Sparse interpolation, the FFT algorithm and FIR filters

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Abstract. In signal processing, the Fourier transform is a popular method to analyze the frequency content of a signal, as it decomposes the signal into a linear combination of complex exponentials with integer frequencies. A fast algorithm to compute the Fourier transform is based on a binary divide and conquer strategy.
In computer algebra, sparse interpolation is well-known and closely related to Prony's method of exponential fitting, which dates back to 1795.
In this paper we develop a divide and conquer algorithm for sparse interpolation and show how it is a generalization of the FFT algorithm.
In addition, when considering an analog as opposed to a discrete version of our divide and conquer algorithm, we can establish a connection with digital filter theory.

1 Sparse interpolation

Let the function $\phi(t)$ be given by

$$\phi(t) = \sum_{i=1}^{n} \alpha_i \exp(2\pi \mathrm{i} \mu_i t)$$

and let us consider the general nonlinear interpolation problem of the samples $\phi(t_j)$, given by

$$\phi(t_j) = \sum_{i=1}^{n} \alpha_i \exp(2\pi \mathrm{i} \mu_i j/M), \qquad j = 0, \ldots, 2n-1, \ldots \tag{1}$$

with

$$\sqrt{-1} = \mathrm{i}, \quad \text{distinct } \mu_i \in \mathbb{C}, \quad \alpha_i \in \mathbb{C} \setminus \{0\}, \quad |\mathrm{Re}(\mu_i)| < M/2, \quad t_j = j/M,$$

where, without loss of generality, $M \in \mathbb{N}$. A solution of this interpolation problem was already presented in 1795 in [1] and can also be found in [2, pp. 378–382]. Let us denote $\Omega_i = \exp(2\pi \mathrm{i} \mu_i/M)$, with $\Omega_i \neq \Omega_k$ when $i \neq k$ because $|\mathrm{Re}(\mu_i)| < M/2$. It is apparent that the data $\phi(t_j)$ are structured, namely

$$\phi(t_j) = \sum_{i=1}^{n} \alpha_i \Omega_i^j, \qquad j = 0, \ldots, 2n-1, \ldots \tag{2}$$

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We now want to obtain the values $\Omega_i, i = 1, \ldots, n$ and $\alpha_i, i = 1, \ldots, n$ from the $2n$ samples $\phi(t_j)$. From Ω_i the value μ_i can easily be deduced because $2\pi|\mathrm{Re}(\mu_i)|/M < \pi$ and hence no periodicity problem arises. Temporarily we assume that n is known. How n can be extracted from the samples is explained in Sect. 2.

Consider the polynomial

$$\prod_{i=1}^{n}(z - \Omega_i) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0 \tag{3}$$

with so far unknown coefficients $b_i, i = 1, \ldots, n$. Since the Ω_i are its zeroes, we find for $k \geq 0$,

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \alpha_i \Omega_i^k (\Omega_i^n + b_{n-1}\Omega_i^{n-1} + \cdots + b_0) \\ &= \sum_{i=1}^{n} \alpha_i \Omega_i^{n+k} + \sum_{j=0}^{n-1} b_j \left(\sum_{i=1}^{n} \alpha_i \Omega_i^{j+k} \right) \\ &= \phi(t_{k+n}) + \sum_{j=0}^{n-1} b_j \phi(t_{k+j}). \end{aligned}$$

In other words, we can conclude that the structured data $\phi(t_j)$ are linearly generated,

$$\begin{pmatrix} \phi(t_0) & \ldots & \phi(t_{n-1}) \\ \vdots & \ddots & \vdots \\ \phi(t_{n-1}) & \ldots & \phi(t_{2n-2}) \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix} = - \begin{pmatrix} \phi(t_n) \\ \vdots \\ \phi(t_{2n-1}) \end{pmatrix}. \tag{4}$$

