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A finite sum representation of the Appell series $F_1(a,b,b';c;x,y)$

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Abstract

We use Picard's integral representation of the Appell series $F_1(a,b,b';c;x,y)$ for $\mathrm{Re}(a)>0$, $\mathrm{Re}(c-a)>0$ to obtain a finite sum algebraic representation of F_1 in the case when a, b, b' and c are positive integers with $c>a$. The series converges for $|x|<1$, $|y|<1$ and we show that $F_1(a,b,b';c;x,y)$ has two overlaying singularities at each of the points $x=1$ and $y=1$, one polar and one logarithmic in nature, when a, b, b', $c\in\mathbb{N}$ with $c>a$.

Keywords: Multiple hypergeometric function; Appell series; Singularity

1. Introduction

Hypergeometric series in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistics, operations research and so on [3]. The ordinary hypergeometric

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series, also called Gauss's hypergeometric series, is defined by

$$ {}_2F_1(a,b;\,c;\,z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{1.1} $$

where $(a)_n$ is the Pochammer symbol

$$ \begin{aligned} (a)_n &:= a(a+1)(a+2)\cdots(a+n-1), \quad n \geqslant 1, \\ (a)_n &:= 1, \quad n = 0 \end{aligned} \tag{1.2} $$

and the parameters a, b, c and z may be real or complex. Hypergeometric series in two or more variables which reduce to the familiar Gaussian series (1.1) whenever only one variable is non-zero, are called multiple Gaussian hypergeometric series. In total, 14 distinct double Gaussian series exist [6]. The first four of these were introduced by Appell by taking the product of two Gaussian functions and replacing one or more of the three pairs of products

$$ (a)_n(a')_m, \qquad (b)_n(b')_m, \qquad (c)_n(c')_m $$

by the corresponding expressions

$$ (a)_{n+m}, \qquad (b)_{n+m}, \qquad (c)_{n+m}. $$

In this paper we shall investigate the first Appell series $F_1(a,b,b';\,c;\,x,y)$ defined by

$$ F_1(a,b,b';\,c;\,x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_i(b')_j x^i y^j}{(c)_{i+j} i!\, j!}. \tag{1.3} $$

This function converges for $|x| < 1$ and $|y| < 1$ [3, p. 25], but the nature and location of its singularities is not obvious. Our main result is an algebraic representation of $F_1(a,b,b';\,c;\,x,y)$ for a, b, b', c positive integers with $c > a$, which explicitly displays the singularities of the function at $x = 1$ and $y = 1$. A useful tool in deriving this algebraic representation is Picard's integral formula [5]

$$ F_1(a,b,b';\,c;\,x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}\,\mathrm{d}u, \tag{1.4} $$

for $\mathrm{Re}(a) > 0$, $\mathrm{Re}(c-a) > 0$. Although the other three Appell series F_2, F_3 and F_4 also have single integral representations [6, pp. 274–283], they are not the simple Eulerian integrals of type (1.4) for all parameters a, b, b' and c.

This work is a part of a larger study, in which we investigate the approximation of the Appell series F_i, $i = 1,2,3,4$ by multivariate Padé approximants. As pointed out in [2], information on the singularities of the function being approximated, such as derived here for $F_1(a,b,b';\,c;\,x,y)$, can be very useful to determine the form of the denominator polynomials in the multivariate Padé approximants.

Our study is motivated by the fact that a great deal is known about Padé approximants for the Gaussian hypergeometric series (1.1). The aim of our study is the generalization of these results to the double Gaussian hypergeometric series F_i. Results on the structure of the table of multivariate Padé approximants for the Appell series are discussed in [1]. Work in progress includes finding compact explicit formulas for the denominators of the multivariate Padé approximants, in analogy with the results obtained in [4,7] for the Gauss function. The paper is organized as follows: Section 2 contains the statement of the main result and Section 3 consists of the proofs of the auxiliary lemmas followed by the proof of the main result.

2. Algebraic representation of $F_1(a,b,b';c;x,y)$

We present the result in increasing order of complexity, starting with the simplest choice of the parameters a, b, b' and c, and ending with the general case.

Theorem 2.1. (a) *For any non-negative integers s, t and $x \neq y$, we have for $|x|<1$, $|y|<1$,*

$$\begin{aligned} &F_1(1,s+1,t+1;2;x,y)\\ &\quad=\frac{(-1)^t x^t y^s}{(y-x)^{s+t+1}}\binom{s+t}{s}[\ln(1-x)-\ln(1-y)]\\ &\quad-\sum_{j=0}^{t-1}\binom{j+s}{s}\frac{(-x)^j y^s[1-(1-y)^{j-t}]}{(y-x)^{s+1+j}(t-j)}\\ &\quad-\sum_{k=0}^{s-1}\binom{k+t}{t}\frac{x^t(-y)^k[1-(1-x)^{k-s}]}{(x-y)^{t+1+k}(s-k)}. \end{aligned} \tag{2.1}$$

