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INSTABILITY AND MODIFICATION OF THIELE INTERPOLATING CONTINUED FRACTIONS

Annie A.M. CUYT *
Department of Mathematics and Computer Science, University of Antwerp (UIA), B-2610 Wilrijk, Belgium

Lisa JACOBSEN
Department of Mathematics and Statistics, University of Trondheim, N-7055 Dragvoll, Norway

Brigitte M. VERDONK
Department of Mathematics and Computer Science, University of Antwerp (UIA), B-2610 Wilrijk, Belgium

1. Stability questions for Thiele interpolation

For the sake of completeness we shall first repeat the construction of rational interpolants using Thiele-type continued fractions. Let a sequence of distinct complex points $(x_i)_{i\in\mathbb{N}}$ be given and let a complex-valued function $f(x)$ be known by its function values $f(x_i)$ which we shall denote by f_i. A continued fraction of the form

$$d_0+\sum_{i=1}^{\infty}\frac{x-x_{i-1}|}{|\ d_i},$$

with $d_i=\varphi_i[x_0,\dots,x_i]$, where the inverse differences $\varphi_i[x_0,\dots,x_i]$ are computed as

$$\begin{aligned}\varphi_0[x_i]&=f_i, & i\geqslant 0,\\ \varphi_i[x_0,\dots,x_i]&=\frac{x_i-x_{i-1}}{\varphi_{i-1}[x_0,\dots,x_{i-2},x_i]-\varphi_{i-1}[x_0,\dots,x_{i-1}]}, & i\geqslant 1,\end{aligned}\tag{1.1}$$

generates rational interpolants if you consider its successive convergents. The nth convergent

$$C_n(x)=\varphi_0[x_0]+\sum_{i=1}^{n}\frac{x-x_{i-1}|}{|\varphi_i[x_0,\dots,x_i]}$$

satisfies

$$C_n(x_i)=f_i,\quad i=0,\dots,n,$$

if $C_n(x_i)$ is defined.

Numerical experiments have shown that in some cases we have little or no control over the inverse differences $\varphi_i[x_0,\dots,x_i]$. The values $\varphi_i[x_0,\dots,x_i]$ do not only depend upon the function values f_i but also highly on the interpolation points x_i. For some sequences $(x_i)_{i\in\mathbb{N}}$ the computation of the inverse differences $\varphi_i[x_0,\dots,x_i]$ can be highly unstable. In Table 1 we computed the inverse differences $\varphi_i[x_0,\dots,x_i]$ both in single and double precision for $f_i=1/\sqrt{i+1}$, once using $x_i=1/(i+1)$ and once using $x_i=3^{-i}$. It is easy to see that the process is much more stable in the second case than it is in the first case.

* Research Assistant NFWO.

Table 1
Inverse differences $\varphi_i[x_0,\dots,x_i]$ for $f_i = 1/\sqrt{i+1}$, $\lim_{i\to\infty} f_i = 0$

i	$x_i = 1/(i+1)$		$x_i = 3^{-i}$	
	56 bit	24 bit	56 bit	24 bit
0	1.00000000	1.00000000	1.00000000	1.00000000
1	1.70710678	1.70710659	2.27614237	2.27614212
2	1.28445705	1.28446293	1.28445705	1.28445530
3	1.07735027	1.07727909	0.16895710	0.16895831
4	0.94721360	0.94756573	2.08340788	2.08326054
5	0.85546189	0.85502332	0.01358570	0.01358765
6	0.78621276	0.80851626	2.11327925	2.11257172
7	0.73151786	0.40478480	0.00184995	0.00185089
8	0.68688672	−0.30210978	1.28884001	1.28764153
9	0.64956112	−0.28439111	0.00040401	0.00040466
10	0.61773931	0.00164387	0.55758997	0.55608720
11	0.59018166	0.41945827	0.00011427	0.00011470
12	0.56608079	−3.41882515	0.20481832	0.20391613
13	0.54418441	−0.00103522	0.00003595	0.00003613
14	0.52725694	3.93979073	0.07061846	0.07017231
15	0.51709121	−0.02712743	0.00001175	0.00001184