This linear system allows us to compute the coefficients $b_i, i = 0, \ldots, n-1$ and actually compose the polynomial (3) having $\Omega_i, i = 1, \ldots, n$ as its zeroes. Let us now denote by $H_n^{(r)}$ the Hankel matrix

$$H_n^{(r)} = \begin{pmatrix} \phi(t_r) & \ldots & \phi(t_{r+n-1}) \\ \vdots & \ddots & \vdots \\ \phi(t_{r+n-1}) & \ldots & \phi(t_{r+2n-2}) \end{pmatrix}$$

and by $H_n^{(0)}(z)$ the Hankel polynomial [3, p. 625]

$$H_n^{(0)}(z) = \begin{vmatrix} \phi(t_0) & \ldots & \phi(t_{n-1}) & \phi(t_n) \\ \vdots & \ddots & \vdots & \vdots \\ \phi(t_{n-1}) & \ldots & \phi(t_{2n-2}) & \phi(t_{2n-1}) \\ 1 & \ldots & z^{n-1} & z^n \end{vmatrix}.$$

Then

$$\prod_{i=1}^{n}(z - \Omega_i) = \frac{H_n^{(0)}(z)}{|H_n^{(0)}|},$$

where $|H_n^{(0)}|$ denotes the determinant of $H_n^{(0)}$. From the matrix factorisations

$$H_n^{(0)} = V_n D_\alpha V_n^T,$$
$$H_n^{(1)} = V_n D_\alpha \begin{pmatrix} \Omega_1 & & \\ & \ddots & \\ & & \Omega_n \end{pmatrix} V_n^T,$$

where V_n and D_α respectively denote the Vandermonde matrix

$$V_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \Omega_1 & \Omega_2 & \dots & \Omega_n \\ \vdots & \vdots & & \vdots \\ \Omega_1^{n-1} & \Omega_2^{n-1} & \dots & \Omega_n^{n-1} \end{pmatrix}$$

and the diagonal matrix

$$D_\alpha = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix},$$

it is easy to see that the polynomial zeroes Ω_i can also be obtained as generalized eigenvalues [4,5]. So the Ω_i also satisfy

$$\det\left(H_n^{(1)} - \Omega_i H_n^{(0)}\right) = 0, \qquad i = 1, \dots, n. \tag{5}$$

The coefficients α_i in the model (1) can be obtained from any set of n interpolation conditions taken from (2),

$$\begin{pmatrix} \Omega_1^j & \dots & \Omega_n^j \\ \vdots & & \vdots \\ \Omega_1^{j+n-1} & \dots & \Omega_n^{j+n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \phi(t_j) \\ \vdots \\ \phi(t_{j+n-1}) \end{pmatrix}, \qquad 0 \le j \le n. \tag{6}$$

With Ω_i computed as above, the remaining equations are linearly dependant.

Whether solving (4) or (5), the Hankel matrices involved tend to become quite ill-conditioned when n increases [6,7]. So in practice, one may be interested in a divide and conquer approach where the full system is divided into several smaller systems, thus keeping the condition number under control. In Sect. 2 we present such an algorithm, which we connect to the traditional FFT in Sect. 3. Our goal is not to incorporate sparsity considerations into the FFT algorithm as in [8], but rather to add the divide and conquer approach of the FFT to sparse interpolation. Related work can be found in [9] where digital filters are used as a splitting technique and Prony's method is used to solve for the non-filtered μ_i.

So here the classical FFT algorithm will appear as a special case, when restricting the μ_i to integer values. In its most general form, with μ_i complex, our formula is related to a comb filter. The former is the subject of the Sect. 2 and 3, while the latter is discussed in the Sect. 4 and 5.

2 Divide and conquer approach

In this section we assume for simplicity that $\mathrm{Re}(\mu_i) \in \mathbb{Z}$ and we introduce $\omega = \exp(2\pi \mathrm{i}/N)$ with the integer $N > 0$. In addition we require that N divides M, thus guaranteeing that $M/N \in \mathbb{N}$. From our samples $\phi(t_j)$ we now deduce N linear combinations $\phi_k(t_j)$ by the construction [10, pp. 15–17]

$$\phi_k(t_j) := \frac{1}{N} \sum_{\ell=0}^{N-1} \omega^{k\ell} \phi(t_j + \ell/N), \qquad k = 0, \ldots, N-1. \tag{7}$$

These $\phi_k(t_j)$ are linear combinations of already collected samples $\phi(t_{j+M\ell/N})$ since $t_j + \ell/N$ can be expressed as $(j + M\ell/N)/M$. Figure 1 graphically illustrates formula (7). Each derived sample contains only some of the original components

