A de Montessus theorem for multivariate homogeneous Padé approximants

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Dedicated to T.J. Rivlin on the occasion of his 70th birthday

We establish a de Montessus theorem for multivariate homogeneous Padé approximants. The interesting feature is that the approximants converge locally uniformly (that is, uniformly in compact sets) away from a certain analytic set, but need not converge locally uniformly in any neighbourhood of any point of the analytic set.

1. Introduction and results

The classical de Montessus theorem for one complex variable asserts that if f is analytic in $|z| < R$, except for poles of total multiplicity n, none lying at 0, then the (m, n) Padé approximant $[m/n](z)$ to $f(z)$ converges to $f(z)$, as $m \to \infty$, uniformly in compact subsets of $|z| < R$ omitting poles of f.

There have been several generalizations of this to multivariate Padé approximants. The latter fall naturally into two categories of approximants: homogeneous and non-homogeneous. Most of the de Montessus type theorems have been given for the latter – see [5,7,8,11,14]. In this paper, we establish a de Montessus theorem for homogeneous Padé approximants.

Recall first the definition of homogeneous Padé approximants: Let $f(\mathbf{z})$ denote a power series of k variables $z_1, z_2, \ldots z_k$, convergent in a neighbourhood of $\mathbf{0}$, where

$$\mathbf{z} = (z_1, z_2, \ldots z_k). \tag{1}$$

We can rearrange the Maclaurin series of f into a homogeneous expansion

$$f(\mathbf{z}) = \sum_{j=0}^{\infty} f_j(\mathbf{z}), \tag{2}$$

where $f_j(\mathbf{z})$ is a *homogeneous polynomial of degree* j, that is,

$$f_j(\mathbf{z}) = \sum_{j_1+j_2+\ldots j_k=j} c_{j_1 j_2 \ldots j_k} z_1^{j_1} z_2^{j_2} \ldots z_k^{j_k}. \tag{3}$$

The homogeneity of degree j is expressed in the identity

$$f_j(uz) = u^j f_j(z), \quad u \in \mathbb{C}. \tag{4}$$

The homogeneous Padé approximant of type (m,n) to f, denoted $[m/n](z) = (P/Q)(z)$, is a rational function of z, where

$$P(z) = \sum_{j=mn}^{mn+m} p_j(z); Q(z) = \sum_{j=mn}^{mn+n} q_j(z) \tag{5}$$

satisfy

$$(fQ - P)(z) = \sum_{j=mn+m+n+1}^{\infty} c_j(z). \tag{6}$$

(Each p_j, q_j, c_j is a homogeneous polynomial of degree j.)

At first sight, the "shift" mn in the homogeneous terms in P, Q and the remainder term in $fQ - P$ is disconcerting. However, it is essential within this framework to guarantee existence of the approximant. P/Q does have an irreducible form, but even this irreducible form need not be analytic at $\mathbf{0}$. Homogeneous Padé approximants are unique – see [6, chapter 2] for further orientation.

The only convergence result for sequences of homogeneous Padé approximants $\{[m/n]\}_{m=1}^{\infty}$ with n fixed, is due to Cuyt [9]. The result involves functions $f(z)$ meromorphic in a ball centre $\mathbf{0}$ in $\mathbb{C}^k$ whose set of singularities is given by

$$\mathcal{S} := \{z \in \mathbb{C}^k : S(z) = 0\}, \tag{7}$$

where

$$S(z) = \sum_{j=0}^{\mu} s_j(z) \tag{8}$$

is a polynomial of degree μ, so that each s_j is homogeneous of degree j and s_μ is not identically zero. We assume that there exist infinitely many m such that we can cancel common factors from both numerator and denominator of $[m/n]$ to obtain a denominator that does not vanish at $\mathbf{0}$. If $n \geqslant \mu$, Cuyt showed that a subsequence of $\{[m/n]\}_{m=1}^{\infty}$ converges locally uniformly (that is, uniformly in compact sets) away from the zero set of a certain polynomial of degree at most n.

We shall show that in the case where $\mu = n$ the full sequence $\{[m/n]\}$ converges away from a certain $(k-1)$-dimensional analytic set, and moreover, the full sequence need not converge locally uniformly in a neighbourhood of any point of the analytic set. Unlike the result in [9], we do not need to assume that $[m/n](z)$ is analytic at $\mathbf{0}$. Our result is largely an application of the classical one variable de Montessus theorem and the crucial projection property of the homogeneous Padé approximant: This property involves the "slice functions"

$$f_{\boldsymbol{\lambda}}(z) := f(\boldsymbol{\lambda} z), \quad z \in \mathbb{C}, \boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_k) \in \mathbb{C}^k \backslash \{\mathbf{0}\}. \tag{9}$$

