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Properties of Multivariate Homogeneous Orthogonal Polynomials

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It is well-known that the denominators of Padé approximants can be considered as orthogonal polynomials with respect to a linear functional. This is usually shown by defining Padé-type approximants from so-called generating polynomials and then improving the order of approximation by imposing orthogonality conditions on the generating polynomials.

In the multivariate case, a similar construction is possible when dealing with the multivariate homogeneous Padé approximants introduced by the second author. Moreover it is shown here, that several well-known properties of the zeroes of classical univariate orthogonal polynomials, in the case of a definite linear functional, generalize to the multivariate homogeneous case. For the multivariate homogeneous orthogonal polynomials, the absence of common zeroes is translated to the absence of common factors.

Key Words: orthogonal polynomials; multivariate; zero properties; Padé-type approximants.

1. THE UNIVARIATE SITUATION

It has been well-known for a long time that denominators of Padé approximants can be considered as orthogonal polynomials with respect to a linear functional. This is usually shown by defining Padé-type approximants from so-called generating polynomials and then improving the order of approximation by imposing orthogonality conditions on the generating polynomials.

Assume you are given a series development

$$f(t) = \sum_{i=0}^{\infty} c_i t^i.$$

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By defining the linear functional c acting on the space of univariate polynomials as

$$c(x^i)=c_i$$

$f(t)$ can formally be rewritten as

$$f(t)=c\left(\frac{1}{1-xt}\right).$$

Now take any polynomial $V_m(t)$ of degree m and define its associated polynomial

$$W_m(t)=c\left(\frac{V_m(x)-V_m(t)}{x-t}\right)$$

which is then a polynomial of degree $m-1$. Then for

$$\tilde{V}_m(t)=t^m V_m(t^{-1})$$
$$\tilde{W}_m(t)=t^{m-1} W_m(t^{-1})$$

the Padé-type approximation conditions

$$(f\tilde{V}_m-\tilde{W}_m)(t)=\sum_{i=m}^{\infty} d_i t^i$$

hold. If we do not choose V_m randomly, but impose the conditions

$$c(x^i V_m(x))=0 \qquad i=0, ..., m-1 \tag{1}$$

then $V_m(t)$ is called the orthogonal polynomial of degree m with respect to the functional c (the conditions under which V_m can be computed from (1) are well-known [3]). For $V_m(t)$ satisfying (1) the Padé approximation conditions

$$(f\tilde{V}_m-\tilde{W}_m)(t)=\sum_{i=2m}^{\infty} d_i t^i$$

hold (here $V_m(t)$ is normalized such that it is monic). The Padé-type and Padé approximants for f of degree $m-1$ in the numerator and m in the denominator are usually denoted by $(m-1/m)^f$ and $[m-1/m]^f$ respectively.

The construction of the Padé-type and Padé approximants $(m+k/m)^f$ and $[m+k/m]^f$ with $k \geqslant -1$ explained next, follows the same lines. The series for $f(t)$ can be written as

$$f(t) = \sum_{i=0}^{k} c_i t^i + t^{k+1} f_k(t)$$

with

$$f_k(t) = \sum_{i=0}^{k} c_{k+1+i} t^i.$$

If we define the functional $c^{(k+1)}$ by

$$c^{(k+1)}(x^i) = c_{k+1+i}$$

and the polynomials

$$W_m^{(k+1)}(t) = c^{(k+1)}\left(\frac{V_m(x) - V_m(t)}{x-t}\right)$$

$$\tilde{W}_m^{(k+1)}(t) = \tilde{V}_m(t) \sum_{i=0}^{k} c_i t^i + t^{k+1} t^{m-1} W_m^{(k+1)}(t^{-1})$$

then

$$(f\tilde{V}_m - \tilde{W}_m^{(k+1)})(t) = \sum_{i=m+k+1}^{\infty} d_i t^i$$

and so everything remains valid with the functional c replaced by $c^{(k+1)}$. Note that

$$(m+k/m)^f(t) = \frac{\tilde{W}_m^{(k+1)}}{\tilde{V}_m}(t) = \sum_{i=0}^{k} c_i t^i + t^{k+1} (m-1/m)^{f_k}(t).$$

The additional conditions on V_m necessary for the construction of the Padé approximant $[m+k/m]^f$ are

$$c^{(k+1)}(x^i V_m(x)) = 0 \qquad i = 0, ..., m-1 \tag{2}$$

and we shall denote polynomials V_m satisfying (2) by $V_m^{(k+1)}$ so that $[m+k/m]^f = \tilde{W}_m^{(k+1)}/\tilde{V}_m^{(k+1)}$. The ratio $\tilde{W}_m/\tilde{V}_m$ which was introduced for the special case $k = -1$, will usually be denoted by $\tilde{W}_m^{(0)}/\tilde{V}_m^{(0)}$. The same holds for the functional c that can be denoted by $c^{(0)}$.

2. THE MULTIVARIATE HOMOGENEOUS SITUATION

In the multivariate case, a similar construction is possible to obtain the multivariate homogeneous Padé approximants $[m+k/m]_H^f$ introduced by Cuyt in [5]. We give a different and slightly more elegant presentation than the one in [1, 2, 8]. We restrict our description to the bivariate case only for the reason of notational simplicity.

Assume you are given a bivariate series development

$$f(t, s) = \sum_{i, j=0}^{\infty} c_{ij} t^i s^j.$$

For completeness we repeat that the multivariate homogeneous Padé approximant $[m-1/m]_H^f$ is defined as the irreducible form of $P_{m-1, m}/Q_{m-1, m}$ with

$$\begin{aligned} P_{m-1, m}(t, s) &= \sum_{i+j=(m-1)m}^{(m-1)m+m-1} a_{ij} t^i s^j \\ Q_{m-1, m}(t, s) &= \sum_{i+j=(m-1)m}^{(m-1)m+m} b_{ij} t^i s^j \\ (fQ_{m-1, m} - P_{m-1, m})(t, s) &= \sum_{i+j=(m-1)m+2m}^{\infty} d_{ij} t^i s^j. \end{aligned} \tag{3}$$

One of the great advantages of this homogeneous definition is that it results in a unique irreducible form, whatever solution of (3) is considered. Note that the numerator and denominator polynomials $P_{m-1, m}(t, s)$ and $Q_{m-1, m}(t, s)$ start with terms of degree $(m-1)m$ instead of with a constant term. When computing $[m-1/m]_H^f$, in other words taking the irreducible form of $P_{m-1, m}/Q_{m-1, m}$, the numerator and denominator polynomials of $[m-1/m]_H^f$ may start with a constant term but this is not guaranteed. If we denote the order of the denominator polynomial of $[m-1/m]_H^f$ (lowest homogeneous degree of its terms) by Δ_m then

$$0 \leqslant \Delta_m \leqslant (m-1)m$$

and we can show that the order of the numerator polynomial of $[m-1/m]_H^f$ is at least Δ_m. In the rest of the text $\Delta_m = (m-1)m$ is used.