That we still have stability in the computation of $C_n(x)$ in most cases is due to the fact that to compute $\varphi_n[x_0,\dots,x_n]$ we in fact solve the equation $f_n = C_n(x_n)$, in other words

$$f_n = \varphi_0[x_0] + \sum_{i=1}^{n} \frac{x_n - x_{i-1} \mid}{\mid \varphi_i[x_0,\dots,x_i]}, \tag{1.2}$$

where $\varphi_n[x_0,\dots,x_n]$ is the only unknown. Any perturbations in the previous $\varphi_i[x_0,\dots,x_i]$ may increase the inaccuracy of $\varphi_n[x_0,\dots,x_n]$ in order to have the equation satisfied, in other words, in order to increase the accuracy of $C_n(x_n)$.

Table 2 shows results of this kind for $x_i = 3^{-i}$ and $f_i = \Sigma_{j=0}^{i}\ 2^j/j!$. Both the inverse differences $\varphi_i[x_0,\dots,x_i]$ and the convergents $C_i(0)$ evaluated at the origin are displayed in single as well as in double precision.

It is not a general rule that the computation of the convergent $C_n(x)$ is more stable for all values of x, but we experienced that in all cases (stable and unstable behaviour), the value $|f_n - C_n(x_n)|$ is of the magnitude of machine precision.

An explicit formula for the solution of (1.2) is

$$\varphi_n[x_0,\dots,x_n] = \frac{x_n - x_{n-1} \mid}{\mid -\varphi_{n-1}[x_0,\dots,x_{n-1}]} + \frac{x_n - x_{n-2} \mid}{\mid -\varphi_{n-2}[x_0,\dots,x_{n-2}]} + \cdots + \frac{x_n - x_0 \mid}{\mid -\varphi_0[x_0] + \varphi_0[x_n]}, \tag{1.3}$$

which makes it clear why the recursive computation scheme for the inverse differences works since

$$\frac{x_n - x_{n-2} \mid}{\mid -\varphi_{n-2}[x_0,\dots,x_{n-2}]} + \cdots + \frac{x_n - x_0 \mid}{\mid -\varphi_0[x_0] + \varphi_0[x_n]} = \varphi_{n-1}[x_0,\dots,x_{n-2},x_n].$$

Table 2
$f_i = \Sigma_{j=0}^{i} 2^j/j!$, $x_i = 3^{-i}$, $\lim_{i\to\infty} f_i = 7.3890561\ldots$

i	$\varphi_i[x_0,\ldots,x_i]$ 56 bit	$\varphi_i[x_0,\ldots,x_i]$ 24 bit	$C_i(0)$ 56 bit	$C_i(0)$ 24 bit
0	1.00000000	1.00000000	1.00000000	1.00000000
1	−0.33333333	−0.33333331	4.00000000	4.00000000
2	−2.00000000	−2.00000000	7.00000000	7.00000095
3	−1.22222222	−1.22224236	7.30000000	7.29999542
4	−0.11764706	−0.11763608	7.41860465	7.41860771
5	2.07845480	2.07713890	7.41079895	7.41079903
6	0.00248986	0.00249246	7.38824763	7.38827038
7	−16.99596714	−17.14669800	7.39077232	7.39076519
8	0.00003679	0.00003627	7.38977832	7.38977909
9	6.46101997	6.48843670	7.38914616	7.38914394
10	0.00012860	0.00013029	7.38907589	7.38907337
11	0.75097220	1.06083012	7.38905889	7.38906384
12	0.00007661	0.00001180	7.38905653	7.38905144
13	0.20302655	−0.30383861	7.38905615	7.38905621
14	0.00003311	−0.00002824	7.38905610	7.38905907
15	0.06793047	−0.00012451	7.38905610	7.38905621

So the right-hand side of (1.3) could also be computed using continued fraction algorithms. Of course using the backward algorithm for formula (1.3) is equivalent to computing $\varphi_{i+1}[x_0,\ldots,x_i, x_n]$ by means of (1.1).