It obviously suffices to consider $\boldsymbol{\lambda}$ with

$$\|\boldsymbol{\lambda}\| = 1, \tag{10}$$

as scaling factors can be absorbed into the variable z. If $[m/n](\mathbf{z})$ denotes the (m, n) homogeneous Padé approximant to f, and $[m/n]_{f_{\boldsymbol{\lambda}}}(z)$ denotes the ordinary one variable Padé approximant to $f_{\boldsymbol{\lambda}}(z)$, then the projection property is the following identity:

$$[m/n](\boldsymbol{\lambda} z) = [m/n]_{f_{\boldsymbol{\lambda}}}(z). \tag{11}$$

This is easily deduced from (5), (6) because z^{mn} factors out from both P and Q. The projection property was studied in detail in [4].

Since the univariate de Montessus theorem applies to balls centre 0, we need the domain B of our $f(\mathbf{z})$ to be such that for each $\boldsymbol{\lambda}$, $\{z : \boldsymbol{\lambda} z \in B\}$ is a ball centre 0. We shall use *positive homogeneous functions* (cf. [12]) to define such regions:

Definition 1
We say that a continuous function $\rho : \mathbb{C}^k \to [0, \infty)$ is a positive homogeneous function if

(i) $\rho(\mathbf{z}) > 0, \mathbf{z} \neq 0; \rho(\mathbf{0}) = \mathbf{0}$;
(ii) $\rho(u\mathbf{z}) = |u|\rho(\mathbf{z}), \forall u \in \mathbb{C}, \mathbf{z} \in \mathbb{C}^k$.

Given $r \in (0, \infty]$, we define the ρ-ball, radius r, to be

$$B(\rho; r) := \{\mathbf{z} \in \mathbb{C}^k : \rho(\mathbf{z}) < r\}. \tag{12}$$

As an example, any norm on $\mathbb{C}^k$ is a positive homogeneous function. Thus the usual Euclidean ball centre $\mathbf{0}$, radius r, is of the above form. It is only slightly less obvious that any polydisc centre $\mathbf{0}$ is also of the above form. If $r_j > 0$, $1 \leqslant j \leqslant k$, and we choose

$$\rho(\mathbf{z}) := \rho(z_1, z_2, \ldots z_k) := \max\{|z_j|/r_j : 1 \leqslant j \leqslant k\},$$

then

$$\{\mathbf{z} : |z_j| < r_j : 1 \leqslant j \leqslant k\} = B(\rho; 1).$$

Note that for fixed $\boldsymbol{\lambda}$, and any ρ as above,

$$\boldsymbol{\lambda} z \in B(\rho; r) \Longleftrightarrow |z| < r/\rho(\boldsymbol{\lambda}).$$

So the ρ-balls have the desired property of being projected onto balls when we take homogeneous coordinates. We shall use the notation

$$B_{\boldsymbol{\lambda}} := \{z : |z| < r/\rho(\boldsymbol{\lambda})\} \tag{13}$$

for this projected ball, once we have fixed ρ and r.

To avoid the complexities of meromorphic functions of several variables, we shall assume that our function has the form

$$f(z) = \frac{g(z)}{S(z)}, \quad z \in B(\rho; r), \tag{14}$$

where $g(z)$ is analytic in $B(\rho; r)$ and $S(z)$ is a polynomial of degree n (so has the form (8) with $\mu = n$) and

$$S(\mathbf{0}) \neq 0; \quad s_n(z) \text{ not identically zero.} \tag{15}$$

Note that then the set of singularities of f is contained in the zero set $\mathcal{S}$ of S. The ordinary one variable de Montessus theorem for $\{[m/n]\}$ is only applicable to $f_{\boldsymbol{\lambda}}$ if it has poles of total multiplicity n. Accordingly we have to distinguish

$$\Lambda := \{\boldsymbol{\lambda} \in \mathbb{C}^k : \|\boldsymbol{\lambda}\| = 1 \text{ and } f_{\boldsymbol{\lambda}} \text{ has less than } n \text{ poles in } B_{\boldsymbol{\lambda}}\}. \tag{16}$$

This is "thin" in $\mathbb{C}^k$, as it has at most $(k-2)$ degrees of (complex) freedom: If we write

$$S(\boldsymbol{\lambda} z) = \sum_{j=0}^{n} s_j^*(\boldsymbol{\lambda}) z^j, \tag{17}$$

then Λ is contained in the analytic set $\{\boldsymbol{\lambda} : \|\boldsymbol{\lambda}\| = 1 \text{ and } s_n^*(\boldsymbol{\lambda}) = 0\}$. It is hardly surprising that we cannot hope for convergence on the set $\mathcal{S}$ of singularities of f, but it is more surprising that we can neither in general hope for convergence on the analytic set

$$E_\Lambda := \{\boldsymbol{\lambda} z : z \in \mathbb{C}, \boldsymbol{\lambda} \in \Lambda\}. \tag{18}$$

It is unfortunate that Λ is never empty and so E_Λ always contains $\mathbf{0}$. Thus we cannot in general hope for uniform convergence of the approximants near $\mathbf{0}$, as we shall see in example 4 below.

