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A COMPARISON OF SOME MULTIVARIATE PADÉ-APPROXIMANTS*

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Abstract. In [5] Levin defined general order Padé-type rational approximants of a function of n variables (here referred to as "type L" approximants). In §1 we repeat briefly the defining equations and the determinant representation for $n=2$. Levin proved that the Chisholm approximants were a special case of his "type L" approximants.

The multivariate Padé-approximants (here referred to as "type C" approximants) were introduced in [1] and [2]; we repeat the definition for $n=2$ in §2. They have several nice properties which often imply numerical advantages; examples of such situations are given in [3] and [4].

In §3 we show that "type C" approximants are also a special case of the "type L" approximants. The explicit determinant formulas are a link between the solution of the Padé approximation problem and the irreducible rational form of the solution. Via the determinant representation we can also see that, in the case of "type C" approximants, we deal with matrices that are near-Toeplitz. This is not true for the Chisholm approximants. A theorem concerning the displacement-rank of the matrix of the homogeneous system, defining the coefficients of the denominator of the "type C" approximant, is proved.

In §4 analogous results are formulated for $n>2$.

1. General order Padé-type rational approximants in two variables (or type L approximants). We repeat some notations and definitions given by Levin.

Let $\mathbf{N}=\{0,1,2,\cdots\}$. Given a subset D of $\mathbf{Z}^2$ we define:

(a) the complement $\bar{D}=\mathbf{Z}^2\setminus D$,

(b) the (i,j)-translation of D as $D_{ij}=\{(k,n)\,|\,(k+i,n+j)\in D\}$,

(c) the nonnegative part of D as $D^+=D\cap\mathbf{N}^2$.

To any subset D such that D^+ is a finite set we associate polynomials

$$\sum_{(i,j)\in D^+} b_{ij}x^iy^j \quad \text{with } b_{ij} \text{ in } \mathbf{R}$$

We call D the *rank* of the polynomials. Given the double power series

$$f(x,y)=\sum_{i,j=0}^{\infty} c_{ij}x^iy^j$$

we will choose three subsets N,D and E of $\mathbf{Z}^2$ and construct an $[N/D]_E$ approximation to $f(x,y)$ as follows:

$$P(x,y)=\sum_{(i,j)\in N^+} a_{ij}x^iy^j \qquad (N\leftarrow\text{numerator}), \tag{1.1a}$$

$$Q(x,y)=\sum_{(i,j)\in D^+} b_{ij}x^iy^j \qquad (D\leftarrow\text{denominator}), \tag{1.1b}$$

$$(f\cdot Q-P)(x,y)=\sum_{(i,j)\in \bar{E}^+} d_{ij}x^iy^j \qquad (E\leftarrow\text{equations}). \tag{1.1c}$$

We select N,D and E such that

(a) $D\subset\mathbf{N}^2$ has m elements, numbered

$$(i_1,j_1),\cdots,(i_m,j_m),$$

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(b) $N \subset E$ and $H = E \setminus N$ has $m-1$ elements in $\mathbf{N}^2$, numbered

$$(h_2,k_2),\cdots,(h_m,k_m) \qquad (H \leftarrow \text{homogeneous equations}).$$

Then $P(x,y)$ and $Q(x,y)$, defined by (1.1c), are given by:

$$P(x,y)=\begin{vmatrix} x^{i_1}y^{j_1}N_{i_1j_1}(x,y) & x^{i_2}y^{j_2}N_{i_2j_2}(x,y) & \cdots & x^{i_m}y^{j_m}N_{i_mj_m}(x,y) \\ c_{h_2-i_1,k_2-j_1} & c_{h_2-i_2,k_2-j_2} & \cdots & c_{h_2-i_m,k_2-j_m} \\ c_{h_3-i_1,k_3-j_1} & c_{h_3-i_2,k_3-j_2} & \cdots & c_{h_3-i_m,k_3-j_m} \\ \vdots & \vdots & & \vdots \\ c_{h_m-i_1,k_m-j_1} & c_{h_m-i_2,k_m-j_2} & \cdots & c_{h_m-i_m,k_m-j_m} \end{vmatrix}$$

