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ABSTRACT PADÉ-APPROXIMANTS FOR THE SOLUTION OF A SYTEM OF NONLINEAR EQUATIONS

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Abstract—Let $F: \mathbf{R}^q \to \mathbf{R}^q$ and let x^* be a simple root of the system of nonlinear equations $F(x) = 0$. We will construct several iterative methods, based on the (n, m)-APA (abstract Padé-approximant) or the (n, m)-ARA (abstract rational approximant)[4] for either F (direct one-point interpolation) or its inverse operator G (inverse one-point interpolation).

The following methods are special cases: $n = 1, m = 0$: Newton-iteration (via direct and via inverse interpolation); inverse interpolation with $n = 2, m = 0$: improvement of the Newton-iteration as indicated by Ehrmann[7]; direct interpolation with $n = 1, m = 1$: method of tangent hyperbolas[10], under certain conditions for the ARA.

Among other new methods an interesting third-order iterative procedure is constructed via inverse interpolation with $n = 1, m = 1$:

$$x_{i+1} = x_i + \frac{a_i^2}{a_i + \frac{1}{2}F_i'^{-1}F_i''a_i^2}$$

with F_i' the 1st Fréchet-derivative of F at x_i, $a_i = -F_i'^{-1}F_i$ the Newton-correction, F_i'' the 2nd Fréchet-derivative of F at x_i and component-wise multiplication and division in $\mathbf{R}^q$.

This method is to be preferred to the method of tangent hyperbolas, which is also of third order, since it requires less numerical calculations. In general, the methods derived from the use of the (n, m)-APA or (n, m)-ARA with $m \geq 1$ are preferable when F or G have singularities in the neighbourhood of x^* or 0 respectively.

1. INTRODUCTION

Let $F: \mathbf{R}^q \to \mathbf{R}^q$ and let $x^* \in \mathbf{R}^q$ be such that $F(x^*) = 0$. In this paper we will construct two classes of iterative methods to find x^*, i.e. starting from an approximation x_0 for x^* a sequence of further approximations $\{x_i\}$ is constructed in such a way that x_{i+1} is computed by means of x_i.

The first class of methods considered in this paper is obtained from approximating F in a neighborhood of x_i by a ratio P_i/Q_i of abstract polynomials. Then x_{i+1} is such that $P_i(x_{i+1}) = 0$. This process is called direct (one-point) interpolation.

In the second class the inverse function G of F is approximated in a neighborhood of $y_i = F(x_i)$, by a ratio of abstract polynomials P_i/Q_i as well. Now $x_{i+1} = (P_i/Q_i)(0)$. This technique is called inverse (one-point) interpolation.

In both cases P_i/Q_i is an abstract Padé approximant (APA) or an abstract rational approximant (ARA) for F or G. In Section 2 the notions APA and ARA are introduced, together with some properties. For different degrees of P_i or Q_i, different methods are obtained. Section 3 contains several examples of iteration functions obtained by inverse interpolation. Direct interpolation is treated in Section 4. Finally, in Section 5, numerical aspects are discussed and numerical examples considered.

2. ABSTRACT RATIONAL APPROXIMANTS (ARA) AND ABSTRACT PADÉ APPROXIMANTS (APA)

Let X be a Banach-space and Y a Banach-algebra. For $X = \mathbf{R}^q = Y$ the multiplication is componentwise and $0 = (0, \ldots, 0)^T$ is the unit for the addition and $1 = (1, \ldots, 1)^T$ for the

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multiplication. We define

$$\mathscr{L}(X^k, Y) = \{L \mid L \text{ is a bounded } k\text{-linear operator } L: X \to \mathscr{L}(X^{k-1}, Y)\}$$
$$\text{and } \mathscr{L}(X^0, Y) = Y.$$

Thus $Lx_1 \ldots x_k = (Lx_1)x_2 \ldots x_k$ with $(x_1, \ldots, x_k) \in X^k$ and $Lx_1 \in \mathscr{L}(X^{k-1}, Y)$. An operator $L \in \mathscr{L}(X^k, Y)$ is called symmetric if the arguments in $Lx_1 \ldots x_k$ may be permuted without effect. An example of a symmetric bounded k-linear operator is $F^{(k)}(x_0)$, the kth Fréchet derivate of F at x_0 ([13], pp. 100–110). The tensor-product $L_1 \otimes L_2$ of $L_1 \in \mathscr{L}(X^i, Y)$ and $L_2 \in \mathscr{L}(X^j, Y)$ is a bounded and $(i+j)$-linear operator defined by ([9], pp. 318)

$$(L_1 \otimes L_2)(x_1 \ldots x_{i+j}) = (L_1x_1 \ldots x_i) \cdot (L_2x_{i+1} \ldots x_{i+j}) \text{ (product in } Y).$$

