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GENERAL ORDER MULTIVARIATE PADÉ APPROXIMANTS FOR PSEUDO-MULTIVARIATE FUNCTIONS

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Abstract. Although general order multivariate Padé approximants were introduced some decades ago, very few explicit formulas for special functions have been given. We explicitly construct some general order multivariate Padé approximants to the class of so-called pseudo-multivariate functions, using the Padé approximants to their univariate versions. We also prove that the constructed approximants inherit the normality and consistency properties of their univariate relatives, which do not hold in general for multivariate Padé approximants. Examples include the multivariate forms of the exponential and the q-exponential functions

$$E\left(x,y\right)=\sum_{i,j=0}^{\infty}\frac{x^{i}y^{j}}{(i+j)!}$$

and

$$E_{q}\left(x,y\right)=\sum_{i,j=0}^{\infty}\frac{x^{i}y^{j}}{[i+j]_{q}!},$$

as well as the Appell function

$$F_{1}\left(a,1,1;c;x,y\right)=\sum_{i,j=0}^{\infty}\frac{(a)_{i+j}\,x^{i}y^{j}}{(c)_{i+j}}$$

and the multivariate form of the partial theta function

$$F\left(x,y\right)=\sum_{i,j=0}^{\infty}q^{-(i+j)^{2}/2}x^{i}y^{j}.$$

1. Introduction

Multivariate Padé approximants have been extensively investigated in the past few decades. The existence, uniqueness and nonuniqueness for homogeneous and general order multivariate Padé approximants and some convergence theorems have been established ([1], [5], [6], [7]). Despite all these activities, there are very few explicit constructions of multivariate Padé approximants. It is noteworthy that much of the difficulty in finding explicit formulae for multivariate Padé approximants lies in the determination of appropriate index sets for the numerator and denominator polynomials. By using the residue theorem and the functional equation method,

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several researchers have successfully constructed multivariate Padé approximants to some functions which satisfy functional equations ([4], [15], [16], [17]). Unfortunately, not many functions satisfy those functional equations. Besides, because the index sets for the numerator and denominator polynomials cannot be chosen freely, most numerators of the approximants look complicated. In this paper, we explicitly construct multivariate Padé approximants to so-called pseudo-multivariate functions, by using the Padé approximants of particular univariate functions which, in most of the cases, are the univariate projections of the pseudo-multivariate functions obtained by letting all but one variable be zero.

In order to avoid notational difficulties, we restrict ourselves to the case of bivariate functions. The generalization of the definitions to more than two variables is straightforward. We first recall the definition of the multivariate Padé approximant and introduce the concept of the pseudo-multivariate function in this section. The main results are proved in Sections 2 and 3. Several examples are given in Section 4.

Definition 1.1. Let

$$F(x,y) := \sum_{(i,j)\in\mathbb{N}^2} c_{ij}x^iy^j, \quad c_{ij}\in\mathbb{C},$$

be a formal power series, and let M, N, E be index sets in $\mathbb{N}\times\mathbb{N} = \mathbb{N}^2$. The (M,N) general order multivariate Padé approximant to $F(x,y)$ on the lattice E is a rational function

$$[M/N]_E(x,y) := \frac{P(x,y)}{Q(x,y)},$$

where the polynomials

$$P(x,y) := \sum_{(i,j)\in M} a_{ij}x^iy^j, \quad a_{ij}\in\mathbb{C},$$

$$Q(x,y) := \sum_{(i,j)\in N} b_{ij}x^iy^j, \quad b_{ij}\in\mathbb{C},$$

are such that

$$(FQ-P)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij}x^iy^j, \quad d_{ij}\in\mathbb{C}, \tag{1.1}$$

with E satisfying the inclusion property

$$(i,j)\in E,\ 0\le k\le i,\ 0\le l\le j \Longrightarrow (k,l)\in E. \tag{1.2}$$

Equation (1.1) translates to the linear system of equations

$$d_{ij} = \sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}b_{i-\mu,j-\nu} - a_{ij} = 0, \qquad (i,j)\in E, \tag{1.3}$$

where $b_{kl} = 0$ for $(k,l)\notin N$ and $a_{kl} = 0$ for $(k,l)\notin M$. Condition (1.2) takes care of the Padé approximation property, provided $Q(0,0)\neq 0$, namely

$$(F - \frac{P}{Q})(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} e_{ij}x^iy^j, \qquad e_{ij}\in\mathbb{C}.$$

The linear system (1.3) can be split in two parts: some of the equations serve to compute the numerator and denominator coefficients a_{ij} and b_{ij}, while the remaining equations are automatically satisfied by $FQ - P$ for the computed P and Q. We refer to the former set of indices (i, j) as C and to the latter as $E \setminus C$. For the class of pseudo-multivariate functions introduced in Definition 1.2, we shall see that very few equations of (1.3) are actually used for the computation of the coefficients. However, in general this is not the case.