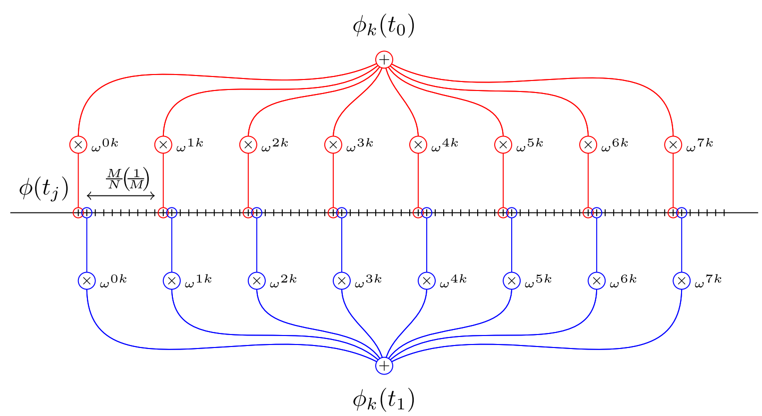

Fig. 1: *Formula* (7) *with* $M = 80$ *and* $N = 8$.

of (1), as can be seen from the rearrangement

$$\begin{aligned}
\phi_k(t_j) &= \frac{1}{N} \sum_{\ell=0}^{N-1} \omega^{k\ell} \phi(t_{j+M\ell/N}) \\
&= \frac{1}{N} \sum_{\ell=0}^{N-1} \omega^{k\ell} \sum_{i=1}^{n} \alpha_i \exp\left(2\pi \mathrm{i} \mu_i (j/M + \ell/N)\right) \\
&= \frac{1}{N} \sum_{\ell=0}^{N-1} \omega^{k\ell} \sum_{i=1}^{n} \alpha_i \exp(2\pi \mathrm{i} \mu_i t_j) \omega^{\ell \mu_i} \\
&= \frac{1}{N} \sum_{i=1}^{n} \alpha_i \exp(2\pi \mathrm{i} \mu_i t_j) \left(\sum_{\ell=0}^{N-1} \omega^{\ell(k+\mu_i)} \right).
\end{aligned} \tag{8}$$

We remark that

$$\sum_{\ell=0}^{N-1} \omega^{\ell(k+\mu_i)} = N \text{ if } \operatorname{mod}(k+\mu_i, N) = 0,$$
$$\sum_{\ell=0}^{N-1} \omega^{\ell(k+\mu_i)} = 0 \text{ otherwise.} \tag{9}$$

So actually, every component of the original exponential sum (1) is present in one and only one linear combination ϕ_k. When $\mathrm{Re}(\mu_i) \in \mathbb{Z}$ formula (7) allows a perfect split of (1) over N smaller sized problems. Since each ϕ_k has the same exponential structure as (1), we can apply (4) or (5) to it and identify the parameters α_i and μ_i present in ϕ_k from the values $\phi_k(t_j)$. And this for each smaller exponential sum $\phi_k, k = 0, \ldots, N-1$.

But (7) also remains valid for general $\mu_i \in \mathbb{C}$ as it is merely a linear combination of the samples taken at equidistant points. In Sect. 4 we see that, what changes when going from $\mathrm{Re}(\mu_i) \in \mathbb{Z}$ to $\mu_i \in \mathbb{C}$, is that the factor

$$\sum_{\ell=0}^{N-1} \omega^{\ell(k+\mu_i)}$$

that accompanies each term in a particular $\phi_k(t_j)$ is replaced by expression (13) of which the behaviour is illustrated in Fig. 2.