(b) *For any non-negative integers a, s, t and $x \neq y$, we have for $|x|<1$, $|y|<1$,*

$$\begin{aligned} &\frac{1}{a+1}F_1(a+1,s+1,t+1;a+2;x,y)\\ &\quad=y^{s-a}\sum_{j=0}^{t}\binom{j+s}{s}\frac{(-x)^j}{(y-x)^{s+j+1}}\left[\sum_{\substack{k=0\\k\neq t-j}}^{a}\binom{a}{k}(-1)^k\frac{[1-(1-y)^{k+j-t}]}{(k+j-t)}\right.\\ &\qquad\qquad\left.-\binom{a}{t-j}(-1)^{t-j}\ln(1-y)\right]\\ &\quad+x^{t-a}\sum_{k=0}^{s}\binom{k+t}{t}\frac{(-y)^k}{(x-y)^{t+k+1}}\left[\sum_{\substack{j=0\\j\neq s-k}}^{a}\binom{a}{j}(-1)^j\frac{[1-(1-x)^{j+k-s}]}{(j+k-s)}\right.\\ &\qquad\qquad\left.-\binom{a}{s-k}(-1)^{s-k}\ln(1-x)\right]. \end{aligned} \tag{2.2}$$

(c) *For any non-negative integers a, s, t, d and $x \neq y$, we have for $|x|<1$, $|y|<1$,*

$$B(a+1,d+1)F_1(a+1,s+1,t+1;\, a+d+2;\, x,y)$$
$$=\sum_{m=0}^{d}\binom{d}{m}(-1)^m\left\{y^{s-a-m}\sum_{j=0}^{t}\binom{j+s}{s}\frac{(-x)^j}{(y-x)^{s+j+1}}A + x^{t-a-m}\sum_{k=0}^{s}\binom{k+t}{t}\frac{(-y)^k}{(x-y)^{t+k+1}}C\right\}, \tag{2.3}$$

where

$$A=\sum_{\substack{k=0\\ k\neq t-j}}^{a+m}\binom{a+m}{k}(-1)^k\frac{[1-(1-y)^{k+j-t}]}{(k+j-t)}-\binom{a+m}{t-j}(-1)^{t-j}\ln(1-y), \tag{2.4}$$

$$C=\sum_{\substack{j=0\\ j\neq s-k}}^{a+m}\binom{a+m}{j}(-1)^j\frac{[1-(1-x)^{j+k-s}]}{(j+k-s)}-\binom{a+m}{s-k}(-1)^{s-k}\ln(1-x). \tag{2.5}$$

Here, $B(\cdot,\cdot)$ is the Beta function, $\binom{n}{k}$ is the binomial coefficient and all sums are understood to be zero where they are not defined.

We observe that (2.3) can also be written in the form

$$B(a+1,d+1)F_1(a+1,s+1,t+1;\, a+d+2;\, x,y)$$
$$=\sum_{m=0}^{d}\binom{d}{m}(-1)^m F_1(a+m+1,s+1,t+1;\, a+m+2;\, x,y).$$

Remark. (1) It is immediately apparent from Theorem 2.1 that for $a, b, b', c \in \mathbb{N}$ with $c>a$, the singularities of $F_1(a,b,b';\, c;\, x,y)$ occur at $x=1$ and $y=1$. There are two overlaying singularities at each of these points, one polar and the other logarithmic. The polar singularity is dominant near $x=1$ and $y=1$, however, the logarithmic singularity causes multi-valuedness in the neighbourhood of each point. It also appears as though there is an infinite set of singularities when $x=y$. However, this is not the case, since whenever $x=y$, the Appell function $F_1(a,b,b';\, c;\, x,x)$ can be expressed as a Gauss hypergeometric function of one variable. We have (cf. [3, p. 30])

$$F_1(a,b,b';\, c;\, x,x)={}_2F_1(a,b+b';\, c;\, x). \tag{2.6}$$

(2) From (2.1) with $s=t=0$, we see that for $|x|<1$, $|y|<1$,

$$F_1(1,1,1;\, 2;\, x,y)=\frac{\ln(1-x)-\ln(1-y)}{y-x}, \quad x\neq y \tag{2.7}$$

and from (2.6), we have for $|x|<1$,

$$F_1(1,1,1;\, 2;\, x,x)={}_2F_1(1,2;\, 2;\, x)={}_1F_0(1;x)=(1-x)^{-1}. \tag{2.8}$$

Further, from (2.1) with $s=t=1$, for $|x|<1$, $|y|<1$, and $x\neq y$,

$$F_1(1,2,2;\,2;\,x,y)=\frac{y^2}{(y-x)^2(1-y)}+\frac{x^2}{(x-y)^2(1-x)}+\frac{2xy}{(y-x)^3}[\ln(1-y)-\ln(1-x)].$$

Also, we have,

$$F_1(1,2,2;\,2;\,x,x)={}_2F_1(1,4;\,2;\,x).$$

3. Proofs

We prepare the proof of Theorem 2.1 with the statements and proofs of two lemmas.