To summarize, since the inverse differences and successive convergents in a Thiele interpolating continued fraction can suffer from instabilities, it is important to obtain the value of that Thiele interpolating continued fraction in as few steps as possible. The next section will indicate several ways to achieve this.

2. Modified Thiele interpolation

Let us now consider a sequence of distinct points $(x_i)_{i\in\mathbb{N}}$ converging to a finite value $z = \lim_{i\to\infty} x_i$. As in the previous section the function $f(x)$ is only known by its funtion values $f(x_i)$ denoted by f_i. An approximation for the value $f(z)$ can be computed by considering consecutive convergents of the interpolating continued fraction

$$\varphi_0[x_0] + \sum_{i=1}^{\infty} \frac{z - x_{i-1} \rfloor}{\lfloor \varphi_i[x_0,\ldots,x_i]}. \tag{2.1}$$

Since a convergent C_n is a rational expression we can write it as $C_n = A_n/B_n$, where

$$\left.\begin{aligned} A_i &= \varphi_i[x_0,\ldots,x_i]A_{i-1} + (z - x_{i-1})A_{i-2}, & A_0 &= \varphi_0[x_0], & A_{-1} &= 1 \\ B_i &= \varphi_i[x_0,\ldots,x_i]B_{i-1} + (z - x_{i-1})B_{i-2}, & B_0 &= 1, & B_{-1} &= 0 \end{aligned}\right\} \quad i = 1,\ldots,n.$$

We know that the limiting value of (2.1), if it exists, is given by

$$\frac{A_n + f^{(n)}A_{n-1}}{B_n + f^{(n)}B_{n-1}},$$

where $f^{(n)}$ is the nth tail of (2.1), given by

$$f^{(n)} = \sum_{i=n+1}^{\infty} \frac{z - x_{i-1} |}{|\varphi_i[x_0, \ldots, x_i]}.$$

Since these tails are not known exactly, the limiting value of (2.1) can only be obtained approximately by computing modified convergents

$$\tilde{C}_n = \frac{A_n + \tilde{f}^{(n)} A_{n-1}}{B_n + \tilde{f}^{(n)} B_{n-1}},$$

where the modifying factor $\tilde{f}^{(n)}$ approximates the tail $f^{(n)}$. The ordinary convergents C_n result by taking $\tilde{f}^{(n)} = 0$. With a good choice for $\tilde{f}^{(n)}$, modified convergents can be more accurate than ordinary convergents [3]. If the asymptotic behaviour of

$$\frac{z - x_{i-1}}{\varphi_i[x_0, \ldots, x_i] \varphi_{i-1}[x_0, \ldots, x_{i-1}]}$$

is known, procedures for choosing $\tilde{f}^{(n)}$ such that

$$\left| \frac{f - \tilde{C}_n}{f - C_n} \right| \to 0$$

exist. For the Thiele interpolating fraction we have $z - x_{i-1} \to 0$, but in general we do not know much about the asymptotic behaviour of $\varphi_i[x_0, \ldots, x_i]$. For the inverse differences it is even true that the instabilities can hide the asymptotic behaviour in the sense that the displayed numerical results consist merely of rounding errors and data perturbations. However, we shall further on suggest some choices for $\tilde{f}^{(n)}$ which apply to the Thiele interpolating case. If we want to obtain as accurate approximations to $f(z)$ as possible, we can use one of the following techniques to improve on the convergents C_n of (2.1).

(A) Starting a convergence acceleration method such as Aitken's Δ^2-process, the ε-algorithm, or the E-algorithm, essentially also reduces to modification. In [2] Brezinski showed that sequence transformations of the form

$$\tilde{C}_n = \frac{C_{n-1} g_n - C_n g_{n-1}}{g_n - g_{n-1}},$$

where $(g_n)_{n \in \mathbb{N}}$ is an auxiliary sequence, can be viewed as modifications of continued fractions with modifying factor