where

$$N_{i_1j_1}(x,y)=\sum_{(i,j)\in N^+_{i_1j_1}} c_{ij}x^iy^j,$$

and

$$(1.1\text{d}) \qquad Q(x,y)=\begin{vmatrix} x^{i_1}y^{j_1} & x^{i_2}y^{j_2} & \cdots & x^{i_m}y^{j_m} \\ c_{h_2-i_1,k_2-j_1} & c_{h_2-i_2,k_2-j_2} & \cdots & c_{h_2-i_m,k_2-j_m} \\ c_{h_3-i_1,k_3-j_1} & c_{h_3-i_2,k_3-j_2} & \cdots & c_{h_3-i_m,k_3-j_m} \\ \vdots & \vdots & & \vdots \\ c_{h_m-i_1,k_m-j_1} & c_{h_m-i_2,k_m-j_2} & \cdots & c_{h_m-i_m,k_m-j_m} \end{vmatrix}.$$

2. Multivariate Padé-approximants for a double power series (or type C approximants). We define a polynomial of degree l in two variables as

$$\sum_{i+j=0}^{l} a_{ij}x^iy^j.$$

A term $a_{ij}x^iy^j$ is said to be of degree $i+j$. The order $\partial_0 P$ and the exact degree ∂P are defined by

$$\partial_0 P=\min\{i+j \mid a_{ij}\neq 0\}, \qquad \partial P=\max\{i+j \mid a_{ij}\neq 0\}.$$

In the Padé-approximation problem of order (l,m) we try to find a pair (P,Q) of two-variable polynomials,

$$(2.1\text{a}) \qquad P(x,y)=\sum_{i+j=lm}^{lm+l} a_{ij}x^iy^j,$$

$$(2.1\text{b}) \qquad Q(x,y)=\sum_{i+j=lm}^{lm+m} b_{ij}x^iy^j,$$

such that

$$(2.1\text{c}) \qquad (f\cdot Q-P)(x,y)=\sum_{i+j=lm+l+m+1}^{\infty} d_{ij}x^iy^j.$$

Equation (2.1c) is equivalent with

$$\partial_0(f\cdot Q-P)\geq lm+l+m+1.$$

A nontrivial $Q(x,y)$, such that (2.1) is satisfied, always exists [2]. If the polynomials $R(x,y)$ and $S(x,y)$ also satisfy (2.1), in other words if $\partial_0(f\cdot S-R)\geq lm+l+m+1$, too, then

$$P(x,y)\cdot S(x,y)=Q(x,y)\cdot R(x,y).$$

This property justifies the following definitions:

(a) Let $(P_*/Q_*)(x,y)$ be the irreducible form of $(P/Q)(x,y)$ such that $Q_*(0,0)=1$; if this form exists we call it the multivariate Padé-approximant of order (l,m) for f.

(b) If the irreducible form $(P_*/Q_*)(x,y)$ is such that $\partial_0 Q_*\geq 1$, then we call P_*/Q_* the multivariate rational approximant of order (l,m) for f.

The (l,m)-multivariate rational approximant is unique up to a multiplicative constant in numerator and denominator. For $P_*(x,y)$ and $Q_*(x,y)$ we define:

$$l'=\partial P_*-\partial_0 Q_*,\qquad m'=\partial Q_*-\partial_0 Q_*.$$

We can prove that

$$l'\leq l,\qquad m'\leq m.$$

It is also easy to verify the following theorem.

THEOREM 2.1. *For the irreducible form P_*/Q_* of P/Q where (P,Q) satisfies* (2.1) *and for every polynomial $R(x,y)=\sum_{i=0}^{s}r_i x^i y^{s-i}$ with $s=lm-\partial_0 Q_*+\min(l-l',m-m')$, $(P_*\cdot R,Q_*\cdot R)$ satisfies* (2.1).

Also $s=lm-\partial_0 Q_*+\min(l-l',m-m')$ is the highest possible degree that allows the construction of a homogeneous polynomial $R(x,y)=\sum_{i=0}^{s}r_i x^i y^{s-i}$ such that (2.1) is satisfied by $(P_*\cdot R,\ Q_*)\cdot R$. From now on the multivariate Padé-approximant as well as the multivariate rational approximant will be called type C approximants.