It is clear (see [5], [10]) that a nontrivial general order multivariate Padé approximant always exists if $\#C$ is at least as large as the number of independent numerator and denominator coefficients (usually this translates to $\#E \geq \#N + \#M - 1$, which if $M \subset E$ simplifies to $\#(E \setminus M) \geq \#N - 1$). It is unique up to a constant factor in the numerator and denominator if the coefficient matrix of the linear system (1.3) has maximal rank. If the rank of the coefficient matrix of (1.3) is less than the maximal rank, then multiple solutions of $Q(x, y)$ and $P(x, y)$ exist, and we refer to Allouche and Cuyt [1] for a detailed discussion of this situation. For all definitions covered by the general definition given here, one cannot guarantee the existence of a unique irreducible form if multiple solutions of (1.3) exist. One may find more properties of general order multivariate Padé approximants in [5], [6], [7].

Definition 1.2. A multivariate function $F(x, y)$ is said to be pseudo-multivariate if the coefficients of its formal power series

$$F(x, y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j$$

satisfy

$$c_{ij} = g(i + j), \quad i, j = 0, 1, \cdots,$$

where $g(k)$ is a certain function of the index k.

For a pseudo-multivariate function $F(x, y)$, if $x \neq y$,

$$\begin{aligned} F(x, y) &= \sum_{k=0}^{\infty} g(k) \sum_{i+j=k} x^i y^j \\ &= \frac{1}{x - y} \left(\sum_{k=0}^{\infty} g(k) x^{k+1} - \sum_{k=0}^{\infty} g(k) y^{k+1} \right), \end{aligned}$$

and if $x = y$,

$$F(x, x) = \sum_{k=0}^{\infty} g(k) \sum_{i+j=k} x^{i+j} = \sum_{k=0}^{\infty} (k + 1) g(k) x^k.$$

If

$$\lim_{k \to \infty} \left| \frac{g(k)}{g(k + 1)} \right| = R < \infty,$$

then the series $\sum_{k=0}^{\infty} g(k) z^k$ has R as its radius of convergence. So if we let

$$h(z) := \sum_{k=0}^{\infty} g(k) z^k,$$

then for $|x|, |y| < R$, if $x \neq y$,

$$F(x, y) = \frac{xh(x) - yh(y)}{x - y},$$

and if $x = y$,

$$F(x, x) = \frac{d}{dx}(xh(x)) = \frac{d}{dy}(yh(y)) = F(y, y).$$

We refer to Section 4 for examples of pseudo-multivariate functions.

2. Padé approximants for pseudo-multivariate functions

In this section, we explicitly construct multivariate Padé approximants to pseudo-multivariate functions using the univariate Padé approximants of their projections. Throughout this paper we let, for integer $k \geq 0$,

$$X^k := \sum_{i+j=k} x^i y^j =: Y^k.$$

Theorem 2.1. *Let*

$$F(x, y) := \sum_{k=0}^{\infty} g(k) X^k$$

be a pseudo-multivariate function. For $m, n \in \mathbb{N}$, let

$$\frac{p_{m,n}}{q_{m,n}}(z) := \frac{\sum_{j=0}^{m} \alpha_j z^j}{\sum_{j=0}^{n} \beta_j z^j}, \qquad \beta_0 = 1, \tag{2.1}$$

be the (m, n) Padé approximant of the function

$$h(z) := \sum_{k=0}^{\infty} g(k) z^k,$$

let $s = \max\{m, n\}$, and let

$$N \quad : \quad = \{(i, j) : 0 \leq i, j \leq n\}, \tag{2.2}$$

$$M \quad : \quad = \{(i, j) : 0 \leq i, j \leq s\} \cap \{(i, j) : 0 \leq i + j \leq m + n\}, \tag{2.3}$$

$$E \quad : \quad = \{(i, j) : 0 \leq i + j \leq m + n, \ i, j \geq 0\} \tag{2.4}$$

be index sets in $\mathbb{N}^2$. Then the (M, N) general order multivariate Padé approximant to $F(x, y)$ on the index set E is

$$[M/N]_E(x, y) = \frac{P(x, y)}{Q(x, y)},$$

where

$$Q(x, y) := q_{m,n}(x)\, q_{m,n}(y) \tag{2.5}$$

and

$$P(x, y) \quad := \quad \sum_{i=0}^{m} \left(\sum_{j=0}^{\min\{i,n\}} \alpha_i \beta_j \left(x^i y^j + x^{i-1} y^{j+1} + \cdots + x^j y^i \right) \right.$$

$$\left. - \sum_{j=\min\{i,n\}+1}^{n} \alpha_i \beta_j \left(x^{i+1} y^{j-1} + x^{i+2} y^{j-2} + \cdots + x^{j-1} y^{i+1} \right) \right). \tag{2.6}$$