Note that for unimodular $u \in \mathbb{C}$ and $\boldsymbol{\lambda} \in \mathbb{C}^k$ with $\|\boldsymbol{\lambda}\| = 1$, we have $f_{u\boldsymbol{\lambda}}(z) = f_{\boldsymbol{\lambda}}(uz)$, so $u\boldsymbol{\lambda}$ gives rise to the same complex line $\{\boldsymbol{\lambda} z : z \in \mathbb{C}\}$ as does $\boldsymbol{\lambda}$. However, this superfluity in the definition of Λ does not affect our results or proofs.

Following is our main result:

Theorem 2

Let ρ be a positive homogeneous function and let $B(\rho; r)$ $(0 < r \leqslant \infty)$ be the ρ-ball radius r. Let f be of the form (14), where S is a polynomial of degree n, satisfying (15) and with zero set $\mathcal{S}$.

(a) For $z \in B(\rho; r) \backslash (E_\Lambda \cup \mathcal{S})$, we have

$$\lim_{m \to \infty} [m/n](z) = f(z). \tag{19}$$

Moreover, if K is a compact subset of $B(\rho; r) \backslash (E_\Lambda \cup \mathcal{S})$, we have

$$\limsup_{m \to \infty} \| f - [m/n] \|_{L_\infty(K)}^{1/m} < 1. \tag{20}$$

Moreover, given $0 < s < r; \varepsilon > 0$ and a compact subset L of the unit ball of $\mathbb{C}^k$ that does not intersect Λ, we have

$$\limsup_{m\to\infty}\left(\max_{\boldsymbol{\lambda}\in L}\max_{\substack{|z|\leqslant s/\rho(\boldsymbol{\lambda})\\ |S(\boldsymbol{\lambda}z)|\geqslant\varepsilon}}\left|f-[m/n]\right|(\boldsymbol{\lambda}z)\right)^{1/m}<1. \tag{21}$$

(b) If L is as above, and for each $\boldsymbol{\lambda} \in L$, the denominator $Q_{m,\boldsymbol{\lambda}}(z)$ in $[m/n]_{f_{\boldsymbol{\lambda}}}(z)$ is normalized to be monic of degree n, we have for each compact set $K \subset \mathbb{C}$,

$$\limsup_{m\to\infty}\left(\max_{\boldsymbol{\lambda}\in L}\left\|Q_{m,\boldsymbol{\lambda}}(z)-S(\boldsymbol{\lambda}z)/s_n^*(\boldsymbol{\lambda})\right\|_{L_\infty(K)}\right)^{1/m}<1. \tag{22}$$

Moreover, we can order the zeros $z_{j,m}(\boldsymbol{\lambda})$ of $Q_{m,\boldsymbol{\lambda}}(z)$ and the zeros $z_j(\boldsymbol{\lambda})$ of $S(\boldsymbol{\lambda}z)$ so that

$$\limsup_{m\to\infty}\left(\max_{\boldsymbol{\lambda}\in L}\left|z_{j,m}(\boldsymbol{\lambda})-z_j(\boldsymbol{\lambda})\right|\right)^{1/m}<1 \tag{23}$$

and each zero of $S(\boldsymbol{\lambda}z)$ attracts zeros of $Q_{m,\boldsymbol{\lambda}}(z)$ according to its multiplicity.

In the case when $r = \infty$, so that f is defined in $\mathbb{C}^k$, we can improve the rate of convergence:

Theorem 3
Assume that $r = \infty$ in theorem 1. Then all the assertions in theorem 1 of the form

$$\limsup_{m\to\infty}[\quad]^{1/m}<1$$

can be replaced by

$$\lim_{m\to\infty}[\quad]^{1/m}=0.$$

As we mentioned earlier, we cannot hope for locally uniform convergence of $[m/n]$ on the set E_Λ:

Example 4
We need not have uniform convergence of $\{[m/n]\}_{m=1}^{\infty}$ in any neighbourhood of any point of E_Λ.

