Good Interpolation Points: Learning from Chebyshev, Fekete, Haar and Lebesgue

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Abstract. The search for sets of good interpolation points is highly motivated by the fact that, due to the finite precision of digital computers, valid results can only be expected when the interpolation problem is well-conditioned. The conditioning of polynomial interpolation and of rational interpolation with preassigned poles is measured by the respective Lebesgue constants. Here we summarize the main results with respect to the Lebesgue constant for polynomial interpolation and we present the best Lebesgue constants in existence for rational interpolation with preassigned poles. The new results are based on a fairly unknown rational analogue of the Chebyshev orthogonal polynomials. We compare with the results obtained in [1] and [2].

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UNIVARIATE POLYNOMIAL INTERPOLATION

Let the function f belong to $C([-1,1])$. When approximating f by an element from a finite-dimensional $V_n = \text{span}\{\phi_0,\dots,\phi_n\}$ with $\phi_i \in C([-1,1])$ for $0 \le i \le n$, we know that there exists at least one element p_n^* in V_n that is closest to f. This element is the unique closest one if the $\phi_0,\dots,\phi_n$ are a Chebyshev system. Since the computation of this element is more complicated than that of the interpolant

$$\sum_{i=0}^{n} \alpha_i \phi_i(x_j) = f(x_j), \qquad j = 0,\dots,n, \qquad -1 \le x_j \le 1,$$

scientists have looked for interpolation points x_j that make the interpolation error

$$\left|\left| f(x) - \sum_{i=0}^{n} \alpha_i \phi_i(x) \right|\right|$$

as small as possible. In this presentation we focus on the infinity or Chebyshev norm on the unit interval $[-1,1]$.

1.1 Minimizing the interpolation error bound

When $\phi_i(x) = x^i$ and f is sufficiently differentiable, then for the interpolant

$$p_n(x) = \sum_{i=0}^{n} \alpha_i x^i,$$

satisfying $p_n(x_j) = f(x_j),\ 0 \le j \le n$, the error $||f - p_n||_\infty$ is bounded by

$$||f - p_n||_\infty \le \max_{x\in[-1,1]} \left(\frac{|f^{(n+1)}(x)|}{(n+1)!} \right) \max_{x\in[-1,1]} \prod_{j=0}^{n} |x - x_j|.$$

It is well-known that the monic $(x-x_0)\cdots(x-x_n)$ is minimal if the x_j are the zeroes of the $(n+1)$-th degree Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos x)$.

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1.2 Minimizing the Lebesgue constant

The procedure that associates with f its interpolant p_n is linear and given by

$$P_n : C([-1,1]) \to V_n : f(x) \to p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$$

where the basic Lagrange polynomials $\ell_i(x)$,

$$\ell_i(x) = \prod_{j=0,\, i\neq j}^{n} \frac{x - x_j}{x_i - x_j},$$

satisfy $\ell_i(x_j) = \delta_{ij}$. Hence another bound for the interpolation error is given by

$$||f - p_n||_\infty \leq (1 + ||P_n||)\, ||f - p_n^*||_\infty, \qquad ||P_n|| = \max_{x\in[-1,1]} \sum_{i=0}^{n} |\ell_i(x)|\,.$$

Here $\Lambda_n := \Lambda_n(x_0,\ldots,x_n) = ||P_n||$ is called the Lebesgue constant and it depends on the location of the interpolation points x_j. An explicit formula for the x_j that minimize the Lebesgue constant is not known, and if no further constraints are imposed on the interpolation points then the solution is not even unique. But it is proved in [3] that the minimal growth of the Lebesgue constant is given by $(2/\pi)\log(n+1) + (2/\pi)(\gamma + \log(4/\pi)) \approx (2/\pi)\log(n+1) + 0.52125\ldots$ with γ the Euler constant.