Proof. From (2.5),

$$Q(x,y)=q_{m,n}(x)q_{m,n}(y)=\sum_{(i,j)\in N} b_{ij}x^iy^j,\quad b_{ij}\in\mathbb{C}, \tag{2.7}$$

where N is defined by (2.2). Now let, for $x\neq y$,

$$P(x,y):=\frac{xp_{m,n}(x)q_{m,n}(y)-yq_{m,n}(x)p_{m,n}(y)}{x-y}. \tag{2.8}$$

If $j<i+1$,

$$\begin{aligned}x^{i+1}y^j-y^{i+1}x^j&=x^jy^j\left(x^{i+1-j}-y^{i+1-j}\right)\\&=x^jy^j\left(x-y\right)\left(x^{i-j}+x^{i-j-1}y+\cdots+xy^{i-j-1}+y^{i-j}\right)\\&=\left(x-y\right)\left(x^iy^j+x^{i-1}y^{j+1}+\cdots+x^{j+1}y^{i-1}+x^jy^i\right),\end{aligned}$$

and if $j\geq i+1$,

$$\begin{aligned}x^{i+1}y^j-y^{i+1}x^j&=x^{i+1}y^{i+1}\left(y^{j-i-1}-x^{j-i-1}\right)\\&=x^{i+1}y^{i+1}\left(y-x\right)\left(y^{j-i-2}+xy^{j-i-3}+\cdots+x^{j-i-3}y+x^{j-i-2}\right)\\&=\left(y-x\right)\left(x^{i+1}y^{j-1}+x^{i+2}y^{j-2}+\cdots+x^{j-2}y^{i+2}+x^{j-1}y^{i+1}\right).\end{aligned}$$

Then

$$\begin{aligned}P(x,y)&=\frac{\sum_{i=0}^m\sum_{j=0}^n\alpha_i\beta_j\left(x^{i+1}y^j-y^{i+1}x^j\right)}{x-y}\\&=\sum_{i=0}^m\sum_{j=0}^n\alpha_i\beta_j\frac{x^{i+1}y^j-y^{i+1}x^j}{x-y}\\&=\sum_{i=0}^m\left(\sum_{j=0}^{\min\{i,n\}}\alpha_i\beta_j\frac{x^{i+1}y^j-y^{i+1}x^j}{x-y}+\sum_{j=\min\{i,n\}+1}^{n}\alpha_i\beta_j\frac{x^{i+1}y^j-y^{i+1}x^j}{x-y}\right)\\&=\sum_{i=0}^m\left(\sum_{j=0}^{\min\{i,n\}}\alpha_i\beta_j\left(x^iy^j+x^{i-1}y^{j+1}+\cdots+x^jy^i\right)\right.\\&\qquad\left.-\sum_{j=\min\{i,n\}+1}^{n}\alpha_i\beta_j\left(x^{i+1}y^{j-1}+x^{i+2}y^{j-2}+\cdots+x^{j-1}y^{i+1}\right)\right)\\&=\sum_{(i,j)\in M}a_{ij}x^iy^j,\qquad a_{ij}\in\mathbb{C},\end{aligned}$$

where M is defined by (2.3). We remark that if $\min\{i,n\}=n$, the second sum in $P(x,y)$ is empty.

For $x \neq y$,

$$F(x,y) = \frac{xh(x) - yh(y)}{x-y},$$

we have

$$\begin{aligned}(FQ-P)(x,y) &= \frac{1}{x-y}\left\{xh(x)\,q_{m,n}(x)\,q_{m,n}(y) - yh(y)\,q_{m,n}(x)\,q_{m,n}(y)\right.\\ &\qquad \left.-xp_{m,n}(x)\,q_{m,n}(y) + yq_{m,n}(x)\,p_{m,n}(y)\right\}\\ &= \frac{1}{x-y}\left\{xq_{m,n}(y)\left(h(x)\,q_{m,n}(x) - p_{m,n}(x)\right)\right.\\ &\qquad \left.-yq_{m,n}(x)\left(h(y)\,q_{m,n}(y) - p_{m,n}(y)\right)\right\}. \end{aligned} \tag{2.9}$$

Recall that $p_{m,n}(z)/q_{m,n}(z)$ is the (m,n) Padé approximant to $h(z)$, i.e.,

$$\begin{aligned} h(x)\,q_{m,n}(x) - p_{m,n}(x) &= \sum_{j\geq m+n+1} \gamma_j x^j, \qquad \gamma_j \in \mathbb{C},\\ h(y)\,q_{m,n}(y) - p_{m,n}(y) &= \sum_{j\geq m+n+1} \gamma_j y^j, \qquad \gamma_j \in \mathbb{C}. \end{aligned}$$

Then

$$\begin{aligned}(FQ-P)(x,y) &= \frac{\sum_{j\geq m+n+1}\gamma_j\left(x^{j+1}q_{m,n}(y) - y^{j+1}q_{m,n}(x)\right)}{x-y}\\ &= \sum_{j\geq m+n+1}\sum_{i=0}^{n}\beta_i\gamma_j\frac{x^{j+1}y^i - x^iy^{j+1}}{x-y}\\ &= \sum_{j\geq m+n+1}\sum_{i=0}^{n}\beta_i\gamma_j\left(x^jy^i + x^{j-1}y^{i+1} + \cdots + x^iy^j\right) \quad (\text{as } i<j)\\ &= \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij}x^iy^j, \qquad d_{ij}\in\mathbb{C}, \end{aligned} \tag{2.10}$$

where E is defined by (2.4). It is easy to see that

$$C = \{(i,0) \mid 0 \leq i \leq n+m\} \cup \{(0,j) \mid 0 \leq j \leq n+m\} \subseteq E.$$

This proves that $P(x,y)/Q(x,y)$ is the (M,N) general order multivariate Padé approximant to $F(x,y)$ for $x \neq y$ on the set E.