We show this for $k = 2$ and $n = 1$. Let h be an entire function, and

$$f(z_1,z_2):=h(z_1)+h(z_2)+\frac{z_2-z_1}{z_1-1}.$$

We note that if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\|\boldsymbol{\lambda}\| = 1$, then it is easy to see that $f_{\boldsymbol{\lambda}}$ has poles of total multiplicity 1 unless $\lambda_1 = \lambda_2$ or $\lambda_1 = 0$, for

$$f_{\lambda}(z) = h(\lambda_1 z) + h(\lambda_2 z) + \frac{z(\lambda_2 - \lambda_1)}{\lambda_1 z - 1}.$$

So

$$\Lambda = \left\{e^{i\theta}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \theta \in [0, 2\pi]\right\} \cup \left\{e^{i\theta}(0, 1), \theta \in [0, 2\pi]\right\}.$$

Then also

$$E_{\Lambda} = \{(z, z) : z \in \mathbb{C}\} \cup \{(0, z) : z \in \mathbb{C}\}.$$

If first $\lambda_1 = \lambda_2$, then

$$f_{\lambda}(z) = 2h(\lambda_1 z)$$

and if $\lambda_1 = 0$, then

$$f_{\lambda}(z) = h(\lambda_2 z) + h(0) - \lambda_2 z.$$

Thus the $\{[m/1]\}$ Padé sequence to f will not converge locally uniformly in any neighbourhood of any point of E_{Λ} provided the ordinary $\{[m/1]_h\}$ Padé approximants to h do not converge locally uniformly in any neighbourhood of any point of $\mathbb{C}$. (The first two coefficients in the second case do not affect this property.) There are many well known examples of such entire h, going back at least to Perron [15]. For those readers unfamiliar with these constructions, we sketch the details: If

$$h(z) = \sum_{j=0}^{\infty} h_j z^j$$

then for those m such that $h_m \neq 0$,

$$[m/1]_h(z) = \sum_{j=0}^{m-1} h_j z^j + \frac{h_m z^m}{1 - h_{m+1} z / h_m}.$$

Thus $[m/1]_h(z)$ has a pole at $z = h_m/h_{m+1}$. Now if $\{\xi_j\}_{j=1}^{\infty}$ is a sequence of non-zero complex numbers that are dense in $\mathbb{C}$ and are such that each point is repeated infinitely often in the sequence, then we can choose a rapidly increasing sequence of positive numbers $\{m_j\}_{j=1}^{\infty}$ such that for $m = m_j$, we have $h_m/h_{m+1} = \xi_j$ so that $[m/1]_h$ has a pole at ξ_j. We can even allow this while ensuring that the coefficients h_m of h approach 0 arbitrarily fast as $m \to \infty$. Then for $\xi \in \{\xi_j : j \geqslant 1\}$, we have that ξ is a pole of $[m/1]_h$ for infinitely many m, so

$$\limsup_{m \to \infty} \left|[m/1]_h(\xi)\right| = \infty.$$

As $\{\xi_j\}_{j=1}^{\infty}$ is dense, we have the desired result. (We note that it is possible to have divergence to ∞ of $\{[m/1]_h\}$ on a set of Hausdorff logarithmic dimension 1 and also positive logarithmic capacity – see [13].)

Remarks

(I) Note that on each complex line $\{\boldsymbol{\lambda} z : z \in \mathbb{C}\}$, the projected approximants $[m/n]_{f_\lambda}(z)$ to $f_\lambda(z)$ converge a.e. with respect to planar Lebesgue measure. (In fact one can say much more: they converge outside a set of Hausdorff dimension 0 and logarithmic dimension at most 1.) It follows that $[m/n](\boldsymbol{z})$ converges a.e.with respect to $2k$-dimensional Lebesgue measure on $B(\rho; r)$ as $m \to \infty$. More generally, this holds for the columns $\{[m/p]\}_{m=1}^{\infty}$ provided $p \geqslant n$.

(II) Buslaev et al. [3] showed that if h is meromorphic in $\mathbb{C}$ with poles of total multiplicity μ, then for $n \geqslant \mu$, a subsequence of $\{[m/n]_h\}_{m=1}^{\infty}$ converges locally uniformly to h away from the poles of h. It follows that under the hypotheses of our theorem 3, once we have fixed $\boldsymbol{\lambda}$, a subsequence of $\{[m/n]_{f_\lambda}(z)\}_{m=1}^{\infty}$ converges uniformly in compact subsets of $\mathbb{C}$ omitting poles of $f_\lambda(z)$. This raises the question of whether a subsequence of $\{[m/n](\boldsymbol{z})\}_{m=1}^{\infty}$exists that converges pointwise throughout E_Λ (and hence $\mathbb{C}^k$) away from singularities of f. That is, we want the subsequence in the Buslaev-Goncar-Suetin result to be independent of $\boldsymbol{\lambda}$. This seems unlikely, but is worth further investigation.

