Applied Numerical Mathematics 4 (1988) 263-271
North-Holland

A REVIEW OF BRANCHED CONTINUED FRACTION THEORY FOR THE CONSTRUCTION OF MULTIVARIATE RATIONAL APPROXIMANTS

Annie A.M. CUYT *
Institut für Angewandte Mathematik der Universität Bonn, D-5300 Bonn 1, Fed. Rep. Germany

Brigitte M. VERDONK
Department of Mathematics and Computer Science, University of Antwerp (UIA), B-2610 Wilrijk, Belgium

1. Introduction

While the history of continued fractions goes back to Euclid's algorithm, branched continued fractions are only twenty years old. The idea to construct them was born in Lvov (U.S.S.R.) in the early sixties. The first and most general form of these fractions was introduced by Skorobogatko in [14] together with Droniuk, Bobyk and Ptashnik.

An ordinary *continued fraction* (CF) is an expression of the form

$$b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cdots}}},$$

which is often written in more convenient way

$$b_0+\sum_{i=1}^{\infty}\frac{a_i|}{|b_i}. \tag{1}$$

Now suppose we substitute for each of the partial denominators b_i an ordinary CF

$$b_{0,i}+\sum_{j=1}^{\infty}\frac{a_{j,i}|}{|b_{j,i}}.$$

Then the CF (1) already looks like

$$b_{00}+\sum_{j=1}^{\infty}\frac{a_{j0}|}{|b_{j0}}+\sum_{i=1}^{\infty}\frac{a_i|}{\left|b_{0i}+\sum_{j=1}^{\infty}\frac{a_{ji}|}{|b_{ji}}\right.}. \tag{2}$$

This process can be repeated where now each b_{ji} is replaced by

$$b_{0,ji}+\sum_{k=1}^{\infty}\frac{a_{k,ji}|}{|b_{k,ji}}.$$

* Research Fellow of the Alexander von Humboldt Foundation.

In this way (2) and hence (1) become

$$b_{000} + \sum_{k=1}^{\infty} \frac{a_{k00}|}{|b_{k00}} + \sum_{j=1}^{\infty} \frac{a_{j0}}{b_{0j0} + \sum_{k=1}^{\infty} \frac{a_{kj0}|}{|b_{kj0}}}$$

$$+ \sum_{i=1}^{\infty} \frac{a_i}{b_{00i} + \sum_{k=1}^{\infty} \frac{a_{k0i}|}{|b_{k0i}} + \sum_{j=1}^{\infty} \frac{a_{ji}}{b_{0ji} + \sum_{k=1}^{\infty} \frac{a_{kji}|}{|b_{kji}}}}.$$

Suppose this recursive computation has been performed $(l-1)$ times. Then the CF (1) is called l-branched. If we proceed with the construction of

nextbranch(l)
  begin
  replace $b_{i_l \dots i_1}$ **by** $b_{0i_l \dots i_1} + \sum_{k=1}^{\infty} \frac{a_{ki_l \dots i_1}|}{|b_{ki_l \dots i_1}}$
  rename $b_{ki_l \dots i_1}$ **as** $b_{i_{l+1} \dots i_1}$
  call nextbranch$(l+1)$
  end

then this recursive scheme defines a *branched continued fraction* (BCF).

The concept was for the first time introduced in [14]. In this paper Skorobogatko chose the terminology branched from looking at the computation scheme for an ordinary CF, see Fig. 1. When this structure is itself copied in each node b_i and this process is performed recursively, one indeed gets a very tree-like picture, see Fig. 2.

For the history of BCF we mainly refer to papers of Bodnar. In the near future he will also publish a book on this subject.

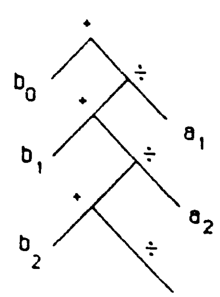

Fig. 1. Computation scheme for an ordinary CF.