Now for the case $x = y$. Since

$$\begin{aligned}
&\lim_{y\to x}\left(F-\frac{P}{Q}\right)(x,y)\\
&\quad=\lim_{y\to x}\frac{xq_{m,n}(y)\,(hq_{m,n}-p_{m,n})(x)-yq_{m,n}(x)\,(hq_{m,n}-p_{m,n})(y)}{(x-y)\,q_{m,n}(x)\,q_{m,n}(y)}\\
&\quad=\lim_{y\to x}\frac{x\,(h-p_{m,n}/q_{m,n})(x)-y\,(h-p_{m,n}/q_{m,n})(y)}{x-y}\\
&\quad=\lim_{y\to x}\sum_{j\ge m+n+1}d_j\frac{x^{j+1}-y^{j+1}}{x-y}\\
&\quad=\sum_{j\ge m+n+1}(j+1)\,d_jX^j,
\end{aligned}$$

$$\begin{aligned}
\lim_{y\to x}\frac{P(x,y)}{Q(x,y)}&=\lim_{y\to x}\frac{xp_{m,n}(x)q_{m,n}(y)-yq_{m,n}(x)p_{m,n}(y)}{(x-y)\,q_{m,n}(x)\,q_{m,n}(y)}\\
&=\lim_{y\to x}\frac{xp_{m,n}(x)\,/q_{m,n}(x)-yp_{m,n}(y)\,/q_{m,n}(y)}{x-y}\\
&=\frac{d}{dx}\left(\frac{xp_{m,n}(x)}{q_{m,n}(x)}\right),
\end{aligned}\tag{2.11}$$

and

$$\frac{d}{dx}\left(\frac{xp_{m,n}(x)}{q_{m,n}(x)}\right)=\frac{\big(p_{m,n}(x)+xp'_{m,n}(x)\big)\,q_{m,n}(x)-xp_{m,n}(x)\,q'_{m,n}(x)}{q^2_{m,n}(x)},$$

we have

$$Q(x,x)=q^2_{m,n}(x),$$

which is (2.5) when $x = y$. Since

$$\begin{aligned}
P(x,x)&=\big(p_{m,n}(x)+xp'_{m,n}(x)\big)\,q_{m,n}(x)-xp_{m,n}(x)\,q'_{m,n}(x)\\
&=\left(\sum_{i=0}^m\alpha_ix^i+x\sum_{i=0}^m i\alpha_ix^{i-1}\right)\sum_{j=0}^n\beta_jx^j-x\left(\sum_{i=0}^m\alpha_ix^i\right)\left(\sum_{j=0}^n j\beta_jx^{j-1}\right)\\
&=\left(\sum_{i=0}^m(i+1)\,\alpha_ix^i\right)\left(\sum_{j=0}^n\beta_jx^j\right)-\sum_{i=0}^m\sum_{j=0}^n j\alpha_i\beta_jx^ix^j\\
&=\sum_{i=0}^m\sum_{j=0}^n(i+1-j)\,\alpha_i\beta_jx^ix^j\\
&=\sum_{i=0}^m\left(\sum_{j=0}^{\min\{i,n\}}+\sum_{j=\min\{i,n\}+1}^n\right)(i+1-j)\,\alpha_i\beta_jx^{i+j}\\
&=\sum_{i=0}^m\sum_{j=0}^{\min\{i,n\}}(i+1-j)\,\alpha_i\beta_jx^{i+j}-\sum_{i=0}^m\sum_{j=\min\{i,n\}+1}^n(j-i-1)\,\alpha_i\beta_jx^{i+j},
\end{aligned}$$

we obtain (2.6) when $x = y$. This proves Theorem 2.1. □

Remark. It is easy to see that $P(x,y)\,/Q(x,y)$ is irreducible, as $p_{m,n}(z)\,/q_{m,n}(z)$ is irreducible.

3. Properties of the approximants

The univariate Padé approximant satisfies a consistency property, meaning that when the given function h is itself rational, then it is reconstructed by $p_{m,n}/q_{m,n}$ when m and n are chosen large enough. This consistency property holds mainly because of the unicity of the irreducible form of $p_{m,n}/q_{m,n}$. For a general order multivariate Padé approximant this is not necessarily the case, because of the possible nonunicity of the irreducible form of the approximant (see [5] for more details). However, the general order multivariate Padé approximants constructed here in Theorem 2.1 have many nice properties. We prove the consistency and normality properties of these approximants, the latter meaning that if the univariate (m,n) Padé approximant $p_{m,n}/q_{m,n}$ to the function h appears only once in the Padé table, then so does its general multivariate counterpart constructed here. In both the univariate and multivariate case the Padé table of the approximants $r_{m,n}(z) = p_{m,n}(z)/q_{m,n}(z)$ and $[M/N]_E(x,y) = P(x,y)/Q(x,y)$ respectively, is defined as a matrix-like structure with row index m and column index n containing all approximants for increasing m and n. A projection property, such as

$$\begin{aligned} \frac{P(x,0)}{Q(x,0)} &= \frac{p_{m,n}(x)}{q_{m,n}(x)}, \\ \frac{P(0,y)}{Q(0,y)} &= \frac{p_{m,n}(y)}{q_{m,n}(y)} \end{aligned}$$

is automatically satisfied because of (2.8), (2.7) and (2.1). At the end of this section we also present a truncation error upperbound.