Several node sets $x_0,\ldots,x_n$ come close to realizing this minimal growth, among which the Chebyshev zeroes from Section 1.1 and the Fekete points from Section 1.3. The node set known in closed form that approximates the optimal node set best is probably the so-called extended Chebyshev node set given by

$$x_j = -\frac{\cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)}{\cos\left(\frac{\pi}{2(n+1)}\right)}, \qquad j = 0,\ldots,n. \tag{1}$$

The division by $\cos(\pi/(2n+2))$ guarantees that $x_0 = -1$ and $x_n = 1$. The growth of the Lebesgue constant for the extended Chebyshev nodes is bounded by [4]

$$\Lambda_n(x_0,\ldots,x_n) < \frac{2}{\pi}\log(n+1) + 0.5829\ldots, \qquad n \geq 4.$$

1.3 Maximizing the Vandermonde determinant

Because the basic Lagrange polynomials are given by the quotient of two Vandermonde determinants, namely

$$\ell_i(x) = \frac{|V(x_0,\ldots,x_{i-1},x,x_{i+1},\ldots,x_n)|}{|V(x_0,\ldots,x_n)|}, \quad V(x_0,\ldots,x_i,\ldots,x_n) = \begin{pmatrix} 1 & x_0 & \ldots & x_0^n \\ \vdots & & \vdots & \\ 1 & x_i & \ldots & x_i^n \\ \vdots & & \vdots & \\ 1 & x_n & \ldots & x_n^n \end{pmatrix},$$

it can be expected that the interpolation points maximizing the Vandermonde determinant $|V(x_0,\ldots,x_n)|$ yield a small Lebesgue constant. This node set is given by

$$(1-x^2)\frac{dL_n}{dx}(x) = 0,$$

or in other words by $x_0 = -1$, $x_n = 1$ and $x_1,\ldots,x_{n-1}$ the extrema of the n-the Legendre polynomial $L_n(x)$ and is known as the Fekete node set.

2. RATIONAL INTERPOLATION WITH FIXED POLES

When moving to rational interpolation, the above conclusions do not hold anymore. For instance, rational interpolation using the Chebyshev nodes may yield worse results than using equidistant interpolation points. As an example we mention $f(x) = \arctan(3x)$ on $[-1,1]$ with the numerator and denominator degrees of the rational interpolant respectively equal to 5 and 4. In addition, the approximation and interpolation problems become nonlinear unless one considers the case of a priori fixed poles as we do in this section. So let $q_m(x) = \prod_{k=0}^{m-1}(1 - x/\xi_k)$ with $\xi_k \notin [-1,1]$ and interpolate

$$p_n(x_j) = f(x_j)q_m(x_j), \qquad j = 0,\dots,n \tag{2}$$

with $p_n(x) \in \text{span}\{1,\dots,x^n\}$. In the sequel we restrict ourselves to polynomials $q_m(x)$ having real coefficients, in other words having poles that are real or appear in complex conjugate pairs.

2.1 Minimizing the interpolation error bound

With $x_j \in [-1,1]$ and $\xi_k \notin [-1,1]$ the rational interpolation error is bounded above by

$$\left|\left| f - \frac{p_n}{q_m} \right|\right|_\infty \le \max_{x\in[-1,1]} \left(\frac{|(fq_m)^{(n+1)}(x)|}{(n+1)!} \right) \max_{x\in[-1,1]} \prod_{j=0}^{n} \frac{|x - x_j|}{|q_m(x)|}.$$

The factor $(x - x_0)\cdots(x - x_n)/q_m(x)$ has minimal absolute value if the x_j are the zeroes of the orthogonal rational function $\mathscr{T}_{n+1}(x)$ that is defined as follows [5]. If $n \ge m$ then we first complement the set of poles ξ_k with $\xi_m = \ldots = \xi_n = \infty$. Consider the Joukowski transform

$$J : \mathbb{C} \to \mathbb{C} : z \to J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