$$\tilde{f}^{(n)} = -\frac{B_n g_n}{B_{n-1} g_{n-1}}. \tag{2.2}$$

For instance, for Aitken's Δ^2-process, $g_n = \Delta C_n$ and $\tilde{f}^{(n)} = (z - x_n) B_n / B_{n+1}$. The advantage of using this modifying factor has been proved if one is dealing with limit periodic continued fractions. It is important to note that this modifying factor $\tilde{f}^{(n)}$ can be computed without much extra effort. If we use the forward algorithm to compute the continued fraction, then B_n is known and B_{n+1} is given by

$$B_{n+1} = \varphi_{n+1}[x_0, \ldots, x_{n+1}] B_n + (z - x_n) B_{n-1}.$$

Otherwise, B_{n+1}/B_n can be computed by the recurrence relation

$$\frac{B_{n+1}}{B_n}=\varphi_{n+1}[x_0,\ldots,x_{n+1}]+(z-x_n)\left(\frac{B_n}{B_{n-1}}\right)^{-1}, \qquad \frac{B_1}{B_0}=\varphi_1[x_0,x_1].$$

(B) Another way to improve the accuracy of the convergents C_n is not to rewrite existing convergence accelerators as modification methods but to introduce new modifying factors for the particular case of a Thiele interpolating continued fraction. The infinite expression (2.1) can be reduced to a finite expression using the definition for the inverse differences $\varphi_i[x_0,\ldots,x_{i-1},z]$:

$$\begin{aligned} f(z)&=\varphi_0[z]\\ &=\varphi_0[x_0]+\frac{z-x_0}{\varphi_1[x_0,z]}\\ &=\varphi_0[x_0]+\sum_{i=1}^{n}\frac{z-x_{i-1}\ |}{|\varphi_i[x_0,\ldots,x_i]}+\frac{z-x_n\ |}{|\varphi_{n+1}[x_0,\ldots,x_n,z]}. \end{aligned} \tag{2.3}$$

It is easy to see that an approximation for $f(z)$ can now also be obtained by plugging in an approximate value $\tilde{\varphi}_{n+1}$ for $\varphi_{n+1}[x_0,\ldots,x_n,z]$. In fact, since $(z-x_n)/\varphi_{n+1}[x_0,\ldots,x_n,z]$ is the nth tail of the continued fraction (2.1) this amounts to approximating the nth tail by $(z-x_n)/\tilde{\varphi}_{n+1}$, which can then be regarded as a modifying factor for the nth convergent of (2.1). Remark that the use of this modifying factor does not affect the interpolation properties in the points $x_0,\ldots,x_n$ which were satisfied by $C_n(x)$, while using the Aitken Δ^2-modifying factor $\tilde{f}^{(n)}=-B_{n+1}g_{n+1}/B_ng_n$ disturbs the interpolation property in x_n. Let us now discuss several approximations for $\varphi_{n+1}[x_0,\ldots,x_n,z]$.

(a) If we put

$$\tilde{\varphi}_{n+1}=\varphi_{n+1}[x_0,\ldots,x_n,x_{n+1}],$$

then

$$\tilde{f}^{(n)}=\frac{z-x_n}{\varphi_{n+1}[x_0,\ldots,x_{n+1}]},$$

and calculating the modified convergent is just taking the next convergent of (2.1).

(b) If we use the fact that $\varphi_{n+1}[x_0,\ldots,x_n,z]$ can be written as

$$\begin{aligned} \varphi_{n+1}[x_0,\ldots,x_n,z]&=\frac{z-x_n\ |}{|-\varphi_n[x_0,\ldots,x_n]}+\frac{z-x_{n-1}\ |}{|-\varphi_{n-1}[x_0,\ldots,x_{n-1}]}\\ &\quad+\cdots+\frac{z-x_0\ |}{|-\varphi_0[x_0]+\varphi_0[z]} \end{aligned} \tag{2.4}$$

and insert the approximation for $\varphi_0[z]=f(z)$ which we got from previous convergents or previous modified convergents, then we must be very careful not to go around in circles. Take for instance the approximation for $\varphi_0[z]$ which you get from the last convergent, i.e. $\varphi_0[z]\simeq C_n(z)$. Then