An abstract polynomial is a non-linear operator $P: X \to Y$ such that $P(x) = A_nx^n + \cdots + A_0$ where $A_i \in \mathscr{L}(X^i, Y)$ are symmetric operators. We also introduce the following notations:

$\partial_0 P$ is the smallest index i for which $A_ix^i \not\equiv 0$,
∂P is the largest index i for which $A_ix^i \not\equiv 0$.

∂P is called the (exact) degree and $\partial_0 P$ the order of P.

We suppose that the operator F of Section 1 is analytic ([13], p. 113) in a neighborhood U of x^* and that x^* is a simple zero of F, i.e. $F'(x^*)$ is a non-singular matrix. Then the inverse operator G exists and is analytic in a neighborhood V of 0. ([2], p. 381).

We will introduce the following definitions for F; the same reasoning may be applied to G. Let $x_0 \in U$. Since F is analytic at x_0,

$$F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(x_0)(x-x_0)^k \text{ for } x \in \mathbf{R}^q \text{ close to } x_0.$$

We say that $F(x) = O((x-x_0)^j)$ if there exists a $0 < J_0 < \infty$ such that $\|F(x)\| \leq J_0\|x-x_0\|^j$ for small $\|x-x_0\|$ $(j \in \mathbf{N})$.

In the abstract Padé approximation problem we try to find a couple of abstract polynomials $(P(x), Q(x))$,

$$P(x) = A_{nm+n}(x-x_0)^{nm+n} + \cdots + A_{nm}(x-x_0)^{nm}$$
$$Q(x) = B_{nm+m}(x-x_0)^{nm+m} + \cdots + B_{nm}(x-x_0)^{nm},$$

such that

$$(F \cdot Q - P)(x) = O((x-x_0)^{nm+n+m+1}). \tag{2.1}$$

The choice of the order and degree of P and Q is justified in [4]. For all non-negative integers n and m a solution of this problem exists.

We define for $Y = \mathbf{R}^q$:

$D(Q) = \{x \in \mathbf{R}^q \mid Q(x) \text{ is a vector with non-zero components}\}$
the operator $\frac{1}{Q}: D(Q) \to \mathbf{R}^q$ by $\frac{1}{Q}(x) = (Q(x))^{-1}$ (the inverse element for the componentwise multiplication in $\mathbf{R}^q$).

We call the abstract rational operator $(1/Q) \cdot P$, the quotient of two abstract polynomials, reducible if there exist polynomials T, P_0 and Q_0 such that $P = T \cdot P_0$, $Q = T \cdot Q_0$, where $\partial T \geq 1$ and T is not a unit in the ring of abstract polynomials (i.e. $1/T$ is not an abstract polynomial). For $X = \mathbf{R}^q = Y$ the uniqueness of the irreducible form of an abstract rational operator is guaranteed [5]. It can also be shown that all the solutions of (2.1) have the same irreducible form.

We can now give the definition of APA and ARA. Let (P, Q) be a couple of abstract polynomials satisfying (2.1) and suppose that $D(Q) \neq \phi$ or $D(P) \neq \phi$. Let $(1/Q_0) \cdot P_0$ be the irreducible form of $(1/Q) \cdot P$ such that x_0 is contained in $D(Q_0)$ and $Q_0(x_0) = I$. If this form exists, we call it the abstract Padé approximant (APA) of order (n, m) for F. If for all solutions (P, Q) of (2.1) with $D(P) \neq \phi$ or $D(Q) \neq \phi$, the irreducible form $(1/Q_0) \cdot P_0$ is such that x_0 is not contained in $D(Q_0)$, then we call $(1/Q_0) \cdot P_0$ the abstract rational approximant (ARA) of order (n, m) for F.

Let $P = T \cdot P_0$ and $Q = T \cdot Q_0$ where $T(x) = \Sigma_k T_k(x - x_0)^k$ is the greatest common divisor of P and Q. Write $t_0 = \partial_0 T$.