For $x = J(z)$ also $x = J(1/z)$ and so we restrict the inverse of the Joukowski transform to $|z| \le 1$. Now take ζ_k, $0 \le k \le n$ such that $\xi_k = J(\zeta_k)$ and define

$$B_0(z) = 1, \qquad B_k(z) = \frac{z - \zeta_{k-1}}{1 - \overline{\zeta}_{k-1}z} B_{k-1}(z), \qquad k = 1,\dots,n,$$

$$\mathscr{T}_0(x) = \sqrt{\frac{1}{\pi}},$$

$$\mathscr{T}_{n+1}(x) = \sqrt{\frac{1 - |\zeta_n|^2}{2\pi}} \left(\frac{z\overline{B}_n(\overline{z})}{1 - \zeta_n z} + \frac{1}{(z - \zeta_n)B_n(z)} \right).$$

This orthogonal Chebyshev rational function has the preassigned poles $\xi_k \notin [-1,1]$ and so is different from the classical Chebyshev rational function with coinciding poles in -1: $\mathscr{T}_{n+1}(x)$ is of the form $p_{n+1}(x)/q_m(x)$.

2.2 Minimizing the Lebesgue constant

The rational interpolant can also be seen as an element of $\text{span}\{1/q_m(x), x/q_m(x),\dots, x^n/q_m(x)\}$. Since $\xi_k \notin [-1,1], 0 \le k \le m-1$ these functions form a Chebyshev system and hence the existence of the unique best approximant and of the interpolant are both guaranteed. The operator R_n that associates with f the rational interpolant p_n/q_m with preassigned poles is linear and so we can define the Lebesgue constant $M_n := M_n(x_0,\dots,x_n;\xi_0,\dots,\xi_{m-1}) = ||R_n||$,

$$M_n = \sup_{||f||_\infty \le 1} ||R_n f||_\infty = \max_{x\in[-1,1]} \sum_{i=0}^{n} \frac{|q_m(x_i)\ell_i(x)|}{|q_m(x)|}.$$

In [1] the authors determine the location of the poles $\xi_k, 0 \le k \le n-1$ that minimize the Lebesgue constant M_n for given interpolation points $x_j, 0 \le j \le n$. In [2] the asymptotic behaviour of the Lebesgue constant M_n is given for equidistant nodes x_j and

$$q_n(x) = \sum_{i=0}^{n} (-1)^i \prod_{j=0, i\neq j}^{n} (x - x_j) \tag{3}$$

also defined in terms of the nodes. In both studies $m = n$ and $q_n(x)$ has real coefficients. When using rational interpolants with preassigned poles, none of the above situations is very practical. The location and the number of the poles is usually determined by the nature of the function f that one is modelling. Hence optimal interpolation points need to be found in terms of the poles and not vice versa.

Another practical drawback is the following. The values for M_n obtained in [1] are optimal in the sense that they are minimal for the considered $(x_0, \ldots, x_n; \xi_0, \ldots, \xi_{n-1})$ combination: changing either the poles or the interpolation points may increase M_n. Hence these values provide the rational analogue of the minimal growth behaviour in the polynomial case. Note that neither these optimal poles $\xi_1, \ldots, \xi_{n-1}$ nor the minimal value for M_n are known by an explicit formula. All are obtained from the solution of a hefty optimization problem.

Our aim is to present a node set that doesn't suffer from the mentioned drawbacks: we give interpolation points that are nearly optimal for given arbitrary poles outside the interval of interpolation instead of vice versa, and our points can easily be obtained from a generalized eigenvalue problem [6]. We make use of the formulas from Section 2.1.

If the preassigned finite poles ξ_k are real or appear in complex conjugate pairs, then for $n+1 \geq m$ the zeroes of $\mathscr{T}_{n+1}(x)$ are real, simple and belong to $(-1,1)$ [5]. These zeroes are the rational counterpart of what the Chebyshev nodes are in the polynomial case and hence are suitable interpolation points for (2). And as with other orthogonal functions, they can be obtained from a generalized eigenvalue problem. Unless there is a pole ξ_k at a very small distance of the interval $[-1,1]$, the maximum value of the Lebesgue function

$$\frac{\sum_{i=0}^{n} |q_m(x_i)\ell_i(x)|}{|q_m(x)|}$$

is not obtained near the endpoints of the interval. Hence extending the points as in the polynomial case to place x_0 in -1 and x_n in $+1$ usually makes no sense.