Let us now discuss the number of terms in each of the ϕ_k and for this we first consider the detection of n in (1) which we didn't touch in Sect. 1. In an exact (noisefree) context, the value of n can simply be detected from the theorems given in [3, p. 603] and [11, pp. 20–31]:

$$\det H_n^{(r)} \neq 0,$$
$$\det H_\nu^{(r)} = 0, \qquad \nu > n,$$

It is analyzed in [12] that when $\nu < n$, the value $\det H_\nu^{(r)}$ is not guaranteed zero as for $\nu > n$, or guaranteed nonzero as for $\nu = n$, but can vanish accidentally when by the choice of M and r one hits a zero of this expression. From these statements the number of components n can be obtained as the rank of $H_\nu^{(r)}$ for $\nu > n$. In order to inspect $|H_\nu^{(r)}|$ for $\nu > n$, additional samples up to $t_{r+2\nu-2}$ need to be provided, in other words at least the additional sample $\phi(t_{2n})$ in case $r = 0$ and $\nu = n + 1$.

The smaller exponential interpolation problems built with the values $\phi_k(t_j)$ for each k separately, may contain less exponential terms and hence their Hankel matrices

$$H_{n,k}^{(r)} = \begin{pmatrix} \phi_k(t_r) & \ldots & \phi_k(t_{r+n-1}) \\ \vdots & \ddots & \vdots \\ \phi_k(t_{r+n-1}) & \ldots & \phi_k(t_{r+2n-2}) \end{pmatrix}$$

may have a rank smaller than n. For each $k = 0, \ldots, N-1$, the rank of $H^{(r)}_{\nu,k}$ is less than or equal to n and the sum of these ranks equals exactly n.

We present a small example to illustrate the principle of (7). Let (1) be defined by the values for α_i and μ_i given in Table 1.

Table 1: *Ill-conditioned example of* (1).

$\mathrm{Re}(\mu_i)$	5	6	7	8	9	45	-10	-33
$\mathrm{Im}(\mu_i)$	0	0	0	0	0	0	0	0
$\mid \alpha_i \mid$	1	1	1	1	1	1	1	1
$\arg(\alpha_i)$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$

With $M = 100$ and $n = 8$ the Hankel matrix $H^{(0)}_n$ has a condition number of the magnitude 7.7×10^9! In [13] oversampling is used as a means to reduce the condition number. Here we use (8) to split the exponential analysis problem and bring the condition number down. We take $N = 5$. Each of the samples $\phi_k(t_j)$ for $k = 0, \ldots, 4$ involves only a subset of the original components $\exp(2\pi \mathrm{i} \mu_i t_j)$, as detailed in Table 2.

Table 2: *Example from Table 1 split into* $N = 5$ *subsets.*

k	$\mathrm{Re}(\mu_i)$			condition nr
0	5	45	-10	2.2×10^0
1	9			1.0×10^0
2	8			1.0×10^0
3	7	-33		1.4×10^0
4	6			1.0×10^0

The major improvement in the conditioning is not only due to the reduction in size of the Hankel matrices involved, but also to a much better disposition in the complex plane of the frequencies μ_i per subsum.

3 The FFT algorithm

An algorithm related to formula (7) is the FFT algorithm which retrieves the coefficients α_i from a set of samples $\phi(t_j), j = 0, \ldots, M-1$ given by

$$\phi(t_j) = \sum_{i=1}^{M} \alpha_i \exp(2\pi \mathrm{i} ij/M). \qquad (10)$$

The difference between (10) and (1) is that now all integer frequencies appear, so $\mu_i = i$, and that therefore the number of terms in the sum equals M, which is also the number of samples. The coefficients α_i in (10) are called Fourier coefficients. In a way, (7) is a generalization of the FFT to sparse interpolation or Prony's algorithm as we now explain in some more detail.

Let $M = N_1 \times \cdots \times N_m$ with all $N_k \in \mathbb{N}$. Then the FFT algorithm breaks down the set of samples (10) into new different sets as follows. We detail the first divide of $\phi(t_j)$ into N_1 smaller exponential sums, starting from (8). For $\mu_i = i$ and $n = M$, we find from (8):

$$\phi_k(t_j) = \frac{1}{N_1} \sum_{i=1}^{M} \alpha_i \exp(2\pi \mathrm{i} i j/M) \left(\sum_{\ell=0}^{N_1-1} \omega^{\ell(k+i)} \right), \qquad k = 0, \ldots, N_1 - 1$$

where

$$\frac{1}{N_1} \sum_{\ell=0}^{N_1-1} \omega^{\ell(k+i)}$$

evaluates to either 0 or 1. Preserving only the terms that are not multiplied by zero, leads to