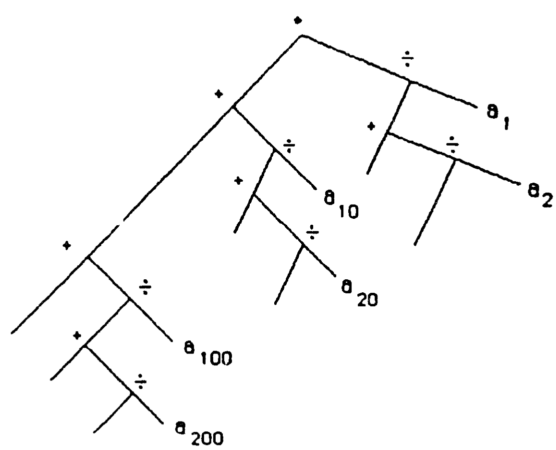

Fig. 2. Tree-like structure of a BCF.

In the beginning branched continued fractions were used to represent the solution of some well-known problems in order to establish new algorithms and new theoretical results. Consider for instance the solution of the system of linear equations [15]

$$\sum_{j=1}^{n} a_{ij}x_j = a_{i,n+1}, \quad i = 1,\ldots, n.$$

From Cramer's rule we have

$$x_1 = \frac{\begin{vmatrix} a_{1,n+1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,n+1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{n,n+1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}},$$

and after expanding numerator and denominator along the first column

$$x_1 = \frac{a_{1,n+1}A_{11} + a_{2,n+1}A_{21} + \cdots + a_{n,n+1}A_{n1}}{a_{1,1}A_{11} + a_{2,1}A_{21} + \cdots + a_{n,1}A_{n1}},$$

where the A_{ij} are the cofactors of the elements a_{ij}. Therefore

$$x_1 = \sum_{i=1}^{n} \frac{a_{i,n+1}}{a_{i,1} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_{j,1}A_{j1}}{A_{i1}}}.$$

Since the A_{ij} are determinants of dimension $n-1$ they can be expanded in the same way and hence a branched continued fraction expression for x_1 can be obtained with $n-k$ branches at the kth stage.

Later on, from the construction of BCF the idea originated to use them for the solution of multivariate problems. However, the most general form for a BCF was not very useful because there was too much freedom in the choice of the partial numerators and denominators [1]. Simplifications seemed to be necessary. If an n-dimensional problem could be solved using n-branched continued fractions instead of infinitely branched continued fractions, then an $(n+1)$-dimensional problem could be attacked by adding a branch to the n-branched continued fraction. In this way bivariate problems could be approached using a BCF of the form (2) because ordinary CFs are often used for univariate problems. We shall see in the next sections that this approach is indeed successful. The sequel of the text is composed so that the level of complexity in the formulas increases steadily. However, if the reader is interested in a chronological approach, then he or she should keep the following historical facts in mind. In [6] Kuchminskaya developed a lot of the formulas mentioned in Sections 3 and 4, while in the western hemispere several other mathematicians worked on the same problem and published their results independently [3,11]. In the middle of this evolution Siemaszko published his own solution to the problem of multivariate approximation [12] and interpolation [13] which we shall present first.

2. Bivariate CF expansions and interpolating CFs introduced by Siemaszko

Given a bivariate power series

$$f(x, y) = \sum_{(i, j) \in \mathbb{N}^2} c_{ij} x^i y^j = \sum_{i \in \mathbb{N}} c_{ii} (xy)^i + \sum_{i>j} c_{ij} x^i y^j + \sum_{i<j} c_{ij} x^i y^j,$$

it is possible to construct a sum of BCFs of the form

$$b_{00} + \sum_{j=1}^{\infty} \frac{xy|}{|b_{j0}} + \sum_{i=1}^{\infty} \frac{x|}{\left|b_{0i} + \sum_{j=1}^{\infty} \frac{xy|}{|b_{ji}}\right.} + \sum_{i=-1}^{-\infty} \frac{y|}{\left|b_{0i} + \sum_{j=1}^{\infty} \frac{xy|}{|b_{ji}}\right.}, \tag{3}$$