$$\tilde{\varphi}_{n+1}=\sum_{i=0}^{n-1}\frac{z-x_{n-i}\ |}{|-\varphi_{n-i}[x_0,\ldots,x_{n-i}]}+\frac{z-x_0\ |}{\left|-\varphi_0[x_0]+\varphi_0[x_0]+\sum_{j=1}^{n}\frac{z-x_{j-1}\ |}{|\varphi_j[x_0,\ldots,x_j]}\right.}=\infty,$$

and hence $\tilde{f}^{(n)} = 0$ which results in no modification at all. Even inserting the approximation for $\varphi_0[z]$ which we got from the last modified convergent, i.e. $\varphi_0[z] \simeq \tilde{C}_{n-1}(z)$, where

$$\tilde{C}_{n-1}(z) = \varphi_0[x_0] + \sum_{i=0}^{n-1} \frac{z - x_{i-1}}{|\varphi_i[x_0,\ldots,x_i]} + \frac{\tilde{f}^{(n-1)}|}{|1},$$

does not help us because then

$$\tilde{\varphi}_{n+1} = \frac{z - x_n}{-\varphi_n[x_0,\ldots,x_n] + \dfrac{z - x_{n-1}}{\tilde{f}^{(n-1)}}},$$

and hence

$$\tilde{f}^{(n)} = -\varphi_n[x_0,\ldots,x_n] + \frac{z - x_{n-1}}{\tilde{f}^{(n-1)}},$$

which produces

$$\begin{aligned}\tilde{C}_n &= \varphi_0[x_0] + \sum_{i=1}^{n} \frac{z - x_{i-1}}{\varphi_i[x_0,\ldots,x_i]} + \frac{\tilde{f}^{(n)}}{1}\\ &= \varphi_0[x_0] + \sum_{i=1}^{n-1} \frac{z - x_{i-1}}{\varphi_i[x_0,\ldots,x_i]} + \frac{\tilde{f}^{(n-1)}}{1}\\ &= \tilde{C}_{n-1}.\end{aligned}$$

This lack of progress is due to the fact that a lot of terms cancel out when we plug in $\varphi_0[z] \simeq C_n(z)$ or $\varphi_0[z] \simeq \tilde{C}_{n-1}(z)$ in (2.4). This can only be helped by replacing z as well as $\varphi_0[z]$ in (2.4), for instance $z \simeq x_{n+k}$ and $\varphi_0[z] \simeq f_{n+k}$ with $k > 0$. In this way

$$\tilde{\varphi}_{n+1} = \varphi_{n+1}[x_0,\ldots,x_n, x_{n+k}]$$

and

$$\tilde{f}^{(n)} = \frac{z - x_n}{\varphi_{n+1}[x_0,\ldots,x_n, x_{n+k}]}. \tag{2.5}$$

Following this idea, we can also approximate the continued fraction (2.4) by a linear expression as given in [4], namely

$$\varphi_{n+1}[x_0,\ldots, x_n, z] = \varphi_{n+1}[x_0,\ldots, x_n, x_{n+k}] + \left.\frac{\partial\varphi_{n+1}[x_0,\ldots, x_n, z]}{\partial z}\right|_{z=x_{n+k}} (z - x_{n+k}).$$

Rather lengthy but simple computations [4] show that

$$\begin{aligned}\left.\frac{\partial\varphi_{n+1}[x_0,\ldots, x_n, z]}{\partial z}\right|_{z=x_{n+k}} &=\\ &= \sum_{i=1}^{n+1} (-1)^{i-1} \frac{\varphi[x_0,\ldots, x_{n-i+1}, x_{n+k}]}{x_{n+k} - x_{n-i+1}} \prod_{j=0}^{i-2} \frac{\varphi^2[x_0,\ldots, x_{n-j}, x_{n+k}]}{x_{n+k} - x_{n-j}}\\ &\quad + (-1)^{n+1}\varphi_0'[x_{n+k}] \prod_{j=0}^{n} \frac{\varphi^2[x_0,\ldots, x_{n-j}, x_{n+k}]}{x_{n+k} - x_{n-j}}.\end{aligned}$$