We mention the following important property[5]: If $(1/Q_0) \cdot P_0$ is the (n, m)-ARA or (n, m)-APA for F at x_0 and if $D(T_{t_0}) \neq \phi$, then

$$(F \cdot Q_0 - P_0)(x) = O((x - x_0)^{nm+n+m+1-t_0}). \tag{2.2}$$

To construct iterative methods for finding x^*, we calculate the (n, m)-APA or (n, m)-ARA $(1/Q_i) \cdot P_i$ for F at x_i in each iteration-step, i.e. we compute a solution (which we denote again by (P, Q)) of the abstract Padé approximation problem for F at x_i and reduce the abstract rational operator $(1/Q) \cdot P$. Let t_i be the order of the greatest common divisor of P and Q. If $D(T_{t_i}) \neq \phi$, then for each i there exists a $0 < J_i < \infty$ such that

$$\|(F \cdot Q_i - P_i)(x)\| \leq J_i \|x - x_i\|^{nm+n+m+1-t_i} \tag{2.3}$$

for small $\|x - x_i\|$.

The condition that J_i must be finite for all i is comparable with conditions mentioned in ([11], p. 148) and ([13], pp. 135–139) to ensure quadratic convergence of the Newton-iteration for operator equations.

If we suppose that, instead of analytic, the operator F is only $(n + m + 1)$ times differentiable in a neighbbourhood V of x^*, then, under the condition that $F^{(n+m+1)}$ is integrable from $x_0 \in V$ to any x close to x_0, we can write ([13], p. 124):

$$F(x) = \sum_{k=0}^{n+m} \frac{1}{k!} F^{(k)}(x_0)(x - x_0)^k + \int_0^1 F^{(n+m+1)}(\theta x + (1 - \theta)x_0)(x - x_0)^{n+m+1} \frac{(1 - \theta)^{n+m}}{(n + m)!} \, d\theta$$

instead of the infinite Taylor series for F.

Since x^* is a simple root, an analogous expression is valid for the inverse operator G ([2], p. 381):

$$G(y) = \sum_{k=0}^{n+m} \frac{1}{k!} G^{(k)}(y_0)(y - y_0)^k + \int_0^1 G^{(n+m+1)}(\theta y + (1 - \theta)y_0)(y - y_0)^{n+m+1} \frac{(1 - \theta)^{n+m}}{(n + m)!} \, d\theta.$$

The abstract Padé approximation problem can be formulated for these operators F and G in the same way as we did before for analytic operators and the next sections also remain valid, but for the sake of simplicity we shall deal with the infinite series from now on.

3. APA AND ARA FOR THE INVERSE FUNCTION G

Let G be the inverse function of F, x_i the ith approximant of the root x^* in an iterative process, and $y_i = F(x_i)$. By $F_i^{(j)}$ and $G_i^{(j)}$ we mean the jth Fréchet-derivative of F and G at x_i and y_i respectively. Note that $F_i^{(j)}$ and $G_i^{(j)}$ are both symmetric j-linear operators. If $j = 1$ or 2, a single or double prime is used instead of the superscript (j). Now, since G is analytic in a neighbourhood of 0,

$$x^* = G(0) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} G_i^{(j)} y_i^j. \tag{3.1}$$

If we denote the Newton-correction $(-F_i'^{-1} F_i)$ by a_i, then

$$(-1)^j G_i^{(j)} y_i^j = G_i^{(j)} (F_i' a_i)^j$$

which can be understood as a j-linear operator E_j evaluated at a_i. Hence

$$x^* = \sum_{j=0}^{\infty} E_j a_i^j.$$

Using the Inversion Theorem ([2], p. 381), we can see that

$$E_0 a_i^0 = x_i$$
$$E_1 a_i = a_i$$
$$E_2 a_i^2 = -\tfrac{1}{2} F_i'^{-1} F_i'' a_i^2$$

so that

$$x^* = x_i + a_i - \tfrac{1}{2} F_i'^{-1} F_i'' a_i^2 + O(a_i^3). \tag{3.2}$$

First we observe that the Newton-iteration is the result of approximating the series in (3.2) by its first two terms (i.e. the (1, 0)-APA):

$$x_{i+1} = x_i + a_i.$$

The (0, 1)-APA gives the following iterative method:

$$x_{i+1} = \frac{x_i^2}{x_i - a_i}.$$

The multiplication and division in $\mathbf{R}^q$ are component by component as indicated in Section 2.

The first three terms in the expansion (3.2), which form in fact the (2, 0)-APA, could also be used to approximate x^*. This leads to the method

$$x_{i+1} = x_i + a_i - \tfrac{1}{2} F_i'^{-1} F_i'' a_i^2 \tag{3.3}$$

which was derived by Ehrmann[7] in a different manner (as an improvement of the Newton-iteration).