Theorem 3.1. *Let M, N, E be defined as in Theorem 2.1. If the pseudo-multivariate function $F(x,y)$ is a rational function, i.e., if $F(x,y)$ has the irreducible form*

$$F(x,y) := \frac{u(x,y)}{v(x,y)} = \frac{\sum_{(i,j)\in M} u_{ij}x^i y^j}{\sum_{(i,j)\in N} v_{ij}x^i y^j}, \qquad u_{ij}, v_{ij} \in \mathbb{C},$$

with $v(0,0) \neq 0$, then the (M,N) general order multivariate Padé approximant to $F(x,y)$ on the index set E given in Theorem 2.1, satisfies

$$\frac{P(x,y)}{Q(x,y)} = F(x,y).$$

Proof. As F is a pseudo-multivariate function, we can write

$$F(x,y) = \sum_{k\geq 0} g(k)\, X^k$$

for some univeriate function $g(k)$, and if we let

$$h(z) := \sum_{k\geq 0} g(k)\, z^k,$$

then $h(z) = F(z,0) = F(0,z)$, and for $x \neq y$,

$$F(x,y) = \frac{u(x,y)}{v(x,y)} = \frac{xh(x) - yh(y)}{x-y}.$$

Because $v_{00} \neq 0$, $F(z,0) = F(0,z)$ implies that $u(z,0) = u(0,z)$ and $v(z,0) = v(0,z)$. From the projection property, the (m,n) Padé approximant to $h(z)$ is

$$h(z) = \frac{u(z,0)}{v(z,0)} = \frac{u(0,z)}{v(0,z)},$$

and from the consistency property of the univariate Padé aproximant, we have $p_{m,n}(z) = u(z,0) = u(0,z)$ and $q_{m,n}(z) = v(z,0) = v(0,z)$. So

$$\begin{aligned}\frac{P(x,y)}{Q(x,y)} &= \frac{xu(x,0)\,v(0,y) - yu(0,y)\,v(x,0)}{(x-y)\,v(x,0)\,v(0,y)} \\ &= \frac{xh(x) - yh(y)}{x-y} \\ &= F(x,y).\end{aligned}$$

For $x = y$,

$$F(x,x) = \frac{d}{dx}(xh(x)) = \frac{d}{dx}\left(\frac{xu(x,0)}{v(x,0)}\right),$$

while (2.11) gives

$$\frac{P(x,x)}{Q(x,x)} = \lim_{y\to x}\frac{P(x,y)}{Q(x,y)} = \frac{d}{dx}\left(\frac{xu(x,0)}{v(x,0)}\right) = F(x,x).$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *For $m,n \in \mathbb{N}$, let M, N, E, $F(x,y)$ and $h(z)$ be defined as in Theorem* 2.1*. If the (m,n) Padé approximant $p_{m,n}(z)/q_{m,n}(z)$ to $h(z)$ is normal, then the (M,N) Padé approximant $P(x,y)/Q(x,y)$ to $F(x,y)$ on the index set E given in Theorem* 2.1*, is also normal.*

Proof. If the (m,n) Padé approximant to $h(z)$,

$$\frac{p_{m,n}}{q_{m,n}}(z) = \frac{\sum_{j=0}^{m}\alpha_j z^j}{\sum_{j=0}^{n}\beta_j z^j}, \qquad \beta_0 = 1,$$

is normal for $m,n \in \mathbb{N}$, then

$$\alpha_m \neq 0, \qquad \beta_n \neq 0,$$

and

$$(hq_{m,n} - p_{m,n})(z) = \sum_{j\geq m+n+1}\gamma_j z^j,$$

with

$$\gamma_{m+n+1} \neq 0.$$

The (M,N) Padé approximant to $F(x,y)$ on the set E is $P(x,y)/Q(x,y)$, where

$$Q(x,y) = q_{m,n}(x)\,q_{m,n}(y) = \sum_{(i,j)\in N} b_{ij}x^i y^j, \;\; b_{ij} \in \mathbb{C},$$