Hence another modifying factor could be

$$\tilde{f}^{(n)} = \frac{z - x_n}{\varphi_{n+1}[x_0, \ldots, x_n, x_{n+k}] + \left.\frac{\partial \varphi_{n+1}[x_0, \ldots, x_n, z]}{\partial z}\right|_{z = x_{n+k}} (z - x_{n+k})}.$$

(c) Another way to proceed is to approximate $\varphi_{n+1}[x_0, \ldots, x_n, z]$ by using a convergence acceleration method for the sequence $(\varphi_{n+1}[x_0, \ldots, x_n, x_{n+i+1}])_{i \in \mathbb{N}}$ because if $\varphi_{n+1}[x_0, \ldots, x_n, z]$ is not undefined, then by a continuity argument

$$\varphi_{n+1}[x_0, \ldots, x_n, z] = \lim_{i \to \infty} \varphi_{n+1}[x_0, \ldots, x_n, x_{n+i+1}].$$

For our choice of a convergence accelerator we keep in mind that $\varphi_{n+1}[x_0, \ldots, x_n, z]$ is a rational expression in z. If we take for instance Aitken's Δ^2-process or the ε-algorithm, we perform in the following computations:

$$\varepsilon_{-1}^{(i)} = 0, \qquad i = 0, \ldots, 2k + l + 1,$$
$$\varepsilon_0^{(i)} = \varphi_{n+1}[x_0, \ldots, x_n, x_{n+i+1}], \quad i = 0, \ldots, 2k + l,$$
$$\varepsilon_{j+1}^{(i)} = \varepsilon_{j-1}^{(i+1)} + \frac{1}{\varepsilon_j^{(i+1)} - \varepsilon_j^{(i)}}, \qquad j = 0, \ldots, 2k - 1, \quad i = 0, \ldots, 2k + l - j - 1.$$

For Aitken's Δ^2-process we take $k = 1$ and $\tilde{\varphi}_{n+1} = \varepsilon_2^{(l)}$. For the ε-algorithm $l = 0$ and $\tilde{\varphi}_{n+1} = \varepsilon_{2k}^{(0)}$. Consequently,

$$\tilde{f}^{(n)} = \frac{z - x_n}{\varepsilon_{2k}^{(l)}}. \tag{2.6}$$

The next theorem will prove the usefulness of this modifying factor if one is dealing with Thiele interpolating continued fractions. We remind the reader that the ε-algorithm and Aitken's Δ^2-process are frequently used convergence accelerators, although conditions under which their application is proved to be successful are very restrictive. The same reasoning holds here.

Theorem 2.1. *Let f be a real-valued function of a real variable x and let the continued fraction* (2.1) *converge for $x = z = \lim_{n \to \infty} x_n$. Let $(\delta_i)_{i \in \mathbb{N}}$ be a monotonically decreasing sequence of strictly positive numbers.*

If for all $n \geq 0$.

$$\varphi_{n+1}[x_0, \ldots, x_n, z] B_n \neq 0,$$

and if for all $n \geq 0$ there exist real numbers $a_n \neq 0$ and b_n such that the sequence

$$(a_n \varphi_{n+1}[x_0, \ldots, x_n, x_{n+i+1}] + b_n)_{i \in \mathbb{N}}$$

is totally oscillating or totally monotone, then for every n there exist k_n and l_n such that for

$$\tilde{f}^{(n)} = \frac{z - x_n}{\varepsilon_{2k_n}^{(0)}} \quad \text{and} \quad \tilde{f}^{(n)} = \frac{z - x_n}{\varepsilon_2^{(l_n)}}$$

we have

$$|f(z) - \tilde{C}_n| \leq \delta_n \cdot |f(z) - C_n|.$$

Proof. We know that

$$\left|\frac{f(z)-\tilde{C}_n}{f(z)-C_n}\right|$$

$$=\left|\frac{f^{(n)}-\tilde{f}^{(n)}}{f^{(n)}}\right|\cdot\left|\frac{B_n}{B_n+\tilde{f}^{(n)}B_{n-1}}\right|$$