By defining the linear functional C acting on the space of bivariate polynomials, as

$$\binom{i+j}{j} C(x^i y^j) = c_{ij}$$

the bivariate series can formally be rewritten as

$$f(t, s) = C\left(\frac{1}{1 - xt - ys}\right).$$

By introducing the notations

$$(t, s) = (\lambda_1 u, \lambda_2 u) \qquad t, s, u \in \mathbb{C} \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \quad \|\lambda\|_p = 1$$

$$C_i(t, s) = \sum_{j=0}^{i} c_{i-j, j} t^{i-j} s^j$$

$$c_i(\lambda) = \sum_{j=0}^{i} c_{i-j, j} \lambda_1^{i-j} \lambda_2^j$$

$$a_i(\lambda) = \sum_{j=0}^{i} a_{i-j, j} \lambda_1^{i-j} \lambda_2^j$$

$$b_i(\lambda) = \sum_{j=0}^{i} b_{i-j, j} \lambda_1^{i-j} \lambda_2^j$$

$$d_i(\lambda) = \sum_{j=0}^{i} d_{i-j, j} \lambda_1^{i-j} \lambda_2^j,$$

where $\|\cdot\|_p$ is one of the Minkowski-norms on $\mathbb{C}^2$, we can rewrite the series development as

$$f(t, s) = \sum_{i=0}^{\infty} C_i(t, s) = \sum_{i=0}^{\infty} c_i(\lambda)\, u^i$$

and (3) as

$$\begin{aligned} P_{m-1, m}(\lambda_1 u, \lambda_2 u) &= \sum_{i=(m-1)m}^{(m-1)m+m-1} a_i(\lambda)\, u^i \\ Q_{m-1, m}(\lambda_1 u, \lambda_2 u) &= \sum_{i=(m-1)m}^{(m-1)m+m} b_i(\lambda)\, u^i \\ (fQ_{m-1, m} - P_{m-1, m})(\lambda_1 u, \lambda_2 u) &= \sum_{i=(m-1)m+2m}^{\infty} d_i(\lambda)\, u^i. \end{aligned} \tag{4}$$

With the introduction of the functional Γ acting on the variable z, as

$$\Gamma(z^i) = c_i(\lambda)$$

the series can now formally also be viewed as

$$f(t, s) = f(\lambda_1 u, \lambda_2 u) = \Gamma\left(\frac{1}{1 - zu}\right). \tag{5}$$

This new view on the multivariate problem in which the cartesian coordinates (t, s) are replaced by the coordinates $\lambda = (\lambda_1, \lambda_2)$ and u, with $\|\lambda\|_p = 1$ will turn out to be a powerful tool in the sequel of the text. It is strongly linked to the following two features of homogeneous multivariate Padé approximants:

- the most striking element in the definition (4) of the homogeneous multivariate Padé approximant $[m+k/m]_H^f$ is that this definition coincides with that of the univariate Padé approximant if you discard the shift in the numerator and denominator degrees and replace the homogeneous expressions by monomials [4, 6];
- the homogeneous Padé approximants apparently satisfy a very strong projection property that we want to exploit here, reducing to univariate Padé approximants on every straight line through the origin [6], namely

$$[m+k/m]_H^f(\lambda_1 t, \lambda_2 t) = [m+k/m]^{f_{\lambda_1, \lambda_2}}(t)$$

with

$$f_{\lambda_1, \lambda_2}(t) = f(\lambda_1 t, \lambda_2 t).$$

In order to clarify the presentation and underline the similarity with the univariate case we again start with the construction of the homogeneous Padé-type approximants $(m-1/m)_H^f$ and shall only afterwards deal with the more general $(m+k/m)_H^f$.

Let us first introduce some notations. We denote by $\mathbb{C}[u]$ the linear space of polynomials in the variable u with complex coefficients, by $\mathbb{C}[\lambda_1, \lambda_2]$ the linear space of bivariate polynomials in λ_1 and λ_2 with complex coefficients, by $\mathbb{C}(\lambda_1, \lambda_2)$ the commutative field of rational functions in λ_1 and λ_2 with complex coefficients, by $\mathbb{C}(\lambda_1, \lambda_2)[u]$ the linear space of polynomials in the variable u with coefficients from $\mathbb{C}(\lambda_1, \lambda_2)$ and by $\mathbb{C}[\lambda_1, \lambda_2][u]$ the linear space of polynomials in the variable u with coefficients from $\mathbb{C}[\lambda_1, \lambda_2]$.

For chosen m and Δ_m defined above, we take any function $V_m(t,s)$ of the form

$$\begin{aligned} V_m(t,s) = \mathcal{V}_m(u) &= \sum_{i=0}^{m} B_{\Delta_m+m-i}(\lambda)\, u^i \\ B_{\Delta_m+m-i}(\lambda) &= \sum_{j=0}^{\Delta_m+m-i} b_{\Delta_m+m-i-j,\, j} \lambda_1^{\Delta_m+m-i-j} \lambda_2^j \end{aligned} \tag{6a}$$

defined parametrically in terms of the coefficients $b_{\ell j}$ with $\ell + j$ ranging from Δ_m to $\Delta_m + m$. The function $V_m(t,s)$ is a polynomial of degree m in u with homogeneous polynomial coefficients from $\mathbb{C}[\lambda_1, \lambda_2]$. We define

$$W_m(t,s) = \mathcal{W}_m(u) = \Gamma\left(\frac{\mathcal{V}_m(z) - \mathcal{V}_m(u)}{z-u}\right)$$

which is then of the form

$$\begin{aligned} W_m(t,s) = \mathcal{W}_m(u) &= \sum_{i=0}^{m-1} A_{\Delta_m+m-1-i}(\lambda)\, u^i \\ A_{\Delta_m+m-1-i}(\lambda) &= \sum_{j=0}^{m-1-i} B_{\Delta_m+m-1-i-j}(\lambda)\, c_j(\lambda). \end{aligned} \tag{6b}$$