and

$$
\begin{aligned}
P(x,y) := \sum_{i=0}^{m} \Bigg(& \sum_{j=0}^{\min\{i,n\}} \alpha_i \beta_j \left(x^i y^j + x^{i-1} y^{j+1} + \cdots + x^j y^i \right) \\
& - \sum_{j=\min\{i,n\}+1}^{n} \alpha_i \beta_j \left(x^{i+1} y^{j-1} + x^{i+2} y^{j-2} + \cdots + x^{j-1} y^{i+1} \right) \Bigg) \\
= & \sum_{(i,j)\in M} a_{ij} x^i y^j, \quad a_{ij} \in \mathbb{C},
\end{aligned}
$$

with

$$
b_{00} = (\beta_0)^2 = 1, \qquad b_{0n} = b_{n0} = \beta_n \beta_0 \neq 0, \qquad b_{nn} = (\beta_n)^2 \neq 0,
$$

$$
a_{mn} = a_{nm} = \alpha_m \beta_n \neq 0, \qquad a_{m0} = a_{0m} = \alpha_m \beta_0 \neq 0.
$$

If

$$
(fQ - P)(x,y) = \sum_{(i,j)\in E} d_{ij} x^i y^j,
$$

then from (2.10), for $i + j = m + n + 1$,

$$
d_{ij} = \beta_0 \gamma_{m+n+1} = \gamma_{m+n+1} \neq 0.
$$

Now assume that for either $m' \neq m$ or $n' \neq n$, $s' = \max\{m', n'\}$ with

$$
\begin{aligned}
N' \; &: = \{(i,j) : 0 \leq i,j \leq n'\}, \\
M' \; &: = \{(i,j) : 0 \leq i,j \leq s'\} \cap \{(i,j) : 0 \leq i+j \leq m'+n'\}, \\
E' \; &: = \{(i,j) : 0 \leq i+j \leq m'+n',\ i,j \geq 0\},
\end{aligned}
$$

the general order multivariate Padé approximant $[M'/N']_{E'}$ for $F(x,y)$ on the set E' equals the same rational function P/Q. Since $\alpha_m = a_{m,0} \neq 0$ and $\beta_n = b_{n0} \neq 0$, this is only possible for $m' \geq m$ and $n' \geq n$. Hence $m'+n'+1 \geq m+n+1$. The fact that $\gamma_{m+n+1} \neq 0$ reduces the occurrence of nonnormality to $m'+n'+1 \leq m+n+1$. Hence $m'+n'+1 = m+n+1$. In combination with $m' \geq m$ and $n' \geq n$ this leads to $m' = m$ and $n' = n$. Since the latter is a contradiction with our assumption that either $m' \neq m$ or $n' \neq n$, normality must hold. $\square$

Theorem 3.3. *Let $M, N, E, F(x,y)$ and $h(z)$ be defined as in Theorem* 2.1*. Let $p_{m,n}(z)/q_{m,n}(z)$ be the (m,n) Padé approximant to $h(z)$ and let $P(x,y)/Q(x,y)$ be the (M,N) Padé approximant, constructed in Theorem* 2.1*, to $F(x,y)$ on E. Then for $x \neq y$,*

$$
\begin{aligned}
& \left| \left(F - \frac{P}{Q} \right)(x,y) \right| \\
& \quad \leq \frac{\sup_{\xi \in [0,\max(x,y)]} \left| (h q_{m,n})^{(m+n+1)}(\xi) \right|}{|x-y|\,(m+n+1)!} \left(\frac{|x|^{m+n+2}}{|q_{m,n}(x)|} + \frac{|y|^{m+n+2}}{|q_{m,n}(y)|} \right),
\end{aligned}
\tag{3.1}
$$

and for $x = y$,

(3.2)
$$\left|F(x,x) - \frac{P(x,x)}{Q(x,x)}\right| \leq \sup_{\xi\in[0,x]} \left|(hq_{m,n})^{(m+n+1)}(\xi)\right| \frac{(m+n+2)\left|q_{m,n}(x)\right| + |x|\left|q'_{m,n}(x)\right|}{\left|q_{m,n}(x)\right|^2 (m+n+1)!}|x|^{m+n+1}.$$

Proof. For the univariate function $h(z)$ and its (m,n) Padé approximant $p_{m,n}/q_{m,n}$, we know that

$$|(hq_{m,n} - p_{m,n})(z)| \leq \sup_{w\in[0,z]} \left|(hq_{m,n})^{(m+n+1)}(w)\right| \frac{|z|^{m+n+1}}{(m+n+1)!}. \tag{3.3}$$

A simple computation using (2.9) gives (3.1), the formula for $x \neq y$. Now if

$$(hq_{m,n} - p_{m,n})(z) = \sum_{j=m+n+1}^{\infty} d_j z^j, \quad d_j \in \mathbb{C},$$

then

$$\begin{aligned}(hq_{m,n} - p_{m,n})'(z) &= \frac{d}{dz}(hq_{m,n} - p_{m,n})(z)\\ &= \sum_{j=m+n+1}^{\infty} j d_j z^{j-1} = \sum_{j=m+n}^{\infty} (j+1)\, d_{j+1} z^j,\end{aligned}$$

and the leading term has degree $m+n$. Therefore

$$\begin{aligned}|(hq_{m,n} - p_{m,n})'(z)| &= \left|(hq_{m,n})'(z) - p'_{m,n}(z)\right|\\ &\leq \sup_{w\in[0,z]} \left|(hq_{m,n})^{(m+n+1)}(w)\right| \frac{|z|^{m+n}}{(m+n)!}.\end{aligned} \tag{3.4}$$