$$=\left|\frac{\dfrac{z-x_n}{\varphi_{n+1}[x_0,\ldots,x_n,z]}-\dfrac{z-x_n}{\varepsilon_{2k}^{(l)}}}{\dfrac{z-x_n}{\varphi_{n+1}[x_0,\ldots,x_n,z]}}\right|\cdot\left|\frac{\varepsilon_{2k}^{(l)}}{\varepsilon_{2k}^{(l)}+(z-x_n)\cdot B_{n-1}/B_n}\right|$$

$$=\left|\frac{\varepsilon_{2k}^{(l)}-\varphi_{n+1}[x_0,\ldots,x_n,z]}{\varepsilon_{2k}^{(l)}-(z-x_n)B_{n-1}/B_n}\right|$$

$$=\left|\frac{\varepsilon_{2k}^{(l)}-\varphi_{n+1}[x_0,\ldots,x_n,z]}{(\varepsilon_{2k}^{(l)}-\varphi_{n+1}[x_0,\ldots,x_n,z])+\varphi_{n+1}[x_0,\ldots,x_n,z]+(z-x_n)B_{n-1}/B_n}\right|.$$

It is easy to see that

$$\varphi_{n+1}[x_0,\ldots,x_n,z]B_n+(z-x_n)B_{n-1}$$
$$=\varphi_{n+1}[x_0,\ldots,x_n,z](\varphi_n[x_0,\ldots,x_{n-1},z]B_{n-1}+(z-x_{n-1})B_{n-2})$$
$$=\prod_{i=0}^{n}\varphi_{i+1}[x_0,\ldots,x_i,z]\neq 0.$$

From this we can conclude that if $(a_n\varphi_{n+1}[x_0,\ldots,x_n,x_{n+i+1}]+b_n)_{i\in\mathbb{N}}$ is a totally monotone or totally oscillating sequence, then for given $\delta_n>0$ there exist k_n and l_n such that [1, p. 83]

$$\left|\varphi_{n+1}[x_0,\ldots,x_n,z]-\varepsilon_{2k_n}^{(0)}\right|\leqslant\frac{1}{2}\left|\frac{1}{B_n}\prod_{i=0}^{n}\varphi_{i+1}[x_0,\ldots,x_i,z]\right|\min(1,\delta_n),$$

$$\left|\varphi_{n+1}[x_0,\ldots,x_n,z]-\varepsilon_{2}^{(l_n)}\right|\leqslant\frac{1}{2}\left|\frac{1}{B_n}\prod_{i=0}^{n}\varphi_{i+1}[x_0,\ldots,x_i,z]\right|\min(1,\delta_n).$$

Hence for $l=0$ and $k=k_n$,

$$\left|\frac{f(z)-\tilde{C}_n}{f(z)-C_n}\right|\leqslant\frac{\left|\varepsilon_{2k_n}^{(0)}-\varphi_{n+1}[x_0,\ldots,x_n,z]\right|}{\left|\dfrac{1}{B_n}\prod_{i=0}^{n}\varphi_{i+1}[x_0,\ldots,x_i,z]\right|-\left|\varepsilon_{2k_n}^{(0)}-\varphi_{n+1}[x_0,\ldots,x_n,z]\right|}\leqslant\delta_n,$$

and analogously for $k=1$ and $l=l_n$. □

Note that in order to prove convergence acceleration we need to impose conditions on the sequences $(\varphi_{n+1}[x_0,\ldots,x_{n+i+1}])_{i\in\mathbb{N}}$. Because of the instabilities in the computation of $\varphi_{n+1}[x_0,\ldots,x_n,x_{n+i+1}]$ it does not make sense to translate these hypotheses to conditions on

Table 3
$f_i = \Sigma_{j=0}^{i} 2^j/j!$, $x_i = 1/\sqrt{i+1}$, $\lim_{i\to\infty} f_i = 7.3890561\ldots$

