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A scale and shift paradigm for sparse interpolation in one and more dimensions

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Sparse interpolation from at least $2n$ uniformly spaced interpolation points t_j can be traced back to the exponential fitting method

$$f(t_j) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i t_j), \qquad \alpha_i, \phi_i \in \mathbb{R}, \qquad t_j \in \mathbb{R} \tag{1}$$

of de Prony from the 18-th century [5]. Almost 200 years later this basic problem is also reformulated as a generalized eigenvalue problem [8]. We generalize (1) to sparse interpolation problems of the form

$$f(t_j) = \sum_{i=1}^{n} \alpha_i g(\phi_i; t_j), \qquad \alpha_i, \phi_i \in \mathbb{C}, \qquad t_j \in \mathbb{R} \tag{2}$$

and some multivariate formulations thereof, from corresponding regular interpolation point patterns. Concurrently we introduce the wavelet inspired paradigm of dilation and translation for the analysis (2) of these complex-valued structured univariate or multivariate samples. The new method is the result of a search on how to solve ambiguity problems in exponential analysis, such as aliasing which arises from too coarsely sampled data, or collisions which may occur when handling projected data.

Fine or coarse sampling is controlled by the choice of a scale or dilation which allows to stretch and shrink the uniform sampling scheme required for exponential analysis. Ambiguity problems can be solved by means of a one- or multidimensional translation of the sampling locations, also called an identification shift [2, 3].

We require the functions $g(\phi_i; t)$ and the set of points t_j to satisfy a discrete generalized eigenfunction relation of the form

$$\sum_{k=-L}^{L} a_k g(\phi_i; t_{j+k}) = \lambda_{ij} \sum_{k=-R}^{R} b_k g(\phi_i; t_{j+k}), \qquad \lambda_{ij} \in \mathbb{C}. \tag{3}$$

This property allows us to split the nonlinear interpolation problem (2) into the separate computation of the nonlinear parameters ϕ_i on the one hand and the linear α_i on the other, as in de Prony's method. As mentioned, the ensemble of sampling points t_j is going to be dictated by scaling factors and shifts.

In the past some attempts have already been made at the identification of functions $g(\phi_i; t)$ and sampling point patterns t_j satisfying (3) with the aim to solve the interpolation problem (2). In computer algebra sparse interpolation is generalized to non-standard bases [9, 6, 1, 7]. In signal processing exponential analysis is generalized to include some additional functions [11, 12, 14, 13]. Here we present a more coherent study undertaken at the occasion of [2, 3]: we solve the nonlinear interpolation problem (2) by reformulating it as a structured generalized eigenvalue problem derived from (3) and one or more structured linear systems of equations.

While by the introduction of a scale factor, we loose the uniqueness of the solution in the nonlinear step of the algorithm, we gain the option to stretch, shrink and eventually translate an otherwise uniform scheme of sample points. The translation of the sample points in the subsequent linear step of the algorithm allows to restore the lost uniqueness.

The list of functions $g(\phi_i; t)$ that the theory covers, includes the exponential function, the trigonometric functions cosine, sine, the hyperbolic cosine and sine functions, the Tchebyshev (1-st, 2-nd, 3-rd, 4-th kind) and spread polynomials, the sinc, gamma and Gaussian function, and several multivariate versions of all of the above.

The new paradigm generalizes the theory of sparse interpolation by extending it to sub-Nyquist sampling [10], by generalizing it to the d-dimensional case with the use of only $(d+1)n$ samples [4], and by offering some new choices for the $g(\phi_i; t)$.

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