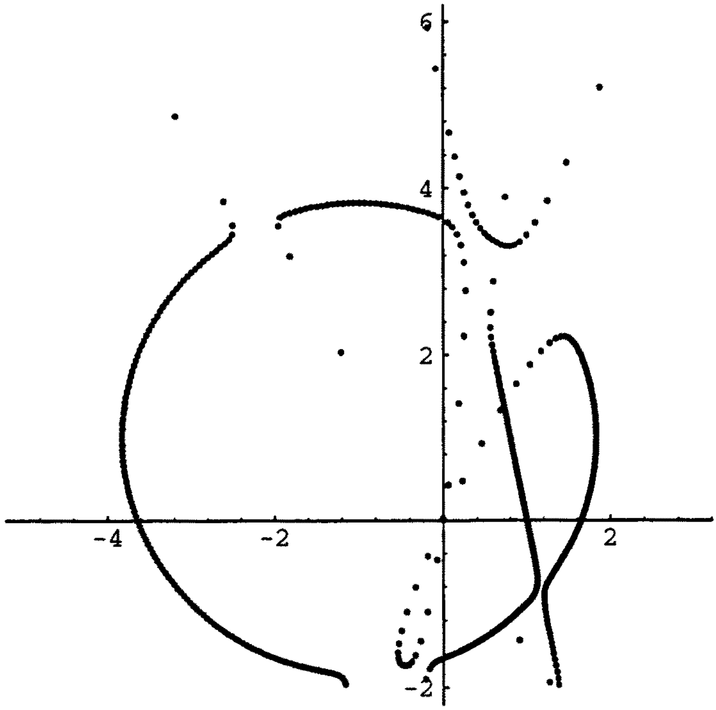

Figure 3. Estimation of all the poles of $f(x, y)$.

$$Q_2^{(24)}(x, \lambda x) = x\frac{\begin{array}{c} + 46.5\lambda^6 + 34.11\lambda^7 + 21.1\lambda^8 + 11.06\lambda^9 + 5.04\lambda^{10} \\ + 1.943\lambda^{11} + 0.6869\lambda^{12} + 0.1865\lambda^{13} + 0.06081\lambda^{14} \\ + 0.006405\lambda^{15} + 0.005508\lambda^{16} + 0.1865\lambda^{13} + 0.06081\lambda^{14} \\ + 0.006405\lambda^{15} + 0.005508\lambda^{16} - 0.001171\lambda^{17} \\ + 0.0007941\lambda^{18} - 0.0002851\lambda^{19} + 0.0001043\lambda^{20} \\ + (10^{-5})(-3.069\lambda^{21} + 0.7565\lambda^{22} - 0.1439\lambda^{23} \\ + 0.01923\lambda^{24} - 0.001382\lambda^{25}) \end{array}}{\begin{array}{c} 1. + 5.667\lambda + 16.18\lambda^2 + 30.25\lambda^3 + 41.44\lambda^4 + 44.01\lambda^5 \\ + 37.69\lambda^6 + 26.57\lambda^7 + 15.79\lambda^8 + 7.911\lambda^9 + 3.452\lambda^{10} \\ + 1.259\lambda^{11} + 0.429\lambda^{12} + 0.106\lambda^{13} + 0.03554\lambda^{14} \\ + 0.002156\lambda^{15} + 0.00327\lambda^{16} - 0.0008024\lambda^{17} \\ + 0.0004399\lambda^{18} - 0.0001453\lambda^{19} + 0.00004598\lambda^{20} \\ + (10)^{-5}(-1.144\lambda^{21} + 0.2249\lambda^{22} - 0.03076\lambda^{23} + 0.002277\lambda^{24}) \end{array}}$$

$$= q_1^{(24)}(\lambda)x.$$

If we compare this convergence result to the one described in [4] for a general order multivariate *qdg*-algorithm, we see that there the algorithm first discovers the lines as a polar factor and then the circle as a polar factor. These factors are identified as separate objects. The price one has to pay for this elegance is that the algorithm must be programmed in order to deal with algebraic expressions instead of with numeric data. The homogeneous *qd*-algorithm delivers the poles point by point (numeric output) while the general order *qdg*-algorithm delivers the poles as algebraic curves (formula output). This means that the general order *qdg*-algorithm is considerably slower than the homogeneous *qd*-algorithm when used for pole detection. However, its reply is considerably more accurate. Combining the figures above into one global picture results in the estimation of all the poles of $f(x, y)$ shown in Figure 3.

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