Another way to approximate x^* is to use the (1, 1)-ARA for the power series (3.2), i.e.

$$x_{i+1} = x_i + \frac{a_i^2}{a_i + \frac{1}{2} F_i'^{-1} F_i'' a_i^2} \tag{3.4}$$

which is a generalization of a formula of Frame[8] and a rediscovery of the Halley-correction (now for systems of equations).

If $F_i'^{-1} F_i'' a_i^2 = (I \otimes L) a_i^2 = a_i \cdot L a_i$ for a certain matrix L (i.e. $L \in \mathscr{L}(\mathbf{R}^q, \mathbf{R}^q)$) where I is the unit matrix, then (3.4) can be reduced to the (1, 1)-APA:

$$x_{i+1} = x_i + \frac{a_i}{1 + \frac{1}{2} L a_i}.$$

For $q = 1$ this reduction can always be performed and results in the Halley-correction.

The iterative procedure (3.4) is closely related to the method of tangent hyperbolas ([10], p. 188):

$$x_{i+1} = x_i + \{I + \tfrac{1}{2} F_i'^{-1} F_i'' a_i\}^{-1} a_i \tag{3.5}$$

or, equivalently (with one matrix-inversion less):

$$x_{i+1} = x_i - \{F_i' + \tfrac{1}{2} F_i'' a_i\}^{-1} F_i. \tag{3.6}$$

Formula (3.5) shows the interrelation with (3.4): instead of solving the system of linear equations in (3.6), both the matrix $I + \frac{1}{2}F_i'^{-1}F_i''a_i$ and the vector a_i are multiplied by a_i; this results in two vectors which are simply divided in (3.4). This technique is similar to a method introduced by Altman to avoid the inversion of matrices[1].

If we use the (0, 2)-APA for the series in (3.2) we get

$$x_{i+1} = \frac{x_i^3}{x_i^2 - x_i a_i + a_i^2 + \frac{1}{2}x_i F_i'^{-1} F_i'' a_i^2}.$$

4. APA AND ARA FOR F

Since F is analytic in a neighbourhood of x^* containing the approximants x_i, we have:

$$F(x) = \sum_{j=0}^{\infty} \frac{1}{j!} F_i^{(j)}(x - x_i)^j. \tag{4.1}$$

To illustrate our technique, we will now calculate the (n, m)-ARA for $n + m \leq 2$.

The (1, 0)-APA consists of the first two terms of (4.1):

$$F_i + F_i'(x - x_i).$$

If x_{i+1} is the zero of this expression, then

$$x_{i+1} = x_i - F_i'^{-1}F_i,$$

which is precisely Newton's method.

The (2, 0)-APA consists of the first three terms of (4.1):

$$F_i + F_i'(x - x_i) + \tfrac{1}{2}F_i''(x - x_i)^2, \tag{4.2}$$

so that x_{i+1} can be obtained by solving a quadratic operator equation. As indicated in [12], solving such an equation is a quite complicated matter. Moreover, the choice of x_{i+1} among distinct solutions of the quadratic equation is also a problem.

However, an approximate solution $\bar{x}_{i+1}$ can be obtained in the following manner[7]: the root x_{i+1} of (4.2) satisfies

$$x_{i+1} = x_i - F_i'^{-1}F_i - \tfrac{1}{2}F_i'^{-1}F_i''(x_{i+1} - x_i)^2.$$

If in the r.h.s. $x_{i+1} - x_i$ is approximated by the Newton-correction a_i, we have an approximation for x_{i+1} which is precisely (3.3):

$$\bar{x}_{i+1} = x_i + a_i - \tfrac{1}{2}F_i'^{-1}F_i''a_i^2.$$

Another way to express x_{i+1} is

$$x_{i+1} = x_i - \{F_i' + \tfrac{1}{2}F_i''(x_{i+1} - x_i)\}^{-1}F_i.$$

If again in the r.h.s. $x_{i+1} - x_i$ is approximated by a_i[6], we get

$$\bar{x}_{i+1} = x_i - \{F_i' + \tfrac{1}{2}F_i''a_i\}^{-1}F_i,$$

which is the method of tangent hyperbolas (3.6).

