Reconstructing Sparse Trigonometric Functions

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Recently, the growing field of compressed sensing has attracted a great deal of attention to sparsity. In many signal processing applications, a compressible signal can be represented as a sparse trigonometric polynomial

$$p(x_1,\ldots,x_n) = \sum_{(\theta_{j_1},\ldots,\theta_{j_n})\in J_1} a_{j_1\cdots j_n}\cos(\theta_{j_1}x_1+\ldots+\theta_{j_n}x_n) + \sum_{(\phi_{j_1},\ldots,\phi_{j_n})\in J_2} b_{j_1\cdots j_n}\sin(\phi_{j_1}x_1+\ldots+\phi_{j_n}x_n), \quad J_1, J_2 \subset \mathbb{Z}^n_{\geq 0},$$

and one seeks to reconstruct a sparse signal from a small number of samples.

The Chebyshev polynomial basis gives a recursive relation among trigonometric functions. In exact arithmetic, Lakshman and Saunders gave a sparse Chebyshev interpolation algorithm [7] that can be regarded as a generalization of the Ben-Or/Tiwari algorithm [1]. In a finite precision environment, a connection between the Ben-Or/Tiwari sparse interpolation and Prony's classical method [9] was observed by Giesbrecht, Labahn, and Lee [4]. A reformulation of Prony's method as a generalized eigenvalue problem [5] leads to a robust numerical sparse interpolation algorithm [4]. A generalized eigenvalue reformulation of sparse Chebyshev interpolation is presented in [3].

Based on Rutishauser's qd-algorithm [10], an iterative algorithm for multivariate sparse polynomial interpolation, in the standard power basis, is developed [2]. This is due to a connection [8] between the generating polynomial computed via the Berlekamp/Massey algorithm in the Ben-Or/Tiwari sparse interpolation and the convergence of Hadamard polynomials, defined in terms of the determinants of the associated Hankel matrices and polynomials [6, p. 625], in the qd-algorithm.

We extend such connection to trigonometric polynomials and the corresponding structured matrices and show how it is used for the reconstruction of a sparse trigonometric polynomial in the form $p(x_1,\ldots,x_n)$. Our trigonometric reconstruction leads to an iterative algorithm for sparse trigonometric interpolation. The Hankel-plus-Toeplitz matrix appearing in the sparse Chebyshev interpolation problem [7] and its generalized eigenvalue reformulation [3], can be factored as a structured matrix product. From the solution of the generalized eigenvalue problem the frequencies $(\theta_{j_1},\ldots,\theta_{j_n}) \in J_1$ and $(\phi_{j_1},\ldots,\phi_{j_n}) \in J_2$ are obtained. The numerical rank of the structured matrices is connected to the number of terms in $p(x_1,\ldots,x_n)$, in other words to the sparsity.

Example Consider the interpolation of

$$p(x) = a_1\cos\theta_1 x + a_2\cos\theta_2 x + b_1\sin\phi_1 x + b_2\sin\phi_2 x$$

from the evaluation of $p(x)$. The values of a_1, a_2, θ_1, θ_2 can be obtained from

$$p(0) = a_1 + a_2, \quad p(1) + p(-1) = a_1(2\cos\theta_1) + a_2(2\cos\theta_2),$$
$$p(2) + p(-2) + 2p(0) = a_1(2\cos\theta_1)^2 + a_2(2\cos\theta_2)^2,$$
$$p(3) + p(-3) + 3\,(p(1) + p(-1)) = a_1(2\cos\theta_1)^3 + a_2(2\cos\theta_2)^3;$$

and b_1, b_2 ϕ_1, ϕ_2 from

$$p(1) - p(-1) = 2b_1 \sin\phi_1 + 2b_2 \sin\phi_2,$$
$$p(2) - p(-2) = 2b_1 \sin\phi_1(2\cos\phi_1) + 2b_2 \sin\phi_2(2\cos\phi_2),$$
$$p(3) + p(-3) + f(1) - f(-1) = 2b_1 \sin\phi_1(2\cos\phi_1)^2 + b_2 \sin\phi_2(2\cos\phi_2)^2,$$
$$p(4) + p(-4) + 2\,(f(2) - f(-2)) = 2b_1 \sin\phi_1(2\cos\phi_1)^3 + 2b_2 \sin\phi_2(2\cos\phi_2)^3.$$

We interpolate $p(x)$ from 9 evaluations: $p(-4), \ldots, p(0), p(1), \ldots, p(4)$. ⊠

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