of which the convergent

$$\frac{p_n(x, y)}{q_n(x, y)} = b_{00} + \sum_{j=1}^{n} \frac{xy|}{|b_{j0}} + \sum_{i=1}^{n} \frac{x|}{\left|b_{0i} + \sum_{j=1}^{n-i} \frac{xy|}{|b_{ji}}\right.} + \sum_{i=-1}^{-n} \frac{y|}{\left|b_{0i} + \sum_{j=1}^{n-i} \frac{xy|}{|b_{ji}}\right.}$$

corresponds with $f(x, y)$ on the subset E of $\mathbb{N}^2$ with E as given in Fig. 3. In other words,

$$(fq_n - p_n)(x, y) = \sum_{(i, j) \in \mathbb{N}^2 \setminus E} d_{ij} x^i y^j = \sum_{\substack{i>n \\ j>n}} d_{ij} x^i y^j,$$

The computation scheme for the coefficients in the BCF (3) is described in detail in [12]: the idea is to use Viscovatov's algorithm along shifted diagonals in $\mathbb{N}^2$. Siemaszko called the convergents of this corresponding BCF bivariate Padé approximants. For a comparison with other definitions of Padé approximants for bivariate power series we refer to [2].

As soon as bivariate Padé approximants exist, one would of course also like to solve the problem of how to interpolate data spread over several points. To this end a Thiele-type BCF of

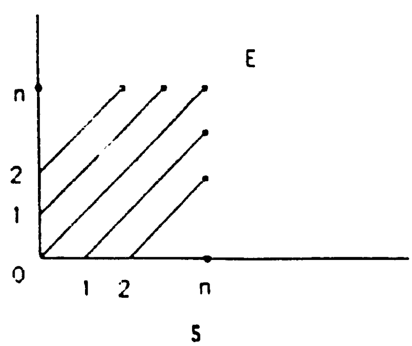

Fig. 3. Index set of correspodence for p_n/q_n.

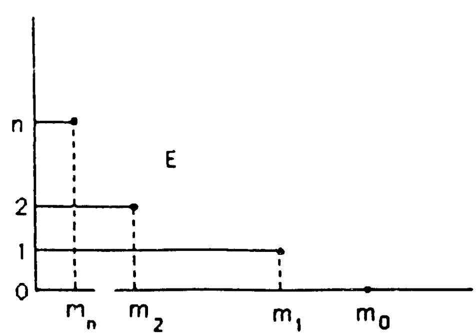

Fig. 4. Index set of interpolation conditions for p_n/q_n.

the form (3) proved to be unsuccessful and hence Siemaszko changed the form of his BCF as follows. Given data $\{f_{ij} \mid (i, j) \in \mathbb{N}^2\}$ in points $\{(x_i, y_j) \mid (i, j) \in \mathbb{N}^2\}$, compute a BCF

$$b_{00} + \sum_{j=1}^{\infty} \frac{(x - x_{j-1}) |}{| b_{j0}} + \sum_{i=1}^{\infty} \frac{(y - y_{i-1}) |}{| b_{0i} + \sum_{j=1}^{\infty} \frac{(x - x_{j-1}) |}{| b_{ji}}} \tag{4}$$

of which the convergent

$$\frac{p_n(x, y)}{q_n(x, y)} = b_{00} + \sum_{j=1}^{m_0} \frac{(x - x_{j-1}) |}{| b_{j0}} + \sum_{i=1}^{n} \frac{(y - y_{i-1}) |}{| b_{0i} + \sum_{j=1}^{m_i} \frac{(x - x_{j-1}) |}{| b_{ji}}}$$

with $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_n$ satisfies

$$(fq_n - p_n)(x_i, y_j) = 0$$

for $(i, j) \in E$ with E as given in Fig. 4.