(III) We have remarked that E_Λ is never empty, and hence we cannot in general guarantee convergence of $\{[m/n]\}_{m=1}^{\infty}$ in a neighbourhood of $\mathbf{0}$, without referring to properties of the one variable Padé approximants for the slice functions f_λ. To see that E_Λ is never empty, note that $\boldsymbol{\lambda} \in \Lambda$ if $S(\boldsymbol{\lambda} z)$ has degree $< n$, that is, if $s_n^*(\boldsymbol{\lambda}) = 0$ (recall the representations (8) and (17)). Since

$$s_n(\boldsymbol{\lambda} z) = z^n s_n^*(\boldsymbol{\lambda}),$$

we see that $s_n^*(\boldsymbol{\lambda})$ is a homogeneous polynomial of degree n in $\boldsymbol{\lambda}$. Hence it has zeros $\boldsymbol{\lambda}$ on the unit ball and in fact its zero set on that unit ball is a non-empty analytic set (the reader to whom this is unfamiliar should examine the case $k = 2$).

We prove the theorems in section 2.

2. Proofs

Throughout we assume that f satisfies the hypotheses of theorem 1. We begin with:

Proof of (19)

Now if $\boldsymbol{z} \in B(\rho; r)\backslash(E_\Lambda \cup \mathcal{S})$, then we can write $\boldsymbol{z} = \boldsymbol{\lambda} z$ for some vector $\boldsymbol{\lambda}$ of unit norm, where f_λ has poles of total multiplicity n in B_λ. Then the projection property (11) and the classical univariate de Montessus theorem give

$$\lim_{m\to\infty}[m/n](\boldsymbol{z}) = \lim_{m\to\infty}[m/n]_{f_\lambda}(z) = f_\lambda(z) = f(\boldsymbol{\lambda} z) = f(\boldsymbol{z}). \qquad \square$$

The classical de Montessus theorem also gives uniform and geometric convergence in $B_{\boldsymbol{\lambda}}$ and this seems the natural starting point to prove the uniformity in theorem 2. We tried this approach, but could not get it to work: the standard tools such as Goncar's Lemma [10], which allow passage from convergence in one dimensional Hausdorff measure to uniform convergence, are inapplicable. So we follow the standard proofs of the de Montessus theorem and establish uniformity in $\boldsymbol{\lambda}$.

We break the proof into several steps.

Step 1: Continuity in $\boldsymbol{\lambda}$

Fix a unit vector $\boldsymbol{\lambda}_0 \notin \Lambda$. Then $f_{\boldsymbol{\lambda}_0}(z)$ has n poles in $B_{\boldsymbol{\lambda}_0} = \{z : |z| < r/\rho(\boldsymbol{\lambda}_0)\}$ and hence $S(\boldsymbol{\lambda}_0 z)$ has zeros of total multiplicity n there. Recall from (17) that we can write for $\boldsymbol{\lambda} \notin \Lambda$,

$$S(\boldsymbol{\lambda} z) = s_n^*(\boldsymbol{\lambda}) \prod_{j=1}^{n} (z - z_j(\boldsymbol{\lambda})).$$

Choose $\sigma < \tau < \omega < r$ such that

$$|z_j(\boldsymbol{\lambda}_0)| < \frac{\sigma}{\rho(\boldsymbol{\lambda}_0)}, \quad 1 \leqslant j \leqslant n.$$

Let $\delta > 0$. We choose $\varepsilon > 0$ so small that for $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\| < \varepsilon$, we have the following four properties:

(I) deg $(S(\boldsymbol{\lambda} z)) = n$ (as a polynomial in z).

(II)

$$|z_j(\boldsymbol{\lambda})| < \frac{\sigma}{\rho(\boldsymbol{\lambda}_0)} < \frac{\tau}{\rho(\boldsymbol{\lambda})}, \quad 1 \leqslant j \leqslant n; \tag{24}$$

$$\frac{\tau}{\rho(\boldsymbol{\lambda}_0)} < \frac{\omega}{\rho(\boldsymbol{\lambda})}. \tag{25}$$

(III) If $z_j(\boldsymbol{\lambda}_0)$ is a zero of $S(\boldsymbol{\lambda}_0 z)$ of multiplicity k, then $S(\boldsymbol{\lambda} z)$ has zeros of total multiplicity k inside the circle centre $z_j(\boldsymbol{\lambda}_0)$, radius δ.

(IV) For $z = z_j(\boldsymbol{\lambda})$, $1 \leqslant j \leqslant n$,

$$|g(\boldsymbol{\lambda} z)| \geqslant C, \tag{26}$$

where $C \neq C(\boldsymbol{\lambda})$.