Note that $V_m(t,s)$ and $W_m(t,s)$ do not necessarily belong to $\mathbb{C}[t,s]$ anymore because the homogeneous degree in λ_1 and λ_2 doesn't equal the degree in u. Instead they belong to $\mathbb{C}[\lambda_1, \lambda_2][u]$. In the remainder of the text we will use both the notations $V_m(t,s)$ and $\mathcal{V}_m(u)$ interchangeably to refer to (6a) and analogously for (6b). For

$$\begin{aligned} \tilde{V}_m(t,s) &= \tilde{\mathcal{V}}_m(u) \\ &= u^{\Delta_m+m} \mathcal{V}_m(u^{-1}) \\ &= \sum_{i=0}^{m} B_{\Delta_m+i}(\lambda)\, u^{\Delta_m+i} \\ &= \sum_{i=0}^{m} \sum_{j=0}^{\Delta_m+i} b_{\Delta_m+i-j,\, j} t^{\Delta_m+i-j} s^j \\ \tilde{W}_m(t,s) &= \tilde{\mathcal{W}}_m(u) \\ &= u^{\Delta_m+m-1} \mathcal{W}_m(u^{-1}) \\ &= \sum_{i=0}^{m-1} A_{\Delta_m+i}(\lambda)\, u^{\Delta_m+i} \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{\Delta_m+i} a_{\Delta_m+i-j,\, j} t^{\Delta_m+i-j} s^j \end{aligned}$$

the Padé-type approximation conditions

$$(f\tilde{V}_m - \tilde{W}_m)(t, s) = (f\tilde{\mathscr{V}}_m - \tilde{\mathscr{W}}_m)(u)$$
$$= \sum_{i=\Delta_m+m}^{\infty} d_i(\lambda)\, u^i$$
$$= \sum_{i=\Delta_m+m}^{\infty} \left(\sum_{j=0}^{i} d_{i-j,\,j} t^{i-j} s^j \right)$$

hold, where as in (6) the subscripted function $d_i(\lambda)$ is a homogeneous function of degree i in λ_1 and λ_2. We remark here that $\tilde{V}_m(t, s)$ and $\tilde{W}_m(t, s)$ again belong to $\mathbb{C}[t, s]$ contrary to $V_m(t, s)$ and $W_m(t, s)$. As in (1), if the function $V_m(t, s)$, say the polynomial $\mathscr{V}_m(u)$, is not chosen randomly, but if it satisfies the additional orthogonality conditions

$$\Gamma(z^i \mathscr{V}_m(z)) = 0 \qquad i = 0, ..., m-1 \tag{7}$$

then the Padé approximation conditions

$$(f\tilde{V}_m(t, s) - \tilde{W}_m(t, s)) = (f\tilde{\mathscr{V}}_m - \tilde{\mathscr{W}}_m)(u)$$
$$= \sum_{i=\Delta_m+2m}^{\infty} d_i(\lambda)\, u^i$$
$$= \sum_{i=\Delta_m+2m}^{\infty} \left(\sum_{j=0}^{i} d_{i-j,\,j} t^{i-j} s^j \right)$$

are satisfied and $\tilde{\mathscr{W}}_m(u)/\tilde{\mathscr{V}}_m(u)$ equals the homogeneous Padé approximant $[m-1/m]_H^f$ [2]. As in the univariate case the orthogonality conditions (10) only determine $\mathscr{V}_m(u)$ up to a kind of normalization: $m+1$ polynomial coefficients $B_{\Delta_m+m-i}(\lambda)$ must be determined from m conditions. How this is solved, is explained below.

With the $c_i(\lambda)$ we now define the polynomial Hankel determinants

$$H_m^{(0)}(\lambda) = \begin{vmatrix} c_0(\lambda) & \cdots & c_{m-1}(\lambda) \\ \vdots & \iddots & c_m(\lambda) \\ & & \vdots \\ c_{m-1}(\lambda) & \cdots & c_{2m-2}(\lambda) \end{vmatrix} \qquad H_0^{(0)}(\lambda) = 1$$

generalizing the classical Hankel determinants as defined in [7]. We also call the functional Γ definite if

$$H_m^{(0)}(\lambda) \not\equiv 0 \qquad m \geqslant 0.$$

In the remainder of the text we shall assume that $\mathscr{V}_m(u)$ satisfies (10) and that Γ is a definite functional. Also we shall assume that $\mathscr{V}_m(u)$ as given by (6a) is primitive, meaning that its polynomial coefficients $B_{\Delta_m+m-i}(\lambda)$ are relatively prime. This last condition can always be satisfied, because for a definite functional Γ solution of (10) is given by [2]

$$\mathscr{V}_m(u) = \frac{1}{p_m^{(0)}(\lambda)} \begin{vmatrix} c_0(\lambda) & \cdots & c_{m-1}(\lambda) & c_m(\lambda) \\ \vdots & \iddots & & \vdots \\ c_{m-1}(\lambda) & & \cdots & c_{2m-1}(\lambda) \\ 1 & u & \cdots & u^m \end{vmatrix} \qquad \mathscr{V}_0(u) = 1, \tag{8}$$

where the polynomial $p_m^{(0)}(\lambda)$ is a polynomial greatest common divisor of the polynomial coefficients of the powers of u. Clearly (8) completely determines $\mathscr{V}_m(u)$ and consequently $V_m(t, s)$.

As in the univariate situation the functional $\Gamma^{(k+1)}$ can be defined by

$$\Gamma^{(k+1)}(z^i) = c_{k+1+i}(\lambda)$$

and Γ can be replaced by $\Gamma^{(k+1)}$ for the construction of homogeneous Padé approximants $[m+k/m]_H^f$ with $k \geqslant -1$. The shift in the numerator and denominator degrees of $[m+k/m]_H^f$ then satisfies

$$0 \leqslant \Delta_m^{(k+1)} \leqslant (m+k)\, m$$

and the numerator and denominator of $[m+k/m]_H^f$ are respectively denoted by $\tilde{W}_m^{(k+1)}(t, s) = \tilde{\mathscr{W}}_m^{(k+1)}(u)$ and $\tilde{V}_m^{(k+1)}(t, s) = \tilde{\mathscr{V}}_m^{(k+1)}(u)$. We denote by $p_m^{(k+1)}(\lambda)$ a polynomial greatest common divisor of the polynomial coefficients of u^i, $i = 0, ..., m$ in the determinant

$$\begin{vmatrix} c_{k+1}(\lambda) & \cdots & c_{k+m}(\lambda) & c_{k+m+1}(\lambda) \\ \vdots & \iddots & & \vdots \\ c_{k+m}(\lambda) & & \cdots & c_{k+2m}(\lambda) \\ 1 & u & \cdots & u^m \end{vmatrix}$$