The values b_{kl} can be computed using a bivariate inverse difference scheme [13],

$$\psi[x_i][y_j] = f_{ij},$$

$$\psi[x_0][y_0, y_1] = \frac{y_1 - y_0}{\psi[x_0][y_1] - \psi[x_0][y_0]},$$

$$\psi[x_0][y_0, \ldots, y_l] = \frac{y_l - y_{l-1}}{\psi[x_0][y_0, \ldots, y_{l-2}, y_l] - \psi[x_0][y_0, \ldots, y_{l-1}]},$$

$$\psi[x_0, x_1][y_0, \ldots, y_l] = \frac{x_1 - x_0}{\psi[x_1][y_0, \ldots, y_l] - \psi[x_0][y_0, \ldots, y_l]},$$

$$\psi[x_0, \ldots, x_k][y_0, \ldots, y_l]$$
$$= \frac{x_k - x_{k-1}}{\psi[x_0, \ldots, x_{k-2}, x_k][y_0, \ldots, y_l] - \psi[x_0, \ldots, x_{k-1}][y_0, \ldots, y_l]} = b_{kl}.$$

From this scheme it is clear that one proceeds along horizontal lines in $\mathbb{N}^2$. Of course the roles of x and y can be interchanged in (4) and then one proceeds along vertical lines in $\mathbb{N}^2$. Remark that interchanging x and y yields different interpolants because the algorithm for the inverse differences is not symmetric.

If $m = m_0 = \cdots = m_n$, then we denote the rational interpolant by $p_{n,m}(x, y)/q_{n,m}(x, y)$ and the following error formula can be proved [13]. For

$$\Phi_{n,m}(x, y) = [(fq_{n,m} - p_{n,m})q_{n,m+1}](x, y),$$

there exist η_0 and ξ_0 in intervals J and I that respectively contain the points $y_0, \ldots, y_n$ and $x_0, \ldots, x_m$ such that for $(x, y) \in I \times J$,

$$\Phi_{n,m}(x, y) = \frac{\prod_{j=0}^{n}(y - y_j)}{(n+1)!} \frac{\partial^{n+1}\Phi_{n,m}(x, y)}{\partial^{n+1} y}\bigg|_{y=\eta_0} + \frac{\prod_{i=0}^{m}(x - x_i)}{(m+1)!} \frac{\partial^{m+1}\Phi_{n,m}(x, y)}{\partial^{m+1} x}\bigg|_{x=\xi_0}.$$

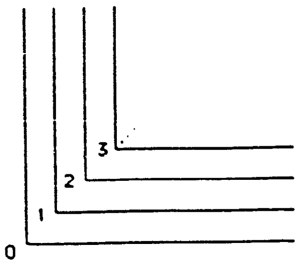

Fig. 5. $\mathbb{N}^2$ as a union of prongs.

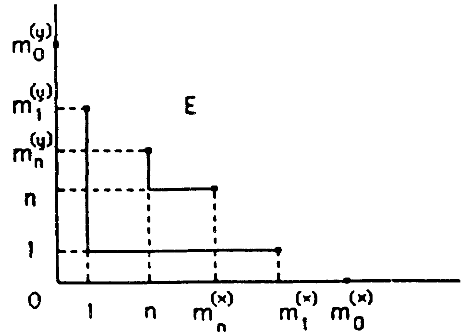

Fig. 6. Index set of interpolation conditions for p_n/q_n.

When all the interpolation points coincide, one needs reciprocal differences for the calculation of the limiting values of the inverse differences. Doing so Siemaszko again gets Padé approximants but now different in form from those given at the beginning of the section.

Up to now we have looked at $\mathbb{N}^2$ as a union of straight lines (diagonal or horizontal or vertical). The next sections will illustrate that one can develop more symmetric formulas when looking at $\mathbb{N}^2$ as a union of prongs. This implies that the partial denominators b_i in (1) are replaced by a sum of ordinary CFs instead of by a single CF.