We indicate briefly how to choose ε. It is easy to see that (I) and (II) follow from (III) and continuity of ρ (provided δ is small enough). Now the principle of the argument shows that if $\eta > 0$ is small enough, and $z_j(\boldsymbol{\lambda}_0)$ is a zero of $S(\boldsymbol{\lambda}_0 z)$ of multiplicity k,

$$\frac{1}{2\pi i} \int_{|t - z_j(\boldsymbol{\lambda}_0)| = \eta} \frac{\frac{d}{dt} S(\boldsymbol{\lambda}_0 t)}{S(\boldsymbol{\lambda}_0 t)} \, dt = k.$$

Continuity of $S(\boldsymbol{\lambda}_0 t)$ and its derivative allow us to preserve this relation for $\boldsymbol{\lambda}$ close to $\boldsymbol{\lambda}_0$. So we have (III). To see (IV), we note that our hypothesis that

$_0(z) = g(\boldsymbol{\lambda}_0 z)/S(\boldsymbol{\lambda}_0 z)$ has n poles forces $g(\boldsymbol{\lambda}_0 z) \neq 0, z = z_j(\boldsymbol{\lambda}_0), 1 \leqslant j \leqslant n$, so we ın choose $C > 0$ such that

$$|g(\boldsymbol{\lambda}_0 z)| \geqslant 2C, \quad z = z_j(\boldsymbol{\lambda}_0), \quad 1 \leqslant j \leqslant n.$$

δ in (III) is small enough, then this last inequality and the continuity of g allow us deduce (26).

ep 2: An estimate on $f - [m/n]$ for $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\| < \varepsilon$
'e set $R := \tau/\rho(\boldsymbol{\lambda}_0)$ (recall $\sigma < \tau < \omega < r$ and (24)). Let us set

$$h_{m,\boldsymbol{\lambda}}(z) := \left[S_{\boldsymbol{\lambda}}(f_{\boldsymbol{\lambda}} Q_{m,\boldsymbol{\lambda}} - P_{m,\boldsymbol{\lambda}})\right](z) = \left[g_{\boldsymbol{\lambda}} Q_{m,\boldsymbol{\lambda}} - S_{\boldsymbol{\lambda}} P_{m,\boldsymbol{\lambda}}\right](z),$$

here we write for the given $\boldsymbol{\lambda}$

$$[m/n](\boldsymbol{\lambda} z) = [m/n]_{f_{\boldsymbol{\lambda}}}(z) = (P_{m,\boldsymbol{\lambda}}/Q_{m,\boldsymbol{\lambda}})(z).$$

ere the numerator and denominator are normalized so that $Q_{m,\boldsymbol{\lambda}}(z)$ is a polyno- ial in z of degree at most n, with

$$Q_{m,\boldsymbol{\lambda}}(z) = \prod_{|z_j(\boldsymbol{\lambda})| \leqslant 2R} \left(z - z_j(\boldsymbol{\lambda})\right) \prod_{|z_j(\boldsymbol{\lambda})| > 2R} \left(1 - z/z_j(\boldsymbol{\lambda})\right). \tag{27}$$

oreover, $S_{\boldsymbol{\lambda}}(z) = S(\boldsymbol{\lambda} z); g_{\boldsymbol{\lambda}}(z) = g(\boldsymbol{\lambda} z)$. Note that then

$$\max_{|t|=R} |Q_{m,\boldsymbol{\lambda}}(t)| \leqslant \left(3 \max\{1, R\}\right)^n.$$

nce $h_{m,\boldsymbol{\lambda}}(z)/z^{m+n+1}$ is analytic in $|z| \leqslant R$, we have for fixed $l \geqslant 0$,

$$h_{m,\boldsymbol{\lambda}}^{(l)}(z) = \frac{1}{2\pi i} \int_{|t|=R} \frac{h_{m,\boldsymbol{\lambda}}(t)}{t^{m+n+1}} \left(\frac{d}{dz}\right)^l \left\{\frac{z^{m+n+1}}{t - z}\right\} dt$$

$$= \frac{1}{2\pi i} \int_{|t|=R} \frac{(g_{\boldsymbol{\lambda}} Q_{m,\boldsymbol{\lambda}})(t)}{t^{m+n+1}} \left(\frac{d}{dz}\right)^l \left\{\frac{z^{m+n+1}}{t - z}\right\} dt.$$

ere using (25), we see that

$$\max_{|t|=R} |g_{\boldsymbol{\lambda}}(t)| = \max_{|t|=R} |g(\boldsymbol{\lambda} t)| \leqslant \max_{z \in B(\rho;\omega)} |g(z)| =: C_1,$$