		56 bit	24 bit
Equation (2.2)	$g_n = \Delta C_n$	$\tilde{C}_8 = 6.492160$	$\tilde{C}_8 = 6.505295$
		$\tilde{C}_{10} = 4.216581$	$\tilde{C}_{10} = 2.237958$
Equation (2.5)	$k = 5$	$\tilde{C}_4 = 6.612449$	$\tilde{C}_4 = 6.612472$
	$k = 7$	$\tilde{C}_4 = 6.885566$	$\tilde{C}_4 = 6.885586$
Equation (2.6)	$k = 1$ and $l = 2$	$\tilde{C}_4 = 7.047279$	$\tilde{C}_4 = 7.047497$
	$k = 1$ and $l = 4$	$\tilde{C}_4 = 7.180756$	$\tilde{C}_4 = 7.180896$
	$l = 0$ and $k = 2$	$\tilde{C}_4 = 7.163015$	$\tilde{C}_4 = 7.165371$
	$l = 0$ and $k = 3$	$\tilde{C}_4 = 7.325809$	$\tilde{C}_4 = 7.353887$
$f^{(n)} = 0$		$C_9 = 6.760714$	$C_9 = 6.782696$
		$C_{11} = 7.939311$	$C_{11} = 8.790054$

the f_i and x_i. The best approach is to check the conditions numerically during the computation of the continued fraction and if these are satisfied we can make $|f - C_n|/|f - \tilde{C}_n|$ small, which is definitely what we want. Because of the fact that the conditions on the sequences $(\varphi_{n+1}[x_0, \ldots, x_n, x_{n+i+1}])_{i\in\mathbb{N}}$ cannot be checked beforehand for all n, it is not possible to claim convergence acceleration in the usual sense, i.e. $|f - C_n|/|f - \tilde{C}_n| \to 0$ as n approaches infinity.

Table 4
$f_i = \Sigma_{j=1}^{i} \left\lfloor \frac{(j+1)^2}{1} \right\rceil$, $x_i = -1/(i+1)^2$, $\lim_{i\to\infty} f_i = 1.25889\ldots$

n	C_n, $f^{(n)} = 0$	$\tilde{C}_n$, (2.2), $g_n = \Delta C_n$	$\tilde{C}_n$, (2.6), $k = 1$, $l = 0$
10	0.7044702	1.411553	1.186592
20	0.8993983	1.312232	1.229184
30	0.9921064	1.286098	1.242457
40	1.046538	1.275415	1.248361
50	1.082410	1.270001	1.251523
60	1.107859	1.266877	1.253424
70	1.126862	1.264911	1.254660
80	1.141600	1.263593	1.255512
90	1.153365	1.262665	1.256125
100	1.162978	1.261988	1.256582
150	1.192955	1.260325	1.257751
200	1.208634	1.259715	1.258206
250	1.218280	1.259426	1.258432
1000	1.248405	1.258926	1.258855
2000	1.253617	1.258900	1.258881
3000	1.255367	1.258895	1.258887
4000	1.256246	1.258894	1.258889

We remark also that it is a good investment to put more effort in the computation of the nth tail for small n than in the further computation of ordinary convergents C_n. The reason is simple. For the former we need the values $(\varphi_{n+1}[x_0, \ldots, x_n, x_{n+i+1}])_{i \in \mathbb{N}}$ which are all as stable (or unstable) as $\varphi_{n+1}[x_0, \ldots, x_{n+1}]$, while for the latter we need inverse differences with a growing number of arguments and hence a growing effect of data perturbations and rounding errors. Of course the method used to accelerate the convergence of $(\varphi_{n+1}[x_0, \ldots, x_n, x_{n+i+1}])_{i \in \mathbb{N}}$ must be more stable than the computation scheme that generates inverse differences with more than $n+2$ arguments.

The results displayed in Table 3 illustrate that when the computation of the inverse differences is highly unstable, in other words if the number of digits common to single and double precision output is small, it may be a good idea to switch to modification. One has more significant digits with $\tilde{f}_n \neq 0$ than with $\tilde{f}_n = 0$. The results displayed in Table 4 illustrate that, even for a minor extra effort, in case of stability modification can be very successful.

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