3. Symmetric corresponding BCFs introduced by Kuchminskaya and Murphy–O'Donohoe

A double power series

$$f(x, y) = \sum_{(i,j)\in\mathbb{N}^2} c_{ij}x^iy^j$$

can be written as

$$f(x, y) = \sum_{i=0}^{\infty}\left(c_{ii}(xy)^i + \sum_{j=1}^{\infty} c_{i+j,i}x^{i+j}y^i + \sum_{j=1}^{\infty} c_{i,i+j}x^iy^{i+j}\right),$$

which means that we locate each point in $\mathbb{N}^2$ on a particular prong, see Fig. 5.

This way to group the terms of $f(x, y)$ suggests to look for a corresponding BCF of the form

$$b_{00} + \sum_{j=1}^{\infty}\frac{x\,|}{|\,b_{j0}} + \sum_{j=1}^{\infty}\frac{y\,|}{|\,b_{0j}} + \sum_{i=1}^{\infty}\frac{xy\,|}{\left|\,b_{ii} + \sum_{j=i+1}^{\infty}\frac{x\,|}{|\,b_{ji}} + \sum_{j=i+1}^{\infty}\frac{y\,|}{|\,b_{ij}}\right.}. \tag{5}$$

In [7] and [11] it is proved independently that the convergent

$$\frac{p_n(x, y)}{q_n(x, y)} = b_{00} + \sum_{j=1}^{n}\frac{x\,|}{|\,b_{j0}} + \sum_{j=1}^{n}\frac{y\,|}{|\,b_{0j}} + \sum_{i=1}^{\lfloor n/2\rfloor}\frac{xy\,|}{\left|\,b_{ii} + \sum_{j=i+1}^{n-2i}\frac{x\,|}{|\,b_{ji}} + \sum_{j=i+1}^{n-2i}\frac{y\,|}{|\,b_{ij}}\right.}$$

satisfies

$$(fq_n - p_n)(x, y) = \sum_{i+j>n} d_{ij} x^i y^j.$$

The b_{kl} are again computed using Viscovatov-like algorithms along each leg of the prong. Remark that (5) and p_n/q_n are symmetric functions of x and y. Also the form (5) is suitable for generalization to bivariate rational interpolation.

4. Symmetric Thiele-type BCFs introduced by Cuyt–Verdonk and Kuchminskaya

Given data $\{f_{ij} \mid (i, j) \in \mathbb{N}^2\}$ at points $\{(x_i, y_j) \mid (i, j) \in \mathbb{N}^2\}$ it is proved independently in [3] and [15] that one can interpolate these data using a Thiele-type BCF of the form

$$b_{00} + \sum_{j=1}^{\infty} \frac{x - x_{j-1}|}{|b_{j0}} + \sum_{j=1}^{\infty} \frac{y - y_{j-1}|}{|b_{0j}} + \sum_{i=1}^{\infty} \frac{(x - x_{i-1})(y - y_{i-1})|}{\left| b_{ii} + \sum_{j=i+1}^{\infty} \frac{x - x_{j-1}|}{|b_{ji}} + \sum_{j=i+1}^{\infty} \frac{y - y_{j-1}|}{|b_{ij}} \right.}.$$

The unknowns b_{kl} are symmetric inverse differences given by

$$\varphi[x_i][y_j] = f_{ij},$$

$$\varphi[x_0,x_1][y_0] = \frac{x_1 - x_0}{\varphi[x_1][y_0] - \varphi[x_0][y_0]},$$

$$\varphi[x_0][y_0,y_1] = \frac{y_1 - y_0}{\varphi[x_0][y_1] - \varphi[x_0][y_0]},$$

$$\varphi[x_0,\ldots, x_k][y_0] = \frac{x_k - x_{k-1}}{\varphi[x_0,\ldots, x_{k-2},t_k][y_0] - \varphi[x_0,\ldots, x_{k-1}][y_0]},$$

$$\varphi[x_0][y_0,\ldots, y_l] = \frac{y_l - y_{l-1}}{\varphi[x_0][y_0,\ldots, y_{l-2},y_l] - \varphi[x_0][y_0,\ldots, y_{l-1}]},$$

$$\varphi[x_0,x_1][y_0,y_1] = \frac{(x_1 - x_0)(y_1 - y_0)}{f_{11} - f_{01} - f_{10} + f_{00}};$$