and identify $\mathscr{W}_m/\mathscr{V}_m$ and Γ respectively with $\mathscr{W}_m^{(0)}/\mathscr{V}_m^{(0)}$ and $\Gamma^{(0)}$. It is then easy to check that $\mathscr{V}_m^{(k+1)}$ given by

$$\mathscr{V}_m^{(k+1)}(u) = \frac{1}{p_m^{(k+1)}(\lambda)} \begin{vmatrix} c_{k+1}(\lambda) & \cdots & c_{k+m}(\lambda) & c_{k+m+1}(\lambda) \\ \vdots & \iddots & & \vdots \\ c_{k+m}(\lambda) & & \cdots & c_{k+2m}(\lambda) \\ 1 & u & \cdots & u^m \end{vmatrix}$$

$$\mathscr{V}_0^{(k+1)}(u) = 1 \tag{9}$$

one has

$$\Gamma^{(k+1)}(u^i \mathscr{V}_m^{(k+1)}(u)) = 0 \qquad i = 0, ..., m-1 \tag{10}$$

$$\Gamma^{(k+1)}(u^m \mathscr{V}_m^{(k+1)}(u))$$

$$= \Gamma^{(k+1)} \left(\frac{1}{p_m^{(k+1)}(\lambda)} \begin{vmatrix} c_{k+1}(\lambda) & \cdots & c_{k+m}(\lambda) & c_{k+m+1}(\lambda) \\ \vdots & \iddots & & \vdots \\ c_{k+m}(\lambda) & & \cdots & c_{k+2m}(\lambda) \\ u^m & u^{m+1} & \cdots & u^{2m} \end{vmatrix} \right)$$

$$= \frac{1}{p_m^{(k+1)}(\lambda)} \begin{vmatrix} c_{k+1}(\lambda) & \cdots & c_{k+m}(\lambda) & c_{k+m+1}(\lambda) \\ \vdots & \iddots & & \vdots \\ c_{k+m}(\lambda) & & \cdots & c_{k+2m}(\lambda) \\ c_{k+m+1}(\lambda) & c_{k+m+2}(\lambda) & \cdots & c_{k+2m+1}(\lambda) \end{vmatrix}$$

$$= \frac{H_{m+1}^{(k+1)}(\lambda)}{p_m^{(k+1)}(\lambda)}. \tag{11}$$

To conclude this section we summarize the most important results.

Summary

(a) *For the bivariate series* $f(t, s)$ *and for* $k \geqslant -1$ *holds*

$$[m + k/m]_H^f(t, s) = \frac{\tilde{W}_m^{(k+1)}(t, s)}{\tilde{V}_m^{(k+1)}(t, s)}.$$

(b) *For the monic univariate polynomial* $V_m(u)$ *satisfying* (1) *and for the bivariate polynomial* $V_m(t, s) = \mathscr{V}_m(u)$ *given by* (8) *with* $(t, s) = (\lambda_1 u, \lambda_2 u)$ *holds*

$$H_m^{(0)}(\lambda_1, \lambda_2)\, V_m(u) = p_m^{(0)}(\lambda_1, \lambda_2)\, V_m(\lambda_1 u, \lambda_2 u) = p_m^{(0)}(\lambda_1, \lambda_2)\, \mathscr{V}_m(u).$$

This last property can be seen as a projection property.

3. PROPERTIES OF THE HOMOGENEOUS ORTHOGONAL POLYNOMIALS

Let us now generalize the well-known univariate property [3, p. 57] that for a definite functional $c^{(k+1)}$ as in (2) the polynomials $V_m^{(k+1)}(t)$ and $V_{m+1}^{(k+1)}(t)$ have no common zeroes. The same is true in the univariate case for the polynomials $W_m^{(k+1)}(t)$ and $W_{m+1}^{(k+1)}(t)$, and the polynomials $V_m^{(k+1)}(t)$ and $W_m^{(k+1)}(t)$. Before we can formulate the multivariate generalization, we first need a number of lemmas and theorems.

In the multivariate discussion we shall often switch between the coordinates (t, s) and the coordinate u in the one-dimensional subspaces spanned by the vectors λ. Remember that $V_m(t, s) = \mathscr{V}_m(u)$ and $W_m(t, s) = \mathscr{W}_m(u)$ do not belong to $\mathbb{C}[t, s]$ but to $\mathbb{C}[\lambda_1, \lambda_2][u]$.

LEMMA 1. *Let the functional $\Gamma^{(k+1)}$ which is defined for $k \geqslant -1$ be definite and let the polynomials $\{\mathscr{V}_m^{(k+1)}(u)\}_m$ satisfy* (10). *Then the $\{\mathscr{V}_m^{(k+1)}(u)\}_m$ are linearly independent in $\mathbb{C}(\lambda_1, \lambda_2)[u]$.*

Proof. Suppose we have coefficients $\eta_0(\lambda), \eta_1(\lambda), ... \in \mathbb{C}(\lambda_1, \lambda_2)$ such that formally

$$\forall u \in \mathbb{C} : \sum_{i=0}^{\infty} \eta_i(\lambda) \, \mathscr{V}_i^{(k+1)}(u) \equiv 0.$$

Then we also have for $j \geqslant 0$ that

$$\sum_{i=0}^{\infty} \eta_i(\lambda) \, \Gamma^{(k+1)}(u^j \mathscr{V}_i^{(k+1)}(u)) \equiv 0.$$

Taking (10) into account, we obtain for $j \geqslant 0$

$$\sum_{i=0}^{j} \eta_i(\lambda) \, \Gamma^{(k+1)}(u^j \mathscr{V}_i^{(k+1)}(u)) \equiv 0.$$

For $j = 0$ this reduces to

$$\eta_0(\lambda) \, \Gamma^{(k+1)}(\mathscr{V}_0^{(k+1)}(u)) \equiv 0$$

which results in $\eta_0(\lambda) = 0$ because $\Gamma^{(k+1)}(\mathscr{V}_0^{(k+1)}(u)) = c_{k+1}(\lambda) = H_1^{(k+1)}(\lambda) \not\equiv 0$. For $j > 1$ the proof that $\eta_j(\lambda) = 0$ is by induction. ∎