3. Connection between the two approaches. First of all we remark that for the case of one variable the type-L approximant [5] as well as the type C approximant [1,2] reduce to the well-known ordinary Padé-approximant. And the polynomials $P(x,y)$ and $Q(x,y)$ satisfying (2.1) do also satisfy (1.1) when the sets N,D and E are chosen as follows:

$$N=\{(i,j)\mid i,j\in\mathbf{N},\ lm\leq i+j\leq lm+l\},$$
$$D=\{(i,j)\mid i,j\in\mathbf{N},\ lm\leq i+j\leq lm+m\},$$
$$E=\{(i,j)\mid i,j\in\mathbf{N},\ lm\leq i+j\leq lm+l+m\}.$$

The set $H=E\setminus N$ has one element less than the set D, as required; but we could also add to E the set $\{(i,j)\mid i,j\in\mathbf{N},\ i+j<lm\}$, since $\partial_0(f\cdot Q-P)\geq lm$ for all polynomials P and Q as in (2.1a) and (2.1b). Doing so we do not impose more conditions on the coefficients a_{ij} and b_{ij}; we write

$$E^{\text{ext}}=\{(i,j)\mid i,j\in\mathbf{N},\ i+j\leq lm+l+m\}.$$

Let us now number the points in D and H, using a diagonal enumeration:

(a)

$$D=\Big\{\underbrace{(lm,0),(lm-1,1),\cdots,(0,lm)}_{\text{first diagonal}},\underbrace{(lm+1,0),\cdots,(0,lm+1)}_{\text{second diagonal}},\cdots,$$
$$\underbrace{(lm+m,0),\cdots,(0,lm+m)}_{\text{last diagonal}}\Big\},$$

(b)

$$H=\Big\{\underbrace{(lm+l+1,0),\,(lm+l,1),\cdots,(0,lm+l+1)}_{\text{first diagonal}},\underbrace{(lm+l+2,0),\cdots,(0,lm+l+2)}_{\text{second diagonal}},\cdots,$$

$$\underbrace{(lm+l+m,0),\cdots,(0,lm+l+m)}_{\text{last diagonal}}\Big\}.$$

When we write down the equations equivalent with condition (2.1c), the set of homogeneous equations in the unknown b_{ij} has a coefficient matrix which is exactly the matrix in (1.1d) after removing the first row. From now on we will call this matrix $\mathcal{H}$; it has $p=\binom{lm+l+m+2}{2}-\binom{lm+l+2}{2}$ rows and one more columns than rows.

THEOREM 3.1. *The rank of the matrix* $\mathcal{H}$ *is at most* $p-(lm-\partial_0 Q_* + \min(l-l', m-m'))$.

Proof. We only have to prove that the dimension of the null-space of $\mathcal{H}$, which is the dimension of the space of solutions of the homogeneous system of equations, is at least $lm-\partial_0 Q_*+\min(l-l',m-m')+1$; in other words that (2.1) admits solutions where at least $lm-\partial_0 Q_*+\min(l-l',m-m')+1$ of the b_{ij} can be freely chosen. Precisely this is formulated in Theorem 2.1.

The number $s=lm-\partial_0 Q_*+\min(l-l',m-m')$ is one less than the number of coefficients in a homogeneous polynomial of degree s in two variables, namely $\binom{s+1}{s}$. The number of coefficients in a homogeneous polynomial of degree s in n variables is $\binom{s+n-1}{s}$. But first of all we are going to take a closer look at the matrix $\mathcal{H}$ for the type C approximants when $n=2$; in the next section we will treat the n-variable case with $n>2$. To examine the special structure of $\mathcal{H}$ we introduce the following notation. For $Q(x,y)=\sum_{i+j=lm}^{lm+m} b_{ij}x^i y^j$ we write

$$B_{lm}=\begin{pmatrix} b_{lm,0}\\ b_{lm-1,1}\\ \vdots\\ b_{0,lm}\end{pmatrix},\quad B_{lm+1}=\begin{pmatrix} b_{lm+1,0}\\ b_{lm,1}\\ \vdots\\ b_{0,lm+1}\end{pmatrix},\quad \cdots,\quad B_{lm+m}=\begin{pmatrix} b_{lm+m,0}\\ b_{lm+m-1,1}\\ \vdots\\ b_{0,lm+m}\end{pmatrix}.$$