$k = l$:
$$\begin{aligned}&\varphi[x_0,\ldots, x_k][y_0,\ldots, y_k]\\ &= (x_k - x_{k-1})(y_k - y_{k-1})\\ &\quad /(\varphi[x_0,\ldots, x_{k-2},x_k][y_0,\ldots, y_{k-2},y_k] - \varphi[x_0,\ldots, x_{k-1}][y_0,\ldots, y_{k-2},y_k]\\ &\qquad - \varphi[x_0,\ldots, x_{k-2},x_k][y_0,\ldots, y_{k-1}] + \varphi[x_0,\ldots, x_{k-1}][y_0,\ldots, y_{k-1}])\\ &= b_{kk};\end{aligned}$$

$k < l$:
$$\begin{aligned}&\varphi[x_0,\ldots, x_k][y_0,\ldots, y_l]\\ &= \frac{y_l - y_{l-1}}{\varphi[x_0,\ldots, x_k][y_0,\ldots, y_{l-2},y_l] - \varphi[x_0,\ldots, x_k][y_0,\ldots, y_{l-1}]}\\ &= b_{kl};\end{aligned}$$

$$k > l: \quad \varphi[x_0,\dots,x_k][y_0,\dots,y_l] = \frac{x_k - x_{k-1}}{\varphi[x_0,\dots,x_{k-2},x_k][y_0,\dots,y_l] - \varphi[x_0,\dots,x_{k-1}][y_0,\dots,y_l]} = b_{kl}.$$

For the convergent

$$\frac{p_n(x,y)}{q_n(x,y)} = b_{00} + \sum_{j=1}^{m_0^{(x)}} \frac{x - x_{j-1}|}{|\, b_{j0}} + \sum_{j=1}^{m_0^{(y)}} \frac{y - y_{j-1}|}{|\, b_{0j}} + \sum_{i=1}^{n} \frac{(x - x_{i-1})(y - y_{i-1})|}{|\, b_{ii} + \sum_{j=i+1}^{m_i^{(x)}} \frac{x - x_{j-1}|}{|\, b_{ji}} + \sum_{j=i+1}^{m_i^{(y)}} \frac{y - y_{j-1}|}{|\, b_{ij}}}$$

it is shown [3] that if $m_0^{(x)} \geqslant m_1^{(x)} \geqslant \cdots \geqslant m_n^{(x)}$ and $m_0^{(y)} \geqslant m_1^{(y)} \geqslant \cdots \geqslant m_n^{(y)}$, then

$$(fq_n - p_n)(x_i, y_j) = 0$$

for $(i, j) \in E$ with E as given in Fig. 6.

If we take $m_0^{(x)} = \cdots = m_n^{(x)} = n$ and $m_0^{(y)} = \cdots = m_n^{(y)} = n$, then it is possible to write down the following error formula which is proved by Kuchminskaya in [8]. Let

$$\Phi_n(x, y) = fq_n(x, y) - p_n(x, y).$$

Then there exist ξ_1, ξ_2 and η_1, η_2 in intervals I and J respectively, containing $x_0,\dots,x_n$ and $y_0,\dots,y_n$ such that for $(x, y) \in I \times J$,

$$\Phi_n(x, y) = \frac{1}{(n+1)!} \left\{ \prod_{i=1}^{n+1} (x - x_i) \frac{\partial^{n+1}\Phi_n(x, y)}{\partial x^{n+1}}\Big|_{x=\xi_1} + \prod_{i=1}^{n+1} (y - y_i) \frac{\partial^{n+1}\Phi_n(x, y)}{\partial y^{n+1}}\Big|_{y=\eta_1} - \frac{1}{n!} \prod_{i=1}^{n+1} (x - x_i)(y - y_i) \frac{\partial^{2n+2}\Phi_n(x, y)}{\partial x^{n+1}\partial y^{n+1}}\Big|_{(\xi_2, \eta_2)} \right\}.$$