THEOREM 1. *Let the functional $\Gamma^{(k+1)}$ which is defined for $k \geqslant -1$ be definite and let the polynomials $\mathscr{V}_m^{(k+1)}(u)$ and $p_m^{(k+1)}(\lambda)$ be defined as in* (9). *Then the polynomials $\{\mathscr{V}_m^{(k+1)}(u)\}_m$ and $\{\mathscr{W}_m^{(k+1)}(u)\}_m$ satisfy the recurrence relations*

$$\begin{aligned} V_{m+1}^{(k+1)}(t, s) = \alpha_{m+1}^{(k+1)}(\lambda)\big[&(u - \beta_{m+1}^{(k+1)}(\lambda)) \, V_m^{(k+1)}(t, s) \\ &- \gamma_{m+1}^{(k+1)}(\lambda) \, V_{m-1}^{(k+1)}(t, s)\big] \end{aligned}$$

$$V_{-1}^{(k+1)}(t, s) = 0 \qquad V_0^{(k+1)}(t, s) = 1$$

$$\begin{aligned} W_{m+1}^{(k+1)}(t, s) = \alpha_{m+1}^{(k+1)}(\lambda)\big[&(u - \beta_{m+1}^{(k+1)}(\lambda)) \, W_m^{(k+1)}(t, s) \\ &- \gamma_{m+1}^{(k+1)}(\lambda) \, W_{m-1}^{(k+1)}(t, s)\big] \end{aligned}$$

$$W_{-1}^{(k+1)}(t, s) = -1 \qquad W_0^{(k+1)}(t, s) = 0$$

with

$$\alpha_{m+1}^{(k+1)}(\lambda) = \frac{p_m^{(k+1)}(\lambda)}{p_{m+1}^{(k+1)}(\lambda)} \frac{H_{m+1}^{(k+1)}(\lambda)}{H_m^{(k+1)}(\lambda)}$$

$$\beta_{m+1}^{(k+1)}(\lambda) = \frac{\Gamma^{(k+1)}(u(V_m^{(k+1)}(t,s))^2)}{\Gamma^{(k+1)}((V_m^{(k+1)}(t,s))^2)}$$

$$\gamma_{m+1}^{(k+1)}(\lambda) = \frac{p_{m-1}^{(k+1)}(\lambda)}{p_m^{(k+1)}(\lambda)} \frac{H_{m+1}^{(k+1)}(\lambda)}{H_m^{(k+1)}(\lambda)} \qquad \gamma_1^{(k+1)}(\lambda) = c_{k+1}(\lambda).$$

Proof. The polynomial $u\mathscr{V}_m^{(k+1)}(u) = uV_m^{(k+1)}(t,s)$ as defined in (6a) can be written as a linear combination

$$uV_m^{(k+1)}(t,s) = \sum_{i=0}^{m+1} \eta_i^{(k+1)}(\lambda)\, V_i^{(k+1)}(t,s),$$

where the $\eta_i^{(k+1)}(\lambda)$ are rational functions of the variable λ. We multiply left and right hand side with $V_j^{(k+1)}(t,s)$ and apply the linear functional $\Gamma^{(k+1)}$ to obtain

$$\eta_i^{(k+1)}(\lambda) = 0 \qquad i = 0, ..., m-2$$

$$\eta_{m-1}^{(k+1)}(\lambda) = \frac{\Gamma^{(k+1)}(uV_{m-1}^{(k+1)}(t,s)\, V_m^{(k+1)}(t,s))}{\Gamma^{(k+1)}((V_{m-1}^{(k+1)}(t,s))^2)} = \gamma_{m+1}^{(k+1)}(\lambda)$$

$$\eta_m^{(k+1)}(\lambda) = \frac{\Gamma^{(k+1)}(u(V_m^{(k+1)}(t,s))^2)}{\Gamma^{(k+1)}((V_m^{(k+1)}(t,s))^2)}$$

$$\eta_{m+1}^{(k+1)}(\lambda) = \frac{\Gamma^{(k+1)}(uV_m^{(k+1)}(t,s)\, V_{m+1}^{(k+1)}(t,s))}{\Gamma^{(k+1)}((V_{m+1}^{(k+1)}(t,s))^2)} = 1/\alpha_{m+1}^{(k+1)}(\lambda).$$

On the other hand we have

$$\mathscr{V}_m^{(k+1)}(u) = V_m^{(k+1)}(t,s) = \frac{H_m^{(k+1)}(\lambda)}{p_m^{(k+1)}(\lambda)}\, u^m + \cdots$$

so that consequently

$$\frac{H_m^{(k+1)}(\lambda)}{p_m^{(k+1)}(\lambda)} = \frac{1}{\alpha_{m+1}^{(k+1)}(\lambda)} \frac{H_{m+1}^{(k+1)}(\lambda)}{p_{m+1}^{(k+1)}(\lambda)}.$$

Using (11) and the fact that

$$\Gamma^{(k+1)}((\mathscr{V}_m^{(k+1)}(u))^2) = \frac{H_m^{(k+1)}(\lambda)}{p_m^{(k+1)}(\lambda)}\, \Gamma^{(k+1)}(u^m \mathscr{V}_m^{(k+1)}(u))$$

the expression for $\gamma_{m+1}^{(k+1)}(\lambda)$ is obtained. For the associated polynomials $\mathscr{W}_m^{(k+1)}(u)$ we have, because $\Gamma^{(k+1)}(\mathscr{V}_m^{(k+1)}(z))=0$,

$$\mathscr{W}_{m+1}^{(k+1)}(u)=\alpha_{m+1}^{(k+1)}(\lambda)\,\Gamma^{(k+1)}\Bigg((u-\beta_{m+1}^{(k+1)}(\lambda))\,\frac{\mathscr{V}_m^{(k+1)}(u)-\mathscr{V}_m^{(k+1)}(z)}{u-z}$$
$$-\gamma_{m+1}^{(k+1)}(\lambda)\,\frac{\mathscr{V}_{m-1}^{(k+1)}(u)-\mathscr{V}_{m-1}^{(k+1)}(z)}{u-z}\Bigg)$$

which gives the desired result. The starting value for $\gamma_1^{(k+1)}(\lambda)$ is easy to verify. ▮