Equations (2.1c) can now be written as

$$\begin{pmatrix} H_{l+1,lm} & H_{l,lm+1} & \cdots & H_{l+1-m,lm+m}\\ H_{l+2,lm} & \cdots & & \\ \vdots & & & \\ H_{l+m,lm} & \cdots & & H_{l,lm+m}\end{pmatrix}\begin{pmatrix} B_{lm}\\ \vdots\\ B_{lm+m}\end{pmatrix}=0,$$

where $H_{i,j}$ is a matrix with $(i+j+1)$ rows and $(j+1)$ columns and the first column equal to the transpose of $(c_{i,0}c_{i-1,1}\cdots c_{1,i-1}c_{0,i}0\cdots 0)$ and the next columns equal to their previous column but with all the elements shifted down one place and a zero added on top.

To calculate the displacement rank $\alpha(\mathcal{H})$ of $\mathcal{H}$, we have to construct the lower shifted difference

$$\mathcal{H}-\overline{\mathcal{H}}=\begin{pmatrix} h_{1,1} & \cdots & h_{1,p+1}\\ \vdots & & \\ h_{p,1} & & h_{p,p+1}\end{pmatrix}-\begin{pmatrix}0&\cdots&&0\\ & h_{1,1}&\cdots&h_{1,p}\\ \vdots&&&\\ &\vdots&&\\ 0&h_{p-1,1}&\cdots&h_{p-1,p}\end{pmatrix}$$

$$=\begin{pmatrix}h_{1,1}&\cdots&h_{1,p+1}\\ \vdots & & \\ & & \delta\mathcal{H}\\ h_{p,1}&&\end{pmatrix}.$$

Now $\alpha(\mathcal{H})=\operatorname{rank}(\delta\mathcal{H})+2$ [6].

THEOREM 3.2. *The displacement-rank of the matrix $\mathcal{H}$ is at most $m+2$.*

Proof. Let us write down the matrix $\mathcal{H}$ more explicitly:

$$\mathcal{H}=\left(\begin{array}{cccc|cccc|c|cccc}
c_{l+1,0} & 0 & \cdots & 0 & c_{l,0} & 0 & \cdots & 0 & & c_{l+1-m,0} & 0 & \cdots & 0\\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots & & \vdots & \ddots & & \vdots\\
c_{0,l+1} & & & 0 & c_{0,l} & & & 0 & \cdots & c_{0,l+1-m} & & & 0\\
0 & & & c_{l+1,0} & 0 & & & c_{l,0} & & 0 & & & c_{l+1-m,0}\\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & & \vdots & & \ddots & \vdots\\
0 & \cdots & 0 & c_{0,l+1} & 0 & \cdots & 0 & c_{0,l} & & 0 & \cdots & 0 & c_{0,l+1-m}\\
\hline
 & & \vdots & & & & & & & & & & \\
\hline
c_{l+m,0} & 0 & \cdots & 0 & & & & & & & & & \\
\vdots & \ddots & & \vdots & & & & & & & & & \\
c_{0,l+m} & & & 0 & & & & & & & & & \\
0 & & & c_{l+m,0} & & & & & & & & & \\
\vdots & & \ddots & \vdots & & & & & & & & & \\
0 & \cdots & 0 & c_{0,l+m} & & & & & & & & &
\end{array}\right)$$

In our case $\delta\mathcal{H}$ has the structure $\delta\mathcal{H}=(\Delta_1\Delta_2\cdots\Delta_{m+1})$, where Δ_1 has $(p-1)$ rows and lm columns, Δ_i has $(p-1)$ rows and $(lm+i)$ columns for $i=2,\cdots,m+1$, and only the first column in Δ_i with $i\geq 2$ may contain nonzero elements; all the other elements in $\delta\mathcal{H}$ equal zero. So $\operatorname{rank}(\delta\mathcal{H})\leq m$ and this proves our theorem.