In Figure 1 we compare

- (full line) the nearly optimal Lebesgue constant $\Lambda_n(x_0, \ldots, x_n)$ for polynomial interpolation using the extended Chebyshev nodes (1),
- ($\circ$) the Lebesgue constant $M_n(x_0, \ldots, x_n; \xi_0, \ldots \xi_{n-1})$ for the Chebyshev nodes $x_j = -\cos((2j+1)\pi/(2n+2))$ with $q_n(x)$ and the ξ_k given by (3),
- ($\square$) the Lebesgue constant $M_n(x_0, \ldots, x_n; \xi_0, \ldots \xi_{n-1})$ for equidistant interpolation points $x_j = -1 + j/n$ and with $q_n(x)$ and ξ_k determined by (3),
- ($+$) the optimal Lebesgue constant obtained in [1] for the case of equidistant interpolation points and optimally associated poles ξ_k,
- and our approach ($\star$), where we take the ξ_k from the same polynomial (3) to be comparable, but take the interpolation points for (2) from $\mathscr{T}_{n+1}(x) = 0$.

In Figure 2 we present, from left to right, the Lebesgue functions for $n = 10$ associated with the Lebesgue constants indicated by $+$, $\star$, $\square$ respectively.

Note that the rational interpolants with preassigned poles all generate Lebesgue constants that are very comparable to the one from the (almost) optimal polynomial interpolant. This comes in addition to the well-known ease of rational interpolation to fit steep changes and asymptotic bahaviour and its tendency to oscillate less inbetween interpolation points than polynomials. And the new technique allows to determine good interpolation points for any set of preassigned poles $\{\xi_0, \ldots, \xi_{m-1}\}$, also for $m < n$ and not only for those determined by (3). Also, the new technique leads to smaller Lebesgue constants M_n than the ones from [2].

The latter is better illustrated in the Tables 1 and 2 where we show the variation between

- on the one hand the Lebesgue constants of the linear rational interpolation (2) using equidistant ($M_n^{\square}$), Chebyshev (M_n°) or extended Chebyshev nodes ($M_n^{\bullet}$), and
- on the other hand the Lebesgue constant from our technique ($M_n^{\star}$) that takes the interpolation points from $\mathscr{T}_{n+1}(x) = 0$.

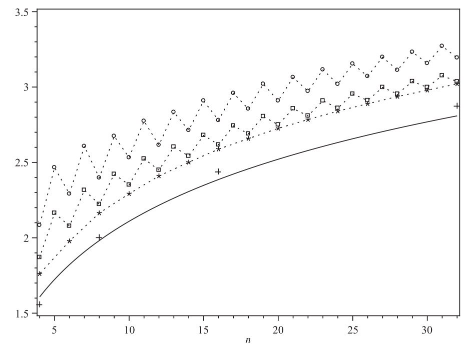

FIGURE 1.

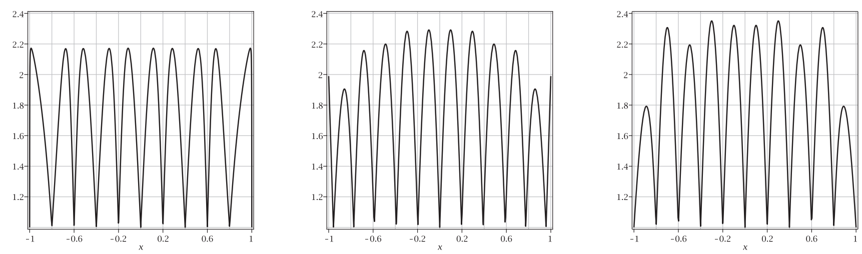

FIGURE 2.