THEOREM 2. *Let the functional $\Gamma^{(k+1)}$ which is defined for $k\geqslant -1$ be definite and let the polynomials $\mathscr{V}_m^{(k+1)}(u)$ and $p_m^{(k+1)}(\lambda)$ be defined as in (9). Then the polynomials $\{\mathscr{V}_m^{(k+1)}(u)\}_m$ and $\{\mathscr{W}_m^{(k+1)}(u)\}_m$ satisfy the identity*

$$\begin{aligned}&\mathscr{V}_m^{(k+1)}(u)\,\mathscr{W}_{m+1}^{(k+1)}(u)-\mathscr{W}_m^{(k+1)}(u)\,\mathscr{V}_{m+1}^{(k+1)}(u)\\&\quad=V_m^{(k+1)}(t,s)\,W_{m+1}^{(k+1)}(t,s)-W_m^{(k+1)}(t,s)\,V_{m+1}^{(k+1)}(t,s)\\&\quad=\frac{[H_{m+1}^{(k+1)}(\lambda)]^2}{p_m^{(k+1)}(\lambda)\,p_{m+1}^{(k+1)}(\lambda)}.\end{aligned}$$

Proof. For simplicity we omit writing the arguments (t,s) in $V_m^{(k+1)}$ and $W_m^{(k+1)}$ and (λ) in $\alpha_m^{(k+1)}$, $\beta_m^{(k+1)}$ and $\gamma_m^{(k+1)}$. The proof makes use of the previous recurrence relations:

$$\begin{aligned}W_m^{(k+1)}\,V_{m+1}^{(k+1)}&=\alpha_{m+1}^{(k+1)}\big[(u-\beta_{m+1}^{(k+1)}V)\,V_m^{(k+1)}\,W_m^{(k+1)}\\&\qquad-\gamma_{m+1}^{(k+1)}\,V_{m-1}^{(k+1)}\,W_m^{(k+1)}\big]\\V_m^{(k+1)}\,W_{m+1}^{(k+1)}&=\alpha_{m+1}^{(k+1)}\big[(u-\beta_{m+1}^{(k+1)})\,W_m^{(k+1)}\,V_m^{(k+1)}\\&\qquad-\gamma_{m+1}^{(k+1)}\,W_{m-1}^{(k+1)}\,V_m^{(k+1)}\big].\end{aligned}$$

By subtracting these expressions one obtains

$$\begin{aligned}&V_m^{(k+1)}\,W_{m+1}^{(k+1)}-W_m^{(k+1)}\,V_{m+1}^{(k+1)}\\&\quad=\alpha_{m+1}^{(k+1)}\gamma_{m+1}^{(k+1)}\cdots\alpha_2^{(k+1)}\gamma_2^{(k+1)}(V_0^{(k+1)}\,W_1^{(k+1)}-W_0^{(k+1)}\,V_1^{(k+1)})\\&\quad=\frac{[H_{m+1}^{(k+1)}]^2}{p_m^{(k+1)}p_{m+1}^{(k+1)}}.\end{aligned}$$ ▮

The preceding theorem shows that the expression

$$\mathscr{V}_m^{(k+1)}(u)\,\mathscr{W}_{m+1}^{(k+1)}(u)-\mathscr{W}_m^{(k+1)}(u)\,\mathscr{V}_{m+1}^{(k+1)}(u)$$

is homogeneous and that if $p_m^{(k+1)}(\lambda)$ and $p_{m+1}^{(k+1)}(\lambda)$ are constants, this homogeneous expression is of degree $(k+m+1)(m+1)$.

Let us now take a closer look at the factorisation of the orthogonal polynomials $\mathscr{V}_m^{(k+1)}(u)$ and their associated polynomials $\mathscr{W}_m^{(k+1)}(u)$ in irreducible factors. This factorisation is unique in $\mathbb{C}[\lambda_1, \lambda_2][u]$ except for multiplicative constants from $\mathbb{C}$ which are the unit multiples in $\mathbb{C}[\lambda_1, \lambda_2]$ and except for the order of the factors. This is because $\mathbb{C}[\lambda_1, \lambda_2][u]$ is a unique factorization domain.

THEOREM 3. *Let the functional $\Gamma^{(k+1)}$ which is defined for $k \geqslant -1$ be definite and let the polynomials $\mathscr{V}_m^{(k+1)}(u)$ and $p_m^{(k+1)}(\lambda)$ be defined as in* (9). *Let $\mathscr{W}_m^{(k+1)}(u)$ be given by* (6b). *Then*

(a) *$\mathscr{V}_m^{(k+1)}(u)$ and $\mathscr{V}_{m+1}^{(k+1)}(u)$ have no common factor*

(b) *$\mathscr{W}_m^{(k+1)}(u)$ and $\mathscr{W}_{m+1}^{(k+1)}(u)$ have no common factor*

(c) *$\mathscr{V}_m^{(k+1)}(u)$ and $\mathscr{W}_m^{(k+1)}(u)$ have no common factor.*

Proof. We only give the proof for (a) since the proof for (b) and (c) is completely similar. The proof is by contradiction. Assume that $\mathscr{V}_m^{(k+1)}(u)$ and $\mathscr{V}_{m+1}^{(k+1)}(u)$ have a common factor. Then, because of theorem 2, it is necessarily a polynomial in λ, different from a complex constant if it is a true common factor. Hence the polynomials $\mathscr{V}_m^{(k+1)}(u)$ and $\mathscr{V}_{m+1}^{(k+1)}(u)$ are not primitive, which is a contradiction. ▮

Let us now restrict ourselves to all variables and coefficients being real and turn to some results for positive definite functionals. The functional $\Gamma^{(k+1)}$ is called positive definite if

$$\forall \lambda \in \mathbb{R}^2 \setminus \{0\}: H_m^{(k+1)}(\lambda) > 0 \qquad m \geqslant 0.$$

LEMMA 2. *For a positive definite functional $\Gamma^{(k+1)}$ and for any polynomial $\mathscr{P}(u) \in \mathbb{R}[\lambda_1, \lambda_2][u]$ holds*

$$\Gamma^{(k+1)}(\mathscr{P}^2(u)) > 0,$$

where the functional $\Gamma^{(k+1)}$ acts on the variable u as defined above.

Proof. Every polynomial $\mathscr{P}(u)$ of degree m in $\mathbb{R}[\lambda_1, \lambda_2][u]$ can be written in the form

$$\mathscr{P}(u) = \sum_{i=0}^{m} \eta_i(\lambda)\, \mathscr{V}_i^{(k+1)}(u),$$

where the $\eta_i(\lambda) \in \mathbb{R}(\lambda_1, \lambda_2)$ are rational functions of the variable λ with real coefficients. From the orthogonality conditions satisfied by $\mathscr{V}_m^{(k+1)}(u)$ we obtain

$$\begin{aligned}\Gamma^{(k+1)}(\mathscr{P}^2(u)) &= \sum_{i=0}^{m} \eta_i^2(\lambda)\, \Gamma^{(k+1)}((\mathscr{V}_i^{(k+1)}(u))^2)\\ &= \sum_{i=0}^{m} \eta_i^2(\lambda) \frac{H_j^{(k+1)}(\lambda)\, H_{j+1}^{(k+1)}(\lambda)}{(p_j^{(k+1)}(\lambda))^2}\\ &> 0. \quad \blacksquare\end{aligned}$$

THEOREM 4. *For a positive definite functional* $\Gamma^{(k+1)}$, *the polynomials* $\mathscr{V}_m^{(k+1)}(u)$ *satisfying* (10) *have no irreducible factors in* $\mathbb{R}[\lambda_1, \lambda_2][u]$ *of multiplicity larger than* 1.