Now for $x = y$,

$$\begin{aligned}&F(x,x) - \frac{P(x,x)}{Q(x,x)}\\ &\quad= \frac{d}{dx}(xh(x)) - \frac{d}{dx}\left(\frac{xp_{m,n}(x)}{q_{m,n}(x)}\right)\\ &\quad= \frac{\left[(h(x)\, q_{m,n}(x) - p_{m,n}(x)) + x\left((h(x)\, q_{m,n}(x))' - p'_{m,n}(x)\right)\right] q_{m,n}(x)}{q^2_{m,n}(x)}\\ &\qquad - \frac{\left[x\,(h(x)\, q_{m,n}(x) - p_{m,n}(x))\right] q'_{m,n}(x)}{q^2_{m,n}(x)}.\end{aligned}$$

Recall that $Q(x,x) = q_{m,n}^2(x)$, and by using (3.3) and (3.4), we have

$$\begin{aligned}
&|Q(x,x)| \left| F(x,x) - \frac{P(x,x)}{Q(x,x)} \right| \\
&\quad \le |h(x)\, q_{m,n}(x) - p_{m,n}(x)|\, |q_{m,n}(x)| \\
&\qquad + |x| \left| \left((h(x)\, q_{m,n}(x))' - p'_{m,n}(x) \right) \right| |q_{m,n}(x)| \\
&\qquad + |x|\, |(h(x)\, q_{m,n}(x) - p_{m,n}(x))| \left| q'_{m,n}(x) \right| \\
&\quad \le \sup_{\xi \in [0,x]} \left| (hq_{m,n})^{(m+n+1)}(\xi) \right| \frac{|x|^{m+n+1} |q_{m,n}(x)|}{(m+n+1)!} \\
&\qquad + \sup_{\xi \in [0,x]} \left| (hq_{m,n})^{(m+n+1)}(\xi) \right| \frac{|x|^{m+n+1} |q_{m,n}(x)|}{(m+n)!} \\
&\qquad + \sup_{\xi \in [0,x]} \left| (hq_{m,n})^{(m+n+1)}(\xi) \right| \frac{|x|^{m+n+2} \left| q'_{m,n}(x) \right|}{(m+n+1)!} \\
&\quad = \sup_{\xi \in [0,x]} \left| (hq_{m,n})^{(m+n+1)}(\xi) \right| \frac{(m+n+2)\, |q_{m,n}(x)| + |x| \left| q'_{m,n}(x) \right|}{(m+n+1)!} |x|^{m+n+1}.
\end{aligned}$$

Dividing by $|Q(x,x)|$, we have (3.2). □

4. Examples

For the sequel we need the standard q-analogues of factorials and binomial coefficients. The q-factorial is defined by

$$[n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n},$$

where $[0]_q! := 1$. The q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]! \cdot [n-k]!}.$$

For any positive integer i,

$$(a)_i := \begin{cases} a(a+1)(a+2)\cdots(a+i-1) & i \ge 1, \\ 1, & i = 0. \end{cases}$$

Throughout the section, we let M, N and E be defined as in Theorem 2.1. The following examples result as applications of Theorem 2.1.

Example 4.1. A multivariate form of the exponential function is

$$E(x,y) = \sum_{i,j=0}^{\infty} \frac{x^i y^j}{(i+j)!}.$$

It is a pseudo-multivariate function with

$$h(z) = \exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

From [2], we have that the (m,n) Padé approximant to $h(z)$ is

$$\frac{p_{m,n}}{q_{m,n}}(z) = \frac{{}_1F_1(-m; -m-n; z)}{{}_1F_1(-n; -m-n; -z)},$$

and then the (M,N) general order multivariate Padé approximant to $E(x,y)$ is $P(x,y)/Q(x,y)$, where

$$\begin{aligned} Q(x,y) &= {}_1F_1(-n;-m-n;-x)\ {}_1F_1(-n;-m-n;-y) \\ &= \sum_{i,j=0}^{n} \frac{(-1)^{i+j}(-n)_i(-n)_j}{(-m-n)_i(-m-n)_j} x^i y^j \end{aligned}$$