In Table 1 we placed two poles at ± 1.001 and we choose the remaining poles randomly in $[-50,-1[\cup]1,50]$. For $M_n^{\star}$ we extended the zeroes to put x_0 in -1 and x_n in $+1$. In Table 2 all poles were complex conjugate pairs with real part in $[-1,1]$ and imaginary parts ± 0.01. Here we did not use extended nodes for $M_n^{\star}$. The displayed results are typical. The rate of growth is different between the situation illustrated in Table 1 and the one illustrated in Table 2. In the former the extended Chebyshev nodes maintain a rather modest rate of growth while the Chebyshev nodes generate a clearly faster growth and the equidistant nodes cause an explosion of the Lebesgue constant. In the latter both Chebyshev sets perform equally bad. We stress that in [2] and [1] the poles ξ_k are preassigned but dictated by the interpolation procedure. In our approach the poles are freely preassigned and the interpolation points are adapted as in Section 2.1. In this more general setting our method offers a clear advantage.

In Figure 3 we graph the Lebesgue functions for $n = 10$ associated with the Lebesgue constants $M_{10}^{\star}, M_{10}^{\bullet}, M_{10}^{\square}$ of Table 1 respectively.

2.3. Maximizing the determinant of the Haar system

The rational interpolant p_n/q_m is a linear combination of the $\phi_i = x^i/q_m(x), 0 \le i \le n$ and therefore can be expressed as

$$\frac{p_n}{q_m}(x) = \sum_{i=0}^{n} f(x_i)\lambda_i(x),$$

TABLE 1.

$M_{10}^{\star}$	2.491×10^0	$M_{20}^{\star}$	3.006×10^0
$M_{10}^{\bullet}$	2.017×10^1	$M_{20}^{\bullet}$	7.743×10^1
M_{10}°	3.586×10^2	M_{20}°	4.846×10^2
$M_{10}^{\square}$	4.943×10^2	$M_{20}^{\square}$	5.354×10^5

TABLE 2.

$M_{10}^{\star}$	3.515×10^{0}
$M_{10}^{\bullet}$	2.714×10^{4}
M_{10}°	2.971×10^{4}
$M_{10}^{\square}$	1.702×10^{9}

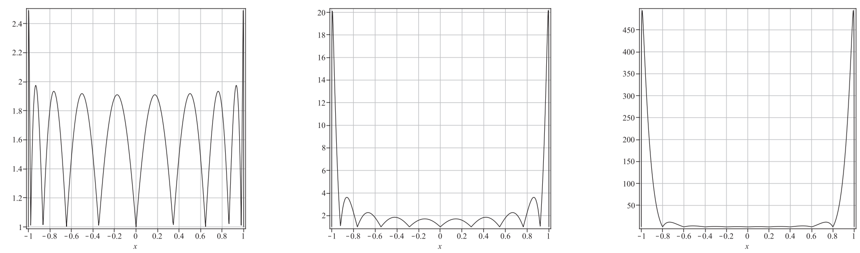

FIGURE 3.

where the basic rational interpolants $\lambda_i(x)$ equal the quotient of determinants

$$\lambda_i(x) = \frac{|H(x_0,\ldots,x_{i-1},x,x_{i+1},\ldots,x_n)|}{|H(x_0,\ldots,x_n)|}, \quad H(x_0,\ldots,x_i\ldots,x_n) = \begin{pmatrix} 1/q_m(x_0) & \ldots & x_0^n/q_m(x_0) \\ \vdots & & \vdots \\ 1/q_m(x_i) & \ldots & x_i^n/q_m(x_i) \\ \vdots & & \vdots \\ 1/q_m(x_n) & \ldots & x_n^n/q_m(x_n) \end{pmatrix}. \tag{4}$$

The rational function $\lambda_i(x)$ satisfies $\lambda_i(x_j) = \delta_{ij}$ and further equals $q_m(x_i)\ell_i(x)/q_m(x)$. Maximizing the value of $|H(x_0,\ldots,x_n|$ is an unsolved problem that may provide another explicitly knowns node set.

3. CONCLUSION

The (extended) zeros of the orthogonal rational function $\mathscr{T}_{n+1}(x)$ constructed in Section 2.1 provide interpolation points for rational interpolation with poles prescribed by $q_m(x) = 0, m \leq n+1$, that are as good as the (extended) Chebyshev zeroes for polynomial interpolation. In the case of poles close to the interval of interpolation, they clearly outperform all other proposed sets of interpolation points.

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