Symbolic-Numeric Sparse Interpolation of Multivariate Rational Functions

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Consider the problem of sparse interpolation for a black-box multivariate rational function

$$f(x_1, \ldots, x_n) = \frac{u(x_1, \ldots, x_n)}{v(x_1, \ldots, x_n)}, \tag{1}$$

in floating point arithmetic, in which both the numerator and denominator

$$\begin{aligned} u(x_1, \ldots, x_n) &= \sum_{j=1}^{\ell} a_j x_1^{d_{j_1}} \cdots x_n^{d_{j_n}}, \quad a_j \neq 0, \\ v(x_1, \ldots, x_n) &= \sum_{k=1}^{m} b_k x_1^{e_{k_1}} \cdots x_n^{e_{k_n}}, \quad b_k \neq 0 \end{aligned} \tag{2}$$

are polynomials with complex coefficients, $u(x_1, \ldots, x_n), v(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$.

That is, to recover coefficients a_j, b_k and multivariate exponents $(d_{j_1}, \ldots, d_{j_n})$, $(e_{k_1}, \ldots, e_{k_n})$ for $1 \le j \le \ell$ and $1 \le j \le m$ in (2) from black-box evaluations of (1) in a finite precision environment.

We present a symbolic-numeric interpolation method that is sensitive to the sparsity of the black-box multivariate rational function. Our method implements the homogenization from [1, 3] and numerically interpolates the modified rational function with respect to the newly introduced homogenizing variable. Then by combining with the numerical sparse polynomial interpolation from [2], we simultaneously recover the multivariate exponents of non-zero terms in both $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$. Once all such non-zero terms are recovered, various techniques can be utilized to determine the corresponding coefficients a_j and b_k, and the given multivariate rational function can be interpolated.

For example, let

$$f(x_1, \ldots, x_n) = \frac{a_1 x_1^{d_{11}} \cdots x_n^{d_{1n}} + \cdots + a_\ell x_1^{d_{\ell 1}} \cdots x_n^{d_{\ell n}}}{1 + b_2 x_1^{e_{21}} \cdots x_n^{e_{2n}} + \cdots + b_m x_1^{e_{m1}} \cdots x_n^{e_{mn}}}$$

be defined at $(0, \ldots, 0)$. By introducing the homogenizing variable z, we obtain a modified rational function $F(z, x_1, \ldots, x_n) = f(x_1 z, \ldots, x_n z)$.

Suppose $p_1, \ldots, p_n \in \mathbb{Z}_{\ge 0}$ are pairwise relatively prime. We fix $(x_1, \ldots, x_n)$ at $(p_1, \ldots, p_n)$ and consider the univariate rational interpolation of $F(z, p_1, \ldots, p_n)$ with respect to z,

$$F(z, p_1, \ldots, p_n) = \frac{A_1(p_1, \ldots, p_n) z^{\delta_1} + \cdots + A_\lambda(p_1, \ldots, p_n) z^{\delta_\lambda}}{1 + B_2(p_1, \ldots, p_n) z^{\epsilon_2} + \cdots + B_\mu(p_1, \ldots, p_n) z^{\epsilon_\mu}}. \tag{3}$$

In (3), the coefficients $A_1(p_1, \ldots, p_n)$, $\ldots$, $A_\lambda(p_1, \ldots, p_n)$, $B_2(p_1, \ldots, p_n)$, $\ldots$, $B_\mu(p_1, \ldots, p_n)$ are multivariate polynomials $A_1(x_1, \ldots, x_n)$, $\ldots$, $A_\lambda(x_1, \ldots, x_n)$, $B_2(x_1, \ldots, x_n)$, $\ldots$, $B_\mu(x_1, \ldots, x_n)$ evaluated at $(p_1, \ldots, p_n)$.

We continue to interpolate the rational functions $F(z, p_1^2, \ldots, p_n^2)$, $F(z, p_1^3, \ldots, p_n^3)$, $\ldots$. From each interpolation, we obtain a set of coefficients that are the corresponding coefficient polynomials evaluated at powers: $A_1(p_1^2, \ldots, p_n^2)$, $\ldots$, $B_\mu(p_1^2, \ldots, p_n^2)$, $A_1(p_1^3, \ldots, p_n^3)$, $\ldots$, $B_\mu(p_1^3, \ldots, p_n^3)$, $\ldots$.

We take a look at the interpolation of polynomial A_1. Using the numerical sparse polynomial interpolation from [2], we can interpolate $A_1(x_1, \ldots, x_n)$ from $A_1(p_1, \ldots, p_n)$, $A_1(p_1^2, \ldots, p_n^2)$, $A_1(p_1^3, \ldots, p_n^3)$, $\ldots$.

For $i = 1, 2, \ldots$, from the coefficients of each interpolated $F(z, p_1^i, \ldots, p_n^i)$ we can simultaneously interpolate coefficient polynomials $A_1, \ldots, A_\lambda, B_2, \ldots, B_\mu$ by the numerical sparse polynomial interpolation [2]. When all the coefficient polynomials $A_1, \ldots, A_\lambda, B_2, \ldots, B_\mu$ are recovered, the original rational function $f(x_1, \ldots, x_n)$ can be reconstructed.

Our rational interpolation is an iterative method and does not require the knowledge on either the degrees or number of terms in the black-box rational function. The number of evaluations and interpolation steps depend on the number of non-zero terms in the given rational function and the accuracy required.

We also investigate other sparse rational interpolation approaches and relevant issues, including the situation when the given function $f(x_1, \ldots, x_n)$ is not defined at the origin $(0, \ldots, 0)$. Some initial tests are demonstrated in Maple.

References

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