We will illustrate the theorems with some very simple examples. Consider $f(x,y)=1+x/(0\cdot 1-y)+\sin(xy)$.

(a) Take $l=1=m$. The type C approximant is

$$\frac{1+10x-10\cdot 1y}{1-10\cdot 1y}$$

with $l'=1=m'$, $\partial_0 Q_*=0$, $s=1$ and $\alpha(\mathcal{H})=3$.

The matrix

$$\mathcal{H}=\begin{bmatrix} 0 & 0 & 10 & 0 & 0 \\ 101 & 0 & 0 & 10 & 0 \\ 0 & 101 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

its rank is 3.

(b) Take $l=4$ and $m=2$. The type C approximant is

$$\frac{1+10x-10y+xy-10xy^2}{1-10y}$$

with $l'=3$, $m'=1$, $\partial_0 Q_*=0$, $s=9$ and $\alpha(\mathcal{H})=4$.

The matrix

$$\mathcal{H}=\begin{bmatrix} H_1 & H_2 & H_3 \\ H_4 & H_5 & H_6 \end{bmatrix},$$

where

$$H_1=10^5[\delta_{i,j+4}], \text{ a } 14\times 9 \text{ matrix,}$$

$$H_2=10^4[\delta_{i,j+3}], \text{ a } 14\times 10 \text{ matrix,}$$

$$H_3=10^3[\delta_{i,j+2}], \text{ a } 14\times 11 \text{ matrix,}$$

$$H_4=10^6[\delta_{i,j+5}]-\tfrac{1}{6}[\delta_{i,j+3}], \text{ a } 15\times 9 \text{ matrix,}$$

$$H_5=10^5[\delta_{i,j+4}], \text{ a } 15\times 10 \text{ matrix,}$$

$$H_6=10^4[\delta_{i,j+3}], \text{ a } 15\times 11 \text{ matrix,}$$

and $\delta_{i,j}$ is the Kronecker symbol (here used in rectangular matrices). $\mathcal{H}$ is a matrix of rank 20.

(c) Take $l=1$ and $m=2$. The type C approximant is

$$\frac{x-1.01y+10y^2+10x^2-20.2xy}{x-1.01y+10y^2-10.1xy+2.01xy^2}$$

with $l'=1$, $m'=2$, $\partial_0 Q_*=1$, $s=1$ and $\alpha(\mathcal{H})=4$. The matrix

$$\mathcal{H}=\left[\begin{array}{ccc|cccc|ccccc} 0 & 0 & 0 & & 10I_4 & & & & & I_5 & & \\ & 101I_3 & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & 10I_5 & & \\ 0 & 0 & 0 & & 10I_4 & & & & & & & \\ & 1000I_3 & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where I_k is the $k\times k$ unit-matrix. The rank of $\mathcal{H}$ is 10.

(d) Consider $f(x,y)=(xe^x-ye^y)/x-y$ and take $l=1=m$. The determinant representations yield

$$P(x,y)=-\tfrac{1}{2}(x+y+0.5x^2+1.5xy+0.5y^2),$$

$$Q(x,y)=-\tfrac{1}{2}(x+y-0.5x^2-0.5xy-0.5y^2),$$

and indeed the type C approximant is

$$\frac{P(x,y)}{Q(x,y)}=\frac{P_*(x,y)}{Q_*(x,y)}$$

with $\partial_0 Q_* = 1$ and $l'=1=m'$. The matrix $\mathcal{H}$ has rank p $(s=0)$ and $\alpha(\mathcal{H})=3$.

4. The multivariate case. Given the power series

$$f(x)=\sum_{k=0}^{\infty} c_k x^k,$$

where $x=(x_1,\cdots,x_n)\in\mathbf{R}^n$, $c_k=c_{k_1\cdots k_n}$, $x^k=x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$ and $\Sigma_{k=0}^{\infty}=\Sigma_{k_1=0}^{\infty}\cdots\Sigma_{k_n=0}^{\infty}$, the Padé-approximation problem of order (l,m) is the following:
find

(4.1a) $$P(x)=\sum_{|i|=lm}^{lm+l} a_i x^i,$$

(4.1b) $$Q(x)=\sum_{|j|=lm}^{lm+m} b_j x^j$$

where $a_i=a_{i_1\cdots i_n}$ and $b_j=b_{j_1\cdots j_n}$, $|i|=i_1+\cdots+i_n$ and $|j|=j_1+\cdots+j_n$, such that

(4.1c) $$\partial_0(f\cdot Q-P)\geq lm+l+m+1$$

where ∂_0 is again the degree of the first nonzero term.