$$\begin{aligned} \phi_k(t_j) &= \sum_{i=1}^{M/N_1} \alpha_{1+(i-1)N_1+k} \exp(2\pi \mathrm{i} j(1+(i-1)N_1+k)/M) \\ &= \sum_{i=1}^{M/N_1} \alpha_{1+(i-1)N_1+k} \exp(2\pi \mathrm{i} j i N_1/M) \exp(2\pi \mathrm{i} j(1-N_1+k)/M) \\ &= \sum_{i=1}^{M/N_1} \alpha_{1+(i-1)N_1+k} \exp(2\pi \mathrm{i} i j/(M/N_1)) \exp(2\pi \mathrm{i} j(1-N_1+k)/M) \\ & \qquad k = 0, \ldots, N_1 - 1 \end{aligned} \tag{11}$$

The subsequent step in which each smaller sum is divided into N_2 new smaller sums is obvious for $k = N_1 - 1$, but the other ϕ_k first need to be multiplied by the so-called twiddle factor $\exp(-2\pi \mathrm{i} j(1-N_1+k)/M)$ in order to bring them in the correct form (1). For the subdivision of each of the N_1 sums into N_2 yet smaller sums, one substitutes in (11) and the expression for the twiddle factors, M by M/N_1 and N_1 by N_2. In this way one continues until the algorithm has created M sums each containing only one component of the form $\alpha_i \exp(2\pi \mathrm{i} i j/M)$. Thus at the final stage each single component immediately reveals the coefficient α_i.

The case where $M = 2^m$ is of particular interest because then (8) and (11) simplify even further ($\omega = \exp(\pi \mathrm{i}) = -1$) into

$$\phi_k(t_j) = \frac{1}{2} \sum_{\ell=0}^{1} (-1)^{\ell k} \phi(t_j + \ell/2), \qquad k = 0, 1.$$

4 An analog version of the splitting technique

We now consider a generalization of (7) when it does not make sense to require that the $\mathrm{Re}(\mu_i)$ be integer, as we did in the discrete case. To this end we introduce, in addition to $\omega = \exp(2\pi \mathrm{i}/N)$,

$$\Omega = \omega\kappa, \qquad ||\kappa|| = 1.$$

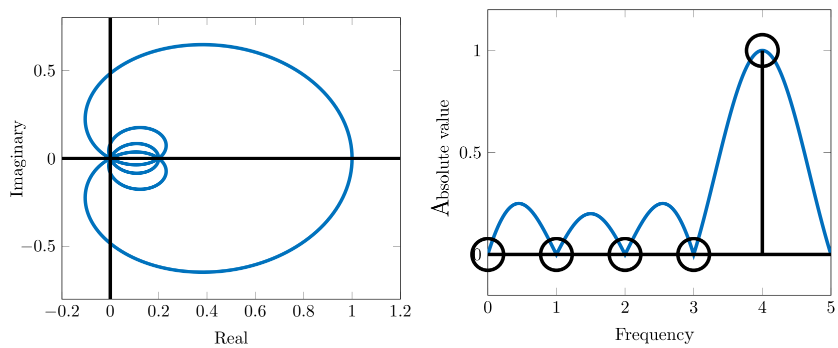

Fig. 2: *The functions* $\mathcal{M}_4(1, \mu)/N$ *(left) and* $|\mathcal{M}_4(1, \mu)/N|$ *(right) for* $\mu \in [0, 5]$.

The samples $\phi_k(t_j)$ derived from the samples $\phi(t_j)$ are then defined by the following continuous analogon of (7):