Cuyt and Verdonk on the other hand have defined in [4] symmetric reciprocal differences to compute limiting values of the b_{kl} in case all the interpolation points coincide. In this case one establishes a link between these rational interpolants and convergents of the corresponding BCF (5).

5. Perspectives

Most of the bivariate rational approximants are now under investigation for their use in numerical applications such as the convergence acceleration of multidimensional tables, the solution of systems of nonlinear equations with singularities in the neighbourhood of the root, the solution of ordinary and partial differential equations, and model reduction in the theory of multidimensional signal processing. Other points of interest are convergence criteria for the BCF similar to the results of van Vleck, Worpitzky, Pringsheim [9,10] and forward evaluation algorithms via the solution of block-tridiagonal linear systems [5].

Since many univariate results do not yet have their multivariate analogon, a lot of work remains to be done.

References

[1] P. Bodnarcuk and W. Skorobogatko, *Branched Continued Fractions and Applications* (Naukowaja Dumka, Kiev, 1974) (in Ukrainian).

[2] A. Cuyt, A review of multivariate Padé approximation theory, *J. Comput. Appl. Math. 12–13* (1985) 221–232.

[3] A. Cuyt and B. Verdonk, Multivariate rational interpolation, *Computing 34* (1985) 41–61.

[4] A. Cuyt and B. Verdonk, Multivariate reciprocal differences for branched Thiele continued fraction expansions, J. Comput. Appl. Math. (to appear).

[5] A. Cuyt and B. Verdonk, Block-tridiagonal linear systems and branched continued fractions, *IMA J. Numer. Anal.* (to appear).

[6] K. Kuchminskaya, Approximation and interpolation of functions by continued and branched continued fractions, Ph.D., Ukr. Akad. Nauk, Lvov, 1976 (in Russian).

[7] K. Kuchminskaya, Corresponding and associated branched continued fractions for double power series, *Dokl. Akad. Nauk Ukrain. SSR Ser. A 7* (1978) 613–617.

[8] K. Kuchminskaya, On approximation of functions by two-dimensional continued fractions, in: J. Gilewicz, M. Pindor and W. Siemaszko, eds., *Rational Approximation and Its Applications in Mathematics and Physics*, Lecture Notes in Mathematics 1237 (Springer, New York, 1987) 207–216.

[9] K. Kuchminskaya, On sufficient conditions for the absolute convergence of two-dimensional continued fractions, *Mat. Metody i Fiz.-Mekh. Polya 20* (1984) 19–23 (in Russian).

[10] K. Kuchminskaya, On the convergence of two-dimensional continued fractions, in: *Constructive Theory of Functions* (Publishing House of the Bulgarian Academy of Sciences, Sofia, 1984) 501–506.

[11] J. Murphy and M. O'Donohoe, A two-variable generalization of the Stieltjes-type continued fraction, *J. Comput. Appl. Math. 4* (1978) 181–190.

[12] W. Siemaszko, Branched continued fractions for double power series, *J. Comput. Appl. Math. 6* (1980) 121–125.

[13] W. Siemaszko, Thiele-type branched continued fractions for two-variable functions, *J. Comput. Appl. Math. 9* (1983) 137–153.

[14] V. Skorobogatko, N. Droniuk, O. Bobyk and B. Ptashnik, Branched continued fractions, *Dokl. Akad. Nauk Ukrain. SSR Ser. A 2* (1966) 131–133 (in Russian).

[15] V. Skorobogatko, *Branched Continued Fractions and Their Applications in Mathematics* (Nauka, Moscow, 1983) (in Russian).