Proof. Assume $\mathscr{V}_m^{(k+1)}(u)$ has an irreducible factor $\mathscr{F}(u)$ of multiplicity $\ell > 1$. Then we can write

$$\mathscr{V}_m^{(k+1)}(u) = \mathscr{F}^{\ell}(u)\, \mathscr{L}(u),$$

where $\mathscr{L}(u)$ is a polynomial in u of degree $m - \ell\, \partial_u \mathscr{F} < m$ where $\partial_u \mathscr{F}$ is the degree of $\mathscr{F}(u)$ as a polynomial in u. If $\ell > 1$ is even then because of Lemma 2

$$\begin{aligned}\Gamma^{(k+1)}(\mathscr{L}(u)\, \mathscr{V}_m^{(k+1)}(u)) &= \Gamma^{(k+1)}(\mathscr{L}^2(u)\, \mathscr{F}^{\ell}(u))\\ &> 0\end{aligned}$$

which is impossible because of the orthogonality conditions satisfied by $\mathscr{V}_m^{(k+1)}$. If $\ell > 1$ is odd then

$$\begin{aligned}\Gamma^{(k+1)}(\mathscr{F}(u)\, \mathscr{L}(u)\, \mathscr{V}_m^{(k+1)}(u)) &= \Gamma^{(k+1)}(\mathscr{L}^2(u)\, \mathscr{F}^{\ell+1}(u))\\ &> 0\end{aligned}$$

which is also a contradiction. $\blacksquare$

Let us illustrate this by considering the following positive definite functional

$$\Gamma(z^i) = c_i(\lambda) = \sum_{j=0}^{i} c_{i-j,\, j} \lambda_1^{i-j} \lambda_2^{j}$$

$$c_{i-j,\, j} = \binom{i}{j} \iint_{\|(x,\, y)\|_2 \leqslant 1} x^{i-j} y^{j}\, dx\, dy.$$

The first few orthogonal polynomials satisfying (7) and having only simple irreducible factors are given by

$$\mathcal{V}_0^{(0)}(u)=1$$

$$\mathcal{V}_1^{(0)}(u)=u$$

$$\mathcal{V}_2^{(0)}(u)=\left(u-\frac{1}{2}\sqrt{\lambda_1^2+\lambda_2^2}\right)\left(u+\frac{1}{2}\sqrt{\lambda_1^2+\lambda_2^2}\right)$$

$$\mathcal{V}_3^{(0)}(u)=2u\left(u-\frac{1}{\sqrt{2}}\sqrt{\lambda_1^2+\lambda_2^2}\right)\left(u+\frac{1}{\sqrt{2}}\sqrt{\lambda_1^2+\lambda_2^2}\right)$$

$$\mathcal{V}_4^{(0)}(u)=16\left(u-\frac{\sqrt{3-\sqrt{5}}}{2\sqrt{2}}\sqrt{\lambda_1^2+\lambda_2^2}\right)\left(u+\frac{\sqrt{3-\sqrt{5}}}{2\sqrt{2}}\sqrt{\lambda_1^2+\lambda_2^2}\right)$$
$$\times\left(u-\frac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}}\sqrt{\lambda_1^2+\lambda_2^2}\right)\left(u+\frac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}}\sqrt{\lambda_1^2+\lambda_1^2}\right)$$

$$\mathcal{V}_5^{(0)}(u)=16u\left(u-\frac{1}{2}\sqrt{\lambda_1^2+\lambda_2^2}\right)\left(u+\frac{1}{2}\sqrt{\lambda_1^2+\lambda_2^2}\right)$$
$$\times\left(u-\frac{\sqrt{3}}{2}\sqrt{\lambda_1^2+\lambda_1^2}\right)\left(u+\frac{\sqrt{3}}{2}\sqrt{\lambda_1^2+\lambda_2^2}\right). \qquad (12)$$

4. COMMON ZEROES INSTEAD OF COMMON FACTORS

From the previous section it is clear that our orthogonal polynomials $\{\mathcal{V}_m^{(k+1)}(u)\}_{m\in\mathbb{N}}$ do hot have any irreducible factors in common in $\mathbb{C}[\lambda_1, \lambda_2][u]$. Since each of these irreducible factors would determine a zero curve, it is also clear that the $\{V_m^{(k+1)}(t, s)\}_{m\in\mathbb{N}}$ do not have any zero curves in common. But since their coefficients belong to the unique factorization domain $\mathbb{C}[\lambda_1, \lambda_2]$, we can use a well-known theorem to detect isolated zeroes for which for instance $\mathcal{V}_m^{(k+1)}(u)=V_m^{(k+1)}(t, s)$ and $\mathcal{V}_n^{(k+1)}(u)=V_n^{(k+1)}(t, s)$ vanish simultaneously.

LEMMA 3. *Let the functional $\Gamma^{(k+1)}$ which is defined for $k\geqslant -1$ be definite. Let the polynomials*

$$\mathcal{V}_m^{(k+1)}(u)=\sum_{i=0}^{m} v_{mi}^{(k+1)}(\lambda)\, u^i$$

satisfy (10). *Then* $\mathcal{V}_m^{(k+1)}(u)$ *and* $\mathcal{V}_n^{(k+1)}(u)$ *have a common zero for* $\lambda = (\lambda_1, \lambda_2)$ *satisfying*

$$R(\lambda) = \left| \begin{matrix} v_{m0}^{(k+1)}(\lambda) & \cdots & v_{mm}^{(k+1)}(\lambda) & & \\ 0 & \ddots & & \ddots & \\ 0 & 0 & v_{m0}^{(k+1)}(\lambda) & \cdots & v_{mm}^{(k+1)}(\lambda) \\ v_{n0}^{(k+1)}(\lambda) & \cdots & v_{nn}^{(k+1)}(\lambda) & & \\ 0 & \ddots & & \ddots & \\ 0 & 0 & v_{n0}^{(k+1)}(\lambda) & \cdots & v_{nn}^{(k+1)}(\lambda) \end{matrix} \right| \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \text{ times} \\ = 0 \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m \text{ times} \end{matrix}$$