$$\begin{aligned}
\phi_k(t_j) &= \frac{1}{N} \sum_{\ell=0}^{N-1} \Omega^{k\ell} \phi(t_j + \ell/N) \\
&= \frac{1}{N} \sum_{\ell=0}^{N-1} \Omega^{k\ell} \sum_{i=1}^{n} \alpha_i \exp\left(2\pi \mathrm{i}\mu_i j/M + 2\pi \mathrm{i}\mu_i \ell/N\right) \\
&= \frac{1}{N} \sum_{\ell=0}^{N-1} \Omega^{k\ell} \sum_{i=1}^{n} \alpha_i \exp(2\pi \mathrm{i}\mu_i j/M) \omega^{\ell\mu_i} \\
&= \frac{1}{N} \sum_{i=1}^{n} \alpha_i \exp(2\pi \mathrm{i}\mu_i j/M) \sum_{\ell=0}^{N-1} \omega^{\ell(k+\mu_i)} \ \kappa^{\ell k} \\
&= \frac{1}{N} \sum_{i=1}^{n} \alpha_i \exp(2\pi \mathrm{i}\mu_i j/M) \mathcal{M}_k(\kappa, \mu_i), \qquad k = 0, \ldots, N-1, \qquad (12)
\end{aligned}$$

where $\mathcal{M}_k(\kappa, \mu)$, for fixed N, is defined by

$$\mathcal{M}_k(\kappa, \mu) := \frac{1 - \left(\omega^{k+\mu} \kappa^k\right)^N}{1 - \omega^{k+\mu} \kappa^k}. \qquad (13)$$

In case $\kappa = 1$ formula (12) coincides with (8). However, the value of (13) does not reduce to 0 or N as in (9). By (12) all integer frequencies $\mathrm{Re}(\mu_i)$ are either zeroed or copied to ϕ_k, as in (9), while the non-integer frequencies inbetween are amplified as in Fig. 2, where we illustrate (12) for $\kappa = 1, N = 5$ and $\mathrm{Re}(\mu_i) \in [0, 5]$. The function $\mathcal{M}_k(\kappa, \mu)$ is periodic, and in Fig. 2 the period equals 5. The effect on the integer frequencies $\mu = i, i = 0, \ldots, 5$ is accentuated in the graph at the bottom in Fig. 2.

The complex number $\kappa = \exp(2\pi \mathrm{i}\theta)$ on the unit circle acts as a continuous shifter of $\mathrm{Re}(\mu_i)$, as shown in Fig. 3. Increasing k to $k + 1$ in (7) can also be achieved by choosing $\kappa = \exp(2\pi \mathrm{i}/N)$ in (12).

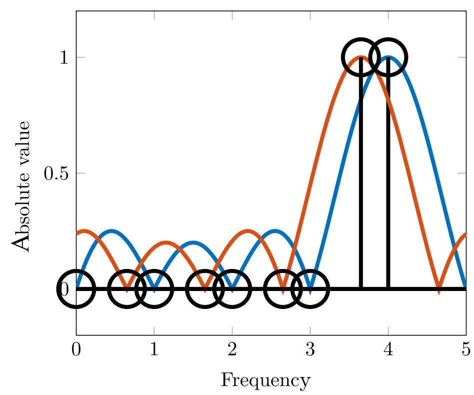

Fig. 3: *Influence of the parameter κ while N and ω are kept equal in both $\mathcal{M}_k$ graphs.*

Table 3: *Analog divide and conquer illustration.*

$\mathrm{Re}(\mu_i)$	5	6	7.3	8	9.5	45	-10	-33
$\mathrm{Im}(\mu_i)$	0	0	-0.1	0	-0.001	0	0	0
$\mid \alpha_i \mid$	1	1	1	1	1	1	1	1
$\arg(\alpha_i)$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$

We repeat the example of Sect. 2 where the data have now been altered so that $\mathrm{Re}(\mu_i) \notin \mathbb{Z}$ and $\mathrm{Im}(\mu_i) \neq 0$. The new data can be found in Table 3. We take a look at $\phi_4(t_j)$ given by (7) and (12) but with the μ_i from Table 3 and with $\kappa = 1$. The components in $\phi_4(t_j)$ are now multiplied by $\mathcal{M}_4(1, \mu_i)/N$. So none of the non-integer frequencies is annihilated. The μ_i with non-integer real parts are weakened in modulus as indicated in Table 4. By repeating the

Table 4: *Analysis of $\phi_4(t_j)$ for μ_i from Table 3.*

$\mathrm{Re}(\mu_i)$	7.3	9.5
$\mathrm{Im}(\mu_i)$	-0.1	-0.001
$\mid \alpha_i \mathcal{M}_4(1,\mu_i)/N \mid$	0.1361	0.6456

mutiplication with $\mathcal{M}_4(1,\mu_i)/N$ this effect is strengthened. In order to retrieve the correct α_i, the coefficient of $\exp(2\pi \mathrm{i} \mu_i j/M)$ in $\phi_4(t_j)$ which can be obtained using a standard exponential analysis needs to be multiplied by $N/\mathcal{M}_4(\kappa,\mu_i)$. The effect of $\mathcal{M}_4(1,\mu)$ is graphically illustrated in Fig. 4.