Lemma 3.1. *Let*

$$\alpha=\frac{y}{y-x},\qquad \beta=\frac{x}{x-y} \tag{3.1}$$

and

$$X=(1-ux)^{-1},\qquad Y=(1-uy)^{-1} \tag{3.2}$$

Then, for any positive integers s and t,

$$X^sY^t=\alpha^s\sum_{j=0}^{t-1}\binom{j+s-1}{s-1}\beta^jY^{t-j}+\beta^t\sum_{k=0}^{s-1}\binom{k+t-1}{t-1}\alpha^kX^{s-k}. \tag{3.3}$$

Proof. For $s=t=1$, we have

$$XY=\frac{1}{(1-ux)}\frac{1}{(1-uy)}=\frac{y}{(y-x)}\frac{1}{(1-uy)}+\frac{x}{(x-y)}\frac{1}{(1-ux)}$$
$$=\alpha Y+\beta X,$$

so that (3.3) holds for $s=t=1$. An inductive argument on the pair of positive integers (s,t), using the identity

$$\sum_{j=0}^{k}\binom{j+p}{p}=\binom{k+p+1}{p+1},$$

yields (3.3). □

Lemma 3.2. *Let X be defined by* (3.2). *Then, for* $p\neq 0$,

$$\int_0^1 u^pX^{q+1}\,\mathrm{d}u=x^{-p-1}\left[\sum_{\substack{j=0\\ j\neq q}}^{p}\binom{p}{j}(-1)^j\frac{[1-(1-x)^{j-q}]}{(j-q)}-\binom{p}{q}(-1)^q\ln(1-x)\right], \tag{3.4}$$

where the obvious terms disappear for $q \geqslant p$. *Also,*

$$\begin{aligned} \int_0^1 X^{q+1}\,\mathrm{d}u &= \frac{-\ln(1-x)}{x}, \quad q=0, \\ \int_0^1 X^{q+1}\,\mathrm{d}u &= \frac{-1}{xq}[1-(1-x)^{-q}], \quad q\neq 0. \end{aligned} \tag{3.5}$$

Proof. We have, for $p \neq 0$,

$$\begin{aligned} \int_0^1 u^p X^{q+1}\,\mathrm{d}u &= \int_0^1 \frac{u^p}{(1-ux)^{q+1}}\,\mathrm{d}u \\ &= x^{-p-1}\int_{1-x}^1 (1-w)^p w^{-q-1}\,\mathrm{d}w \\ &= x^{-p-1}\sum_{k=0}^{p}\binom{p}{k}(-1)^k\int_{1-x}^1 w^{k-q-1}\,\mathrm{d}w, \end{aligned}$$

from which (3.4) follows. For $p = 0$, the integration is similarly straightforward and yields (3.5). □

Remark. Replacing X by Y in (3.4) and (3.5) yields the same formulas with x replaced by y.

Proof of Theorem 2.1. (a) Putting $a=1$, $b=s+1$, $b'=t+1$ and $c=2$ in the integral representation (1.4), we obtain

$$F_1(1,s+1,t+1;\,2;\,x,y) = \int_0^1 X^{s+1}Y^{t+1}\,\mathrm{d}u, \tag{3.6}$$

where X, Y are defined in (3.2). Now, by (3.3), we have

$$\int_0^1 X^{s+1}Y^{t+1}\,\mathrm{d}u = \alpha^{s+1}\sum_{j=0}^{t}\binom{j+s}{s}\beta^j\int_0^1 Y^{t+1-j}\,\mathrm{d}u + \beta^{t+1}\sum_{k=0}^{s}\binom{k+t}{t}\alpha^k\int_0^1 X^{s+1-k}\,\mathrm{d}u \tag{3.7}$$

with α and β given by (3.1). Applying (3.5) to the right-hand side of (3.7), we obtain (2.1).

(b) Replacing a by $a+1$, b by $s+1$, b' by $t+1$ and c by $a+2$ in (1.4), we obtain

$$F_1(a+1,s+1,t+1;\,a+2;\,x,y) = \frac{\Gamma(a+2)}{\Gamma(a+1)}\int_0^1 u^a X^{s+1}Y^{t+1}\,\mathrm{d}u. \tag{3.8}$$

Now, from (3.3), we have

$$\int_0^1 u^a X^{s+1}Y^{t+1}\,\mathrm{d}u = \alpha^{s+1}\sum_{j=0}^{t}\binom{j+s}{s}\beta^j\int_0^1 u^a Y^{t+1-j}\,\mathrm{d}u + \beta^{t+1}\sum_{k=0}^{s}\binom{k+t}{t}\alpha^k\int_0^1 u^a X^{s+1-k}\,\mathrm{d}u. \tag{3.9}$$

Applying (3.4) and (3.5) to the right-hand side of (3.9), yields (2.2).

(c) Replacing a by $a+1$, b by $s+1$, b' by $t+1$ and c by $a+d+2$ in (1.4), and noting that $d \geqslant 0$, we have

$$F_1(a+1,s+1,t+1;\,a+d+2;\,x,y) = \frac{\Gamma(a+d+2)}{\Gamma(a+1)\Gamma(d+1)}\int_0^1 u^a(1-u)^d X^{s+1}Y^{t+1}\,\mathrm{d}u. \tag{3.10}$$

Expanding $(1-u)^d$ in its bionomial expansion and using (3.4) and (3.5), we obtain (2.3) from (3.10). □

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