The (1, 1)-ARA for (4.1) at x_i is

$$\frac{F_iF_i'(x - x_i) + (F_i'(x - x_i))^2 - \frac{1}{2}F_iF_i''(x - x_i)^2}{F_i'(x - x_i) - \frac{1}{2}F_i''(x - x_i)^2}. \tag{4.3}$$

Let x_{i+1} be such that the numerator of (4.3) for $x = x_{i+1}$ vanishes; in other words, we have to solve the problem

$$F_i + F_i'(x_{i+1} - x_i) = \frac{1}{2}\frac{F_i F_i''(x_{i+1} - x_i)^2}{F_i'(x_{i+1} - x_i)}.$$

If we approximate in the r.h.s. $x_{i+1} - x_i$ by a_i we get the approximate solution

$$\bar{x}_{i+1} = x_i + a_i - \tfrac{1}{2}F_i'^{-1}F_i''a_i^2$$

which coincides with (3.3) again.

If $F_i''(x - x_i)^2 = (F_i' \otimes L)(x - x_i)^2 = F_i'(x - x_i) \cdot L(x - x_i)$ for a certain matrix L (i.e. L in $\mathscr{L}(\mathbf{R}^q, \mathbf{R}^q)$), then (4.3) can be reduced to the (1, 1)-APA

$$\frac{F_i + F_i'(x - x_i) - \frac{1}{2}F_i \otimes L(x - x_i)}{I - \frac{1}{2}L(x - x_i)} = \left(\frac{1}{Q_i} \cdot P_i\right)(x).$$

For $q = 1$ this reduction can always be performed.

Let us calculate x_{i+1} such that $P_i(x_{i+1}) = 0$.

Hence

$$x_{i+1} = x_i - (F_i' - \tfrac{1}{2}F_i \otimes L)^{-1}F_i. \tag{4.4}$$

The iterands x_{i+1} from (4.4) and (3.6) now coincide since

$$F_i' + \tfrac{1}{2}F_i''a_i = F_i' + \tfrac{1}{2}F_i'a_i \otimes L = F_i' - \tfrac{1}{2}F_i \otimes L.$$

In conclusion we can say that the methods derived by direct interpolation are either too complicated (when we calculate the exact solution x_{i+1}), or similar to methods from Section 3 (when we calculate an approximate solution $\bar{x}_{i+1}$). This justifies the fact that only techniques from Section 3 will be treated in the sequel of the text.

5. NUMERICAL ASPECTS

One of the main drawbacks to the use of (n, m)-APA or (n, m)-ARA is the computational cost of evaluating higher derivatives of F. However, in some cases these derivatives can be computed quite easily, e.g. if F satisfies a certain differential equation (so that the derivatives can be computed from this equation rather than from F itself) or if F is a composition of polynomials, trigonometric or exponential functions.

Let us now consider a convergent iterative procedure for the calculation of a simple root x^* of F: $\lim_{i\to\infty} x_i = x^*$.

We define the method to be of order at least p if:

$\forall i \in \mathbf{N}, \exists p_1 \geq 0, p_2 > 0$ and abstract power series
$\sum_{j=p_1}^{\infty} C_{1,j}(x^* - x_i)^j$ with $C_{1,p_1}(x^* - x_i)^{p_1} \not\equiv 0$ and
$\sum_{j=p_2}^{\infty} C_{2,j}(x^* - x_i)^j$ satisfying
$$\left(\sum_{j=p_1}^{\infty} C_{1,j}(x^* - x_i)^j\right) \cdot (x^* - x_{i+1}) = \sum_{j=p_2}^{\infty} C_{2,j}(x^* - x_i)^j$$
and $p_2 - p_1 = p$.

In classical definitions of order of an iterative process, the abstract power series on the left hand side of the equality is missing. Its presence here is due to the order $\partial_0 P$ and $\partial_0 Q$ of $P(x)$ and $Q(x)$ in the abstract Padé approximation problem. Nevertheless this definition is an extension of the well-known definition because for $C_{1,0}$ containing nonzero components and $p_1 = 0$ we can prove that there exists $0 < J_i < \infty$ such that $\|x^* - x_{i+1}\| \leq J_i\|x^* - x_i\|^p$. This definition coincides with that given by ([11], p. 148). For the method of tangent hyperbolas (3.5) we have $p_1 = 0$, $C_{1,0} = I$ and $p_2 = 3$, so that the order $p = 3$.

THEOREM

Let $(1/Q_i) \cdot P_i$ be the (n, m)-ARA or (n, m)-APA for G in y_i (inverse interpolation), deduced from a solution (P, Q) of (2.1). The order of the iterative procedure $x_{i+1} = ((1/Q_i) \cdot P_i)(0)$ is at least $nm + n + m + 1 - \partial_0 Q$ $(0 \in D(Q_i))$.