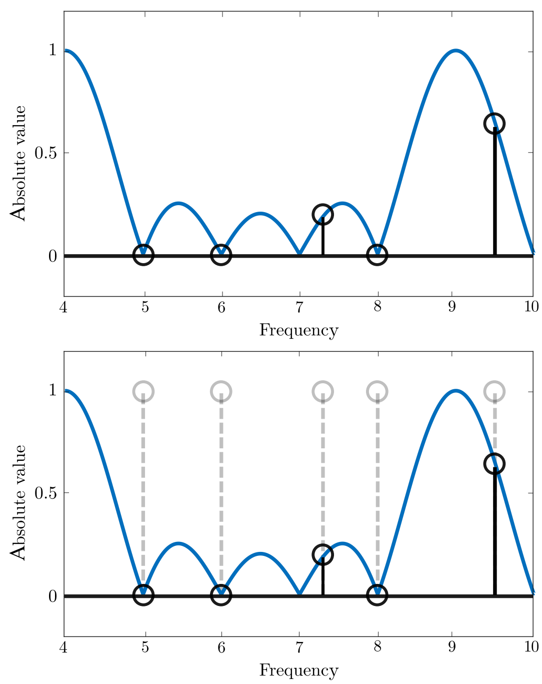

Fig. 4: *Effect of the function $\mathcal{M}_4(1,\mu)/N$ on the frequencies in Table 3.*

5 Connection to FIR filters

We want to illustrate how formula (12) can be interpreted as the result of a digital filter. In general, a digital filter takes a set of samples as input, applies a transform and delivers another set of samples as output. In a finite impulse response or FIR filter the output samples are a linear combination of the present and previous input samples. If we denote the filter coefficients by β_ℓ and the sampling distance is $1/M$, then the filtered signal $\psi(t_j)$ equals

$$\psi(t_j) = \sum_{\ell=0}^{L-1} \beta_\ell \phi(t_j - \ell/M).$$

When the input signal is the unit impulse $\delta(\cdot)$ where δ is the Kronecker delta function, then the output signal is called the impulse response $h(t_j)$ given by

$$h(t_j) = \sum_{\ell=0}^{L-1} \beta_\ell \delta(t_j - \ell/M) = \beta_j, \qquad t_j = j/M.$$

The transfer function associated with the FIR filter ψ equals

$$H(z) = \sum_{\ell=0}^{L-1} \beta_\ell z^{-\ell}.$$

In order to establish a link with formula (12), we define for k fixed and $\ell = 0, \ldots, L-1 = M-1$, (remember that N divides M),

$$\beta_{\ell k} := \begin{cases} \dfrac{1}{N}\ \Omega^{k(N-(\ell+1)/(M/N))}, & (\ell+1)/(M/N) \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

When putting the $\beta_{\ell k}$ for fixed k in a vector, they are structured in N blocks of size M/N, each block containing $M/N - 1$ zeroes and one power of Ω^k:

$$\frac{1}{N}\left(0, \ldots, 0, \Omega^{(N-1)k}, 0, \ldots, 0, \Omega^k, 0, \ldots, 0, \Omega^0\right)$$

Since formula (12) is based on the current and future samples, we also need to shift the signal in order to fit the filter description:

$$\overline{\phi}(t_j) := \phi(t_j + (1 - 1/M)).$$

Then

$$\psi(t_j) = \phi_k(t_j) = \sum_{\ell=0}^{M-1} \beta_{\ell k} \overline{\phi}(t_j - \ell/M). \tag{14}$$

The impulse response of the filter (12), rewritten as (14), is given by

$$h_k(t_j) = \beta_{jk}.$$

The filter (12) gets a crisper look, meaning that it is flatter in the neighborhood of the zeroes and exhibits a sharper peak where it attains one, when applied iteratively. In Fig. 5 we show the result of (12) applied once (as in Fig. 2), twice and five times, reminding us more and more of a comb filter [14, p. 474].