After calculation of the nontrivial solution of (4.1) [2] we can proceed as in §2 and define the multivariate Padé-approximant of order (l,m) and the multivariate rational approximant of order (l,m).

The integers l' and m' are defined as in the two-variable case and it is easy to prove the following n-dimensional analogue of Theorem 2.1.

THEOREM 4.1. *For the irreducible form P_*/Q_* of P/Q where (P,Q) satisfies* (4.1) *and for every polynomial $R(x)=\Sigma_{|i|=s} r_i x^i$ with $s=lm-\partial_0 Q_*+\min(l-l',m-m')$, $(P_*\cdot R, Q_*\cdot R)$ satisfies* (4.1).

Let us again study the connection with the approach of Levin. Condition (4.1c) results in $\binom{n+lm+l+m}{lm+\ l+m}$ equations: the first $\binom{n+lm+l}{lm\ +l}$ equations express the a_i as linear combinations of the b_j and the remaining equations form an overdetermined homogeneous linear system in the unknown b_j [2]; there are

$$p+1=\binom{n+lm+m}{lm+m}-\binom{n+lm-1}{lm-1}$$

unknown coefficients b_j. The b_j can be found by solving a homogeneous subsystem of p equations, having the rank of the overdetermined system.

Choose the sets N, D and H as follows:

- $N=\{i=(i_1,\cdots,i_n)\mid i\in\mathbf{N}^n,\ lm\leq|i|\leq lm+l\}$;
- $D=\{i=(i_1,\cdots,i_n)\mid i\in\mathbf{N}^n,\ lm\leq|i|\leq lm+m\}$;
- select a particular b_j and let $c_{h(k)-j}$ be the coefficient of b_j in the kth equation of the homogeneous subsystem we have to solve $(k=1,\cdots,p)$,
$$H=\{h(k)=(h_1(k),\cdots,h_n(k))\mid k=1,\cdots,p\}.$$

We call the coefficient matrix of the homogeneous subsystem again $\mathcal{H}$. It is easy to prove the following n-dimensional analogue of Theorem 3.1.

THEOREM 4.2. *The rank of the matrix* $\mathcal{H}$ *is at most* $p-\binom{s+n-1}{s}+1$ *with* $s=lm-\partial_0 Q_*+\min(l-l',m-m')$.

If we use an enumeration of the points in D and H, similar to the one described in §3, it is obvious that in the multivariate case $\mathcal{H}$ is also a matrix with low displacement rank.

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REFERENCES

[1] ANNIE A. M. CUYT, *Abstract Padé-approximants in operator theory*, Lecture Notes in Mathematics, 765 L. Wuytack ed., Springer, Berlin, 1979, pp. 61–87.

[2] ______, *Multivariate Padé-approximants*, J. Math. Anal. Appl., to appear.

[3] ______, *Numerical comparison of abstract Padé-approximants and abstract rational approximants with other generalizations of the classical Padé-approximant*, Lecture Notes in Mathematics 888, H. van Rossum and M. de Bruin, eds., Springer, Berlin, 1981, pp. 137–157.

[4] ANNIE A. M. CUYT AND P. VAN DER CRUYSSEN, *Abstract Padé-approximants for the solution of a system of nonlinear equations*, Computers and Mathematics with Applications, to appear.

[5] D. LEVIN, *General order Padé-type rational approximants defined from double power series*, J. Inst. Math. Appl., 18 (1976), pp. 1–8.

[6] M. MORF, S. KUNG AND T. KAILATH, *Displacement ranks of matrices and linear equations*, J. Math. Anal. Appl., 68 (1979), pp. 395–407.