ıere $C_1 \neq C_1(\boldsymbol{\lambda})$. We deduce that for $|z| \leqslant \sigma/\rho(\boldsymbol{\lambda}_0) = (\sigma/\tau)R$,

$$\left|h_{m,\boldsymbol{\lambda}}^{(l)}(z)\right| \leqslant C_2(\sigma/\tau)^m, \tag{28}$$

ıere $C_2 \neq C_2(\boldsymbol{\lambda}, m, z)$. (Recall that R is independent of $\boldsymbol{\lambda}$.)

ep 3: The estimate (28) simplified at zeros of S
ıroughout this step, we fix a zero $z_j(\boldsymbol{\lambda})$ of $S_{\boldsymbol{\lambda}}(z)$ of order k say. Since $(S_{\boldsymbol{\lambda}} P_{m,\boldsymbol{\lambda}})(z)$ s a zero of order k at this point, (28) gives for $l = 0, 1, 2..., k - 1$,

$$\left|(g_{\boldsymbol{\lambda}} Q_{m,\boldsymbol{\lambda}})^{(l)}(z_j(\boldsymbol{\lambda}))\right| \leqslant C_2(\sigma/\tau)^m. \tag{29}$$

(Recall that by choice $|z_j(\boldsymbol{\lambda})| \leqslant \sigma/\rho(\boldsymbol{\lambda}_0)$.) For $l = 0$, this and (26) give

$$\left|Q_{m,\boldsymbol{\lambda}}(z_j(\boldsymbol{\lambda}))\right| \leqslant [C_2/C](\sigma/\tau)^m.$$

Leibniz's formula gives

$$Q_{m,\boldsymbol{\lambda}}^{(l)}(z_j(\boldsymbol{\lambda}))g_{\boldsymbol{\lambda}}(z_j(\boldsymbol{\lambda})) = (g_{\boldsymbol{\lambda}}Q_{m,\boldsymbol{\lambda}})^{(l)}(z_j(\boldsymbol{\lambda})) - \sum_{p=0}^{l-1}\binom{l}{p}Q_{m,\boldsymbol{\lambda}}^{(p)}(z_j(\boldsymbol{\lambda}))g_{\boldsymbol{\lambda}}^{(l-p)}(z_j(\boldsymbol{\lambda})).$$

Applying (29), (26) to this and induction on l, we obtain for $l = 0, 1, 2, \dots, k-1$,

$$\left|Q_{m,\boldsymbol{\lambda}}^{(l)}(z_j(\boldsymbol{\lambda}))\right| \leqslant C_3(\sigma/\tau)^m \tag{30}$$

where $C_3 \neq C_3(\boldsymbol{\lambda}, m, j)$ (recall that k is at most n). We also need to use that $g_{\boldsymbol{\lambda}}$ and its derivatives up to a fixed order are uniformly bounded in $\boldsymbol{\lambda}$. We distinguished above between zeros of $Q_{m,\boldsymbol{\lambda}}(z)$ inside and outside $|z| < 2R$. We now fix a small $\delta > 0$ and distinguish three types of zeros. Write

$$Q_{m,\boldsymbol{\lambda}}(z) = \prod_{\substack{|z_l(\boldsymbol{\lambda})| \leq 2R \\ |z_l(\boldsymbol{\lambda}) - z_j(\boldsymbol{\lambda})| \leqslant \delta}} (z - z_l(\boldsymbol{\lambda})) \prod_{\substack{|z_l(\boldsymbol{\lambda})| \leq 2R \\ |z_l(\boldsymbol{\lambda}) - z_j(\boldsymbol{\lambda})| > \delta}} (z - z_l(\boldsymbol{\lambda})) \prod_{|z_l(\boldsymbol{\lambda})| > 2R} (1 - z/z_l(\boldsymbol{\lambda}))$$
$$=: (U_{m,\boldsymbol{\lambda}}VW)(z).$$

We omit the dependence on $m, \boldsymbol{\lambda}$ in V, W. The crucial thing is that for $|z - z_j(\boldsymbol{\lambda})| \leqslant \delta/2$, we have

$$|VW|(z) \geqslant C_4,$$

where $C_4 \neq C_4(\boldsymbol{\lambda}, m, j)$ (but depends on δ). We also have obvious upper bounds on VW and its derivatives. We can strip off the factors VW from the estimate (30) using Leibniz's formula and induction on l, exactly as we stripped off $g_{\boldsymbol{\lambda}}$ from (29). We then obtain for $l = 0, 1, 2, .., k-1$,

$$\left|U_{m,\boldsymbol{\lambda}}^{(l)}(z_j(\boldsymbol{\lambda}))\right| \leqslant C_5(\sigma/\tau)^m. \tag{31}$$