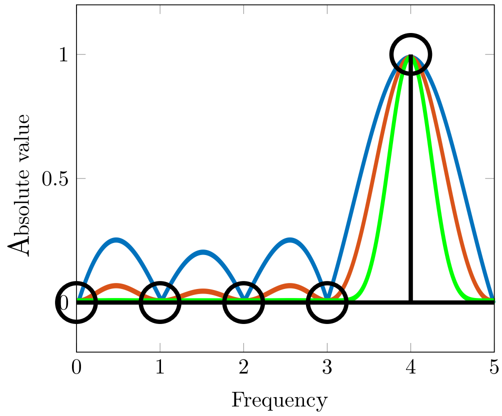

Fig. 5: *FIR filter* (12) *applied once, twice and five times.*

6 Conclusion

Sparse interpolation, which is a special case of multi-exponential analysis, can be combined with a divide and conquer technique which is a direct generalization of the fast Fourier transform algorithm in case the frequencies belong to a discrete set. This connection opens up new computational possibilities in the fitting of sparse models to data.

An analog version of our general divide and conquer method is related to digital filter theory, more precisely FIR filter theory.

References

1. de Prony, R.: Essai expérimental et analytique sur les lois de la dilatabilité des fluides élastiques et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures. J. Ec. Poly. **1** (1795) 24–76
2. Hildebrand, F.: Introduction to numerical analysis. Mc Graw Hill, New York (1956)
3. Henrici, P.: Applied and computational complex analysis I. John Wiley & Sons, New York (1974)
4. Hua, Y., Sarkar, T.K.: Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise. IEEE Transactions on Acoustics, Speech, and Signal Processing **38**(5) (1990) 814–824
5. Golub, G., Milanfar, P., Varah, J.: A stable numerical method for inverting shape from moments. SIAM J. Sci. Comput. **21** (1999) 1222–1243
6. Beckermann, B.: The condition number of real Vandermonde, Krylov and positive definite Hankel matrices. Numer. Math. **85** (2000) 553–577
7. Beckermann, B., Golub, G., Labahn, G.: On the numerical condition of a generalized Hankel eigenvalue problem. Numer. Math. **106**(1) (2007) 41–68
8. Potts, D., Tasche, M., Volkmer, T.: Efficient spectral estimation by MUSIC and ESPRIT with application to sparse FFT. Frontiers in Applied Mathematics and Statistics **2** (2016)

9. Heider, S., Kunis, S., Potts, D., Veit, M.: A sparse Prony FFT. Proc. 10th International Conference on Sampling Theory and Applications (SAMPTA) (2013) 572–575
10. Cuyt, A., Lee, W.-s.: Smart sampling and sparse reconstruction. GB Priority 1114255.1, (filed on 18.08.2011, published on 21.02.2013). WIPO Patentscope, `https://patentscope.wipo.int/search/docservicepdf_pct/id00000020191665/PDOC/WO2013024177.pdf`
11. Baker, Jr., G., Graves-Morris, P.: Padé approximants (2nd Ed.). Volume 59 of Encyclopedia of Mathematics and its Applications. Cambridge University Press (1996)
12. Kaltofen, E., Lee, W.-s., Lobo, A.A.: Early termination in Ben-Or/Tiwari sparse interpolation and a hybrid of Zippel's algorithm. In: Proceedings of the 2000 International Symposium on Symbolic and Algebraic Computation, New York, NY, USA, ACM (2000) 192–201
13. Potts, D., Tasche, M.: Parameter estimation for nonincreasing exponential sums by Prony-like methods. Linear Algebra and its Applications **439**(4) (2013) 1024 – 1039 17th Conference of the International Linear Algebra Society, Braunschweig, Germany, August 2011.
14. Schlichthärle, D.: Digital Filters. Springer Berlin Heidelberg, Berlin, Heidelberg (2011) DOI: 10.1007/978-3-642-14325-0.