and

$$\begin{aligned} P(x,y) = {} & \frac{1}{x-y}\left(x\,{}_1F_1(-m;-m-n;x)\ {}_1F_1(-n;-m-n;-y)\right. \\ & \left. -y\,{}_1F_1(-n;-m-n;-x)\ {}_1F_1(-m;-m-n;y)\right) \\ = {} & \frac{1}{x-y}\left(\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(-1)^j(-m)_i(-n)_j}{(-m-n)_i(-m-n)_j}\left(x^{i+1}y^j - x^j y^{i+1}\right)\right) \\ = {} & \sum_{i=0}^{m}\left(\sum_{j=0}^{\min\{i,n\}}\frac{(-1)^j(-m)_i(-n)_j}{(-m-n)_i(-m-n)_j}\frac{x^{i+1}y^j - y^{i+1}x^j}{x-y}\right. \\ & \left. + \sum_{j=\min\{i,n\}+1}^{n}\frac{(-1)^j(-m)_i(-n)_j}{(-m-n)_i(-m-n)_j}\frac{x^{i+1}y^j - y^{i+1}x^j}{x-y}\right) \\ = {} & \sum_{i=0}^{m}\left(\sum_{j=0}^{\min\{i,n\}}\frac{(-1)^j(-m)_i(-n)_j}{(-m-n)_i(-m-n)_j}\left(x^i y^j + x^{i-1}y^{j+1} + \cdots + x^j y^i\right)\right. \\ & - \sum_{j=\min\{i,n\}+1}^{n}\frac{(-1)^j(-m)_i(-n)_j}{(-m-n)_i(-m-n)_j} \\ & \left. \cdot\left(x^{i+1}y^{j-1} + x^{i+2}y^{j-2} + \cdots + x^{j-1}y^{i+1}\right)\right). \end{aligned}$$

Example 4.2. A multivariate form of the q-exponential function is

$$E_q(x,y) = \sum_{i,j=0}^{\infty}\frac{x^i y^j}{[i+j]_q!}, \quad |q|>1.$$

It is a pseudo-multivariate function with

$$h(z) = E_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{[k]!}, \quad |q|>1.$$

From [3] combined with [16], we find that the (m,n) Padé approximant to $E_q(z)$ is $p_{m,n}/q_{m,n}$, with

$$\begin{aligned} q_{m,n}(z) &:= \frac{1}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2-2nk}\prod_{j=1}^{k}\left(1+(q-1)q^{j-1}z\right), \\ p_{m,n}(z) &:= \frac{(-1)^{n+1}q^{-n(n+1)/2}}{[n]!}\sum_{k+l=n,k,l\geq 0} q^{-nl}[n-k]\begin{bmatrix} n+l \\ l \end{bmatrix} z^k, \end{aligned}$$

and hence the (M,N) general order multivariate Padé approximant can be constructed.

Example 4.3. The natural generalizations of the Gauss hypergeometric function to two variables are called Appell functions (see [8], [13] for more details). The Appell function

$$F_1\left(a,1,1;c;x,y\right)=\sum_{i,j=0}^{\infty}\frac{\left(a\right)_{i+j}x^iy^j}{\left(c\right)_{i+j}}$$

is a pseudo-multivariate function with

$$h\left(z\right)={}_2F_1\left(a,1;c;z\right)=\sum_{i=0}^{\infty}\frac{\left(a\right)_i z^i}{\left(c\right)_i}.$$

We introduce the notation $\Pi_k(f)$ to denote the partial sum of degree k of the Mac Laurin series development of the function $f(z)$. From [14, 12] we know that for $n \le m+1$ and $c \notin \mathbb{Z}^-$ the (m,n) Padé approximant to ${}_2F_1\left(a,1;c;z\right)$ is $p_{m,n}\left(z\right)/q_{m,n}\left(z\right)$, where

$$\begin{aligned}p_{m,n}\left(z\right)&=\Pi_m\left({}_2F_1(a,1;c;z)\,{}_2F_1(-a-m,-n;-c-m-n+1;z)\right)\\ q_{m,n}\left(z\right)&={}_2F_1\left(-a-m,-n;-c-m-n+1;z\right),\end{aligned}$$

and then the (M,N) general order multivariate Padé approximant can be obtained.

Example 4.4. The multivariate form of the partial theta function (also see [16], [11], [3]) is

$$T_q\left(x,y\right)=\sum_{i,j=0}^{\infty}q^{(i+j)(i+j+1)/2}x^iy^j,\qquad |q|<1.$$

It is a pseudo-multivariate function with

$$h(z)=T\left(z\right)=\sum_{k=0}^{\infty}q^{k(k+1)/2}z^k.$$

The (m,n) Padé approximant to $T\left(z\right)$, constructed in [3], equals $p_{m,n}\left(z\right)/q_{m,n}\left(z\right)$, with

$$q_{m,n}\left(z\right)=\sum_{k=0}^{n}\left(-1\right)^k\begin{bmatrix}n\\k\end{bmatrix}q^{mk}z^k.$$

Using the technique developed in [3] and [16], we have that

$$\begin{aligned}p_{m,n}\left(z\right)\quad&=\quad\sum_{k=0}^{n}\left(-1\right)^k\begin{bmatrix}n\\k\end{bmatrix}q^{m(n-k)}\sum_{i=0}^{k-1}q^{i(i-1)/2}z^{n+i-k}\\ &\quad+\left(-1\right)^{n+1}q^{n(n+1)/2}\sum_{k+l=m-n,k,l\ge 0}q^{k(k+1)/2-nl}\begin{bmatrix}n+l\\l\end{bmatrix}z^{n+k}.\end{aligned}$$

From this information again the (M,N) general order multivariate Padé approximant can be given.

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