Here $C_5 \neq C_5(\boldsymbol{\lambda}, m, j)$. Since $U_{m,\boldsymbol{\lambda}}(z)$ is monic, we see that for $m \geqslant m_0$, it has degree at least k, where m_0 is independent of $\boldsymbol{\lambda}, m, j$. (For if it has degree l, its lth derivative is identically $l!$, which does not decay to 0. We obtain a contradiction if $l \leqslant k-1$.) As such an estimate holds for each zero of $S(\boldsymbol{\lambda}z)$ we deduce that $U_{m,\boldsymbol{\lambda}}(z)$ has degree exactly k. Taylor series expansion at $z_j(\boldsymbol{\lambda})$ gives

$$U_{m,\boldsymbol{\lambda}}(z) - (z - z_j(\boldsymbol{\lambda}))^k = \sum_{l=0}^{k-1} \frac{U_{m,\boldsymbol{\lambda}}^{(l)}(z_j(\boldsymbol{\lambda}))}{l!}(z - z_j(\boldsymbol{\lambda}))^l.$$

(Note that the left-hand side has degree at most $k-1$.) Applying (31) in this last estimate gives for each $s > 0$,

$$\max_{|z| \leqslant s}\left|U_{m,\boldsymbol{\lambda}}(z) - (z - z_j(\boldsymbol{\lambda}))^k\right| \leqslant C_6(\sigma/\tau)^m. \tag{32}$$

Here $C_6 \neq C_6(\boldsymbol{\lambda}, m, j)$ (but depends on s).

Step 4: Proof of theorem 2(b)
Note first that any compact subset L of the unit ball of $\mathbb{C}^k$ that does not intersect Λ can be covered by finitely many neighbourhoods of the form $\{\boldsymbol{\lambda} : \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\| < \varepsilon\}$. From (32) we deduce (23) and also our proof above showed that any δ neighbourhood of any zero of $S(\boldsymbol{\lambda}z)$ attracts zeros of $Q_{m,\boldsymbol{\lambda}}(z)$ according to its multiplicity. Finally, multiplying the estimate (32) over each zero of $S(\boldsymbol{\lambda}z)$ easily yields (22): recall that $s_n^*(\boldsymbol{\lambda})$ is the leading coefficient of $S(\boldsymbol{\lambda}z)$ expressed as a polynomial in z.

Step 5: Proof of (21) of theorem 2(a)
Note that for large m, the normalization (27) adopted in step 2 above and theorem 2(b) actually ensure that $Q_{m,\boldsymbol{\lambda}}(z)$ will be monic. We apply (28) with $l = 0$ to deduce that for $|z| \leqslant \sigma/\rho(\boldsymbol{\lambda}_0)$ (and hence for $|z| \leqslant \sigma_1/\rho(\boldsymbol{\lambda})$ if $\sigma_1 < \sigma$)

$$|f - [m/n]|(\boldsymbol{\lambda}z) \leqslant C_2(\sigma/\tau)^m / |S_{\boldsymbol{\lambda}}(z)Q_{m,\boldsymbol{\lambda}}(z)| \leqslant C_3(\sigma/\tau)^m$$

provided $m \geqslant m_0$ and for a fixed ε, $|S(\boldsymbol{\lambda}z)/s_n^*(\boldsymbol{\lambda})| \geqslant \varepsilon$. Here we are applying (22). Of course, $C_3 \neq C_3(\boldsymbol{\lambda}, m, z)$.

Step 6: Proof of (20) of theorem 2(a)
Suppose that K is a compact subset of $B(\rho; r)$ not intersecting $\mathcal{S}$ or E_Λ. Recall that $E_\Lambda$is non-empty (see remark (III) in section I), so we cannot have $\mathbf{0} \in K$. Then for each $z_0 \in K$, we can write $z_0 = \boldsymbol{\lambda}_0 z_0$ where $z_0 \neq 0, S(\boldsymbol{\lambda}_0 z_0) \neq 0$ and $\deg(S(\boldsymbol{\lambda}_0 z))$ $= n$. Our proof above shows that we have uniform and geometric convergence for $|z| \leqslant \sigma/\rho(\boldsymbol{\lambda})$ and $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\| < \varepsilon$. (Formally, we showed this for $|z| \leqslant \sigma/\rho(\boldsymbol{\lambda}_0)$ but this does not make a difference as we can make σ slightly larger.) It is not difficult to see that if η is small enough and as $z_0 \neq 0$, then for $\|z - z_0\| < \eta$, we have $z = \boldsymbol{\lambda}z$, where $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\| < \varepsilon$. So we have uniform and geometric convergence in a neighbourhood of z_0. As K can be covered by finitely many such neighbourhoods, we have (20). □

Proof of theorem 3
It is clear that if $r = \infty$, we can choose σ/τ arbitrarily small in the above arguments. □

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