Proof. From (2.2) we get

$$(G \cdot Q_i - P_i) = O((y - y_i)^{nm+n+m+1-t_i}).$$

Since $G(0) = x^*$ and $x_{i+1} = P_i(0)/Q_i(0)$ for $y = 0$, we have

$$Q_i(0) \cdot (x^* - x_{i+1}) = (G \cdot Q_i - P_i)(0).$$

Hence

$$\left(\sum_{j=\partial_0 Q_i}^{\partial Q_i} D_{1,j}(-y_i)^j \right) \cdot (x^* - x_{i+1}) = \sum_{j=nm+n+m+1-t_i}^{\infty} D_{2,j}(-y_i)^j$$

for certain j-linear operators $D_{1,j}$ and $D_{2,j}$. Note that there exists a linear operator L such that $-y_i = L(x^* - x_i)$. Indeed

$$\begin{aligned} -y_i &= F(x^*) - F(x_i) \\ &= \left\{ \int_0^1 F'(\theta x^* + (1-\theta)x_i)\, d\theta \right\}(x^* - x_i) \\ &= L(x^* - x_i) \end{aligned}$$

Therefore

$$\left(\sum_{j=\partial_0 Q_i}^{\partial Q_i} C_{1,j}(x^* - x_i)^j \right) \cdot (x^* - x_{i+1}) = \sum_{j=nm+n+m+1-t_i}^{\infty} C_{2,j}(x^* - x_i)^j,$$

where $C_{k,j}(x^* - x_i)^j = D_{k,j}(L(x^* - x_i))^j \in \mathscr{L}((\mathbf{R}^q)^j, \mathbf{R}^q)$ $(k = 1, 2)$ ([3], p. 289). Since $D(T_{t_i}) \neq \phi$ (which we assumed in Section 2), $t_i + \partial_0 Q_i = \partial_0 Q$ and this proves the theorem.

For the iterative scheme (3.4) $p_1 = 1$ and $p_2 = 4$; consequently the use of the (1, 1)ARA for G provides a method of order 3.

Since the classical method of tangent hyperbolas is also of order 3, it would be interesting to compare the numerical effort per iteration for (3.4) and (3.6). They both have to solve two systems of linear equations. However, in (3.4) these two systems have the same coefficient-matrix F'_i, so that for the second system the elimination-part of the Gauss-method does not have to be repeated, while (3.6) requires the solution of two linear systems with matrices F'_i and $F'_i + \frac{1}{2}F''_i a_i$, so that the entire Gauss-method has to be performed twice. So we can conclude that (3.4) is to be preferred over (3.6).

Moreover, numerical experiments have confirmed the fact that both methods are of the same order, in the sense that the number of iterations to achieve x^* is comparable. As an example, the results obtained by (3.4) and (3.6) for the function

$$F: \mathbf{R}^2 \to \mathbf{R}^2: (u, v) \to (\exp(-u + v) - 0.1, \exp(-u - v) - 0.1),$$

which has a zero in $(-\ln(0.1), 0)$, are listed.

A last remark concerns the numerical behaviour of the entire class of methods that we have considered in Section 3. The methods derived from the (n, m)-ARA with $m \neq 0$ are more suitable for functions for which the inverse function G has singularities in the neighbourhood of

0. Our example also illustrates this fact:

$$G: (z, w) \to ((-\ln(z+0.1) - \ln(w+0.1))/2, (\ln(z+0.1) - \ln(w+0.1))/2)$$

has singularities for $w = -0.1$ or $z = -0.1$. If (3.3) is used, starting from the same initial point (u_0, v_0), then the sequence of iterands diverges, while (3.4) and (3.6) converge rapidly.

Table A.

Results obtained by (3.4)		Results obtained by (3.6)	
(u_i, v_i) i=0,...,5		(u_i, v_i) i=0,...,5	
.4300000000000000 (01)	.2000000000000000 (01)	.4300000000000000 (01)	.2000000000000000 (01)
.3336155282457216 (01)	.1035972419924183 (01)	.3337356399057231 (01)	.1034771307502802 (01)
.2560818009367738 (01)	.2596797949731372 (00)	.2561541506081360 (01)	.2589564130873139 (00)
.2308175634684460 (01)	.5683785304496196(-02)	.2308222334300647 (01)	.5637241306601315(-02)
.2302585151186788 (01)	.6120489087942105(-07)	.2302585152707625 (01)	.5971357897526734(-07)
.2302585092994046 (01)	-.3759322471455472(-17)	.2302585092994046 (01)	.1443269364993953(-16)

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