Proof. The $(n+m)\times(n+m)$ determinant $R(\lambda)$ is the resultant of the polynomials $\mathcal{V}_m^{(k+1)}(u)$ and $\mathcal{V}_n^{(k+1)}(u)$ and this proves the lemma [9, pp. 23–30]. ▮

Taking the positive definite functional from the previous section again, we see that the resultant of the orthogonal polynomials $\mathcal{V}_2^{(0)}(u)$ and $\mathcal{V}_5^{(0)}(u)$ given in (12) is identically zero, meaning that common zeroes can be found for any value of λ_1 and λ_2. Indeed, these polynomials have two common factors (note that Theorem 3 only applies to polynomials of which the degree differs at most 1). We can further illustrate this occurrence of common zeroes with an example involving a nontrivial resultant. Consider the functional

$$\Gamma^{(0)}(z^i) = c_i(\lambda) = \frac{1}{i!}(\lambda_1^i + \lambda_1^{i-1}\lambda_2 + \cdots + \lambda_2^i).$$

The orthogonal polynomials $\mathcal{V}_1^{(0)}(u)$ and $\mathcal{V}_2^{(0)}(u)$ satisfying (7) are then given by

$$\begin{aligned} \mathcal{V}_1^{(0)}(u) &= -(\lambda_1+\lambda_2)+u \\ \mathcal{V}_2^{(0)}(u) &= -\tfrac{1}{12}(\lambda_1^4 + 2\lambda_1^3\lambda_2 + 5\lambda_1^2\lambda_2^2 + 2\lambda_1\lambda_2^3 + \lambda_2^4) \\ &\quad + \tfrac{1}{6}(2\lambda_1^3 + 5\lambda_1^2\lambda_2 + 5\lambda_1\lambda_2^2 + 2\lambda_2^3)\,u - \tfrac{1}{2}(\lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2)\,u^2. \end{aligned}$$

The resultant of $\mathcal{V}_1^{(0)}(u)$ and $\mathcal{V}_2^{(0)}(u)$ equals (we present the resultant in factored form because we need this form afterwards)

$$R(\lambda) = -\tfrac{1}{64}((3+\sqrt{5})\,\lambda_1 + 2\lambda_2)^2\,((3-\sqrt{5})\,\lambda_1 + 2\lambda_2)^2.$$

Consequently $V_1^{(0)}(t,s) = \mathcal{V}_1^{(0)}(u)$ and $V_2^{(0)}(t,s) = \mathcal{V}_2^{(0)}(u)$ have a common zero for λ satisfying

$$\begin{cases} R(\lambda) = 0 \\ \|\lambda\|_2 = 1. \end{cases}$$

This is for

$$\lambda = \lambda^{(1)} = \left(\frac{-1}{3} \sqrt{\frac{6}{3+\sqrt{5}}}, \sqrt{\frac{3+\sqrt{5}}{6}} \right)$$

and

$$\lambda = \lambda^{(2)} = \left(\frac{-1}{3} \sqrt{\frac{6}{3-\sqrt{5}}}, \sqrt{\frac{3-\sqrt{5}}{6}} \right)$$

or in terms of t and s, for

$$(t, s) = (t^{(1)}, s^{(1)}) = \left(-\frac{1+\sqrt{5}}{9+3\sqrt{5}}, \frac{1+\sqrt{5}}{6} \right)$$

and

$$(t, s) = (t^{(2)}, s^{(2)}) = \left(-\frac{1-\sqrt{5}}{9-3\sqrt{5}}, \frac{1-\sqrt{5}}{6} \right).$$

To further illustrate the above, we have plotted the zeroes of $V_1^{(0)}(t, s)$ and $V_2^{(0)}(t, s)$ in Fig. 1. For $V_1^{(0)}(t, s)$ for instance, these are given by the ellipse in Fig. 1:

$$V_1^{(0)}(t, s) = 0 \Leftrightarrow \begin{cases} t = \lambda_1 u = \lambda_1(\lambda_1 + \lambda_2) \\ s = \lambda_2 u = \lambda_2(\lambda_1 + \lambda_2) \end{cases} \qquad \|\lambda\|_2 = 1.$$

For $V_2^{(0)}(t, s)$ they are given by the hyperbola. The vectors (λ_1, λ_2) for which the ellipse and the hyperbola are tangent, were computed from equating the resultant to zero. The tangent points seen in the figure, are the vectors (t, s) computed above.

Let us at the same time illustrate that $\widetilde{\mathcal{W}}_2^{(0)}(u)/\widetilde{\mathcal{V}}_2^{(0)}(u) = \widetilde{W}_2^{(0)}(t, s)/\widetilde{V}_2^{(0)}(t, s)$ equals the homogeneous Padé approximant $[1/2]_H^f$ for the series

$$f(t, s) = \sum_{i=0}^{\infty} c_i(\lambda)\, u^i$$

$$= \frac{t \exp(t) - s \exp(s)}{t - s} = \sum_{i,\, j=0}^{\infty} \frac{1}{(i+j)!}\, t^i s^j.$$

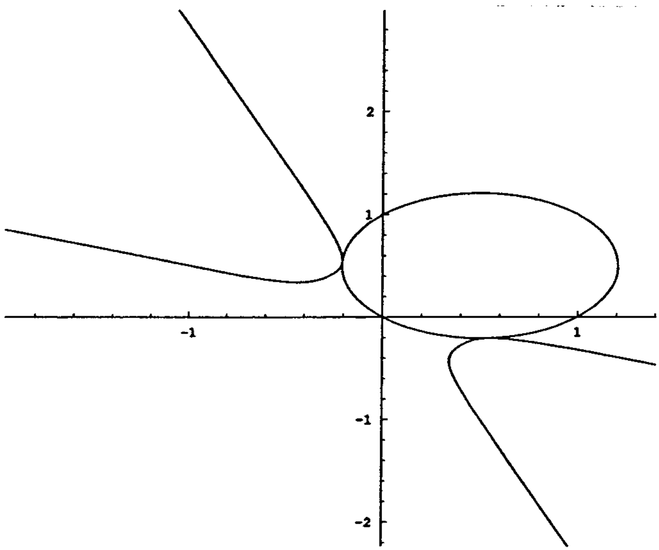

FIGURE 1

From $\mathscr{V}_2^{(0)}(u) = V_2^{(0)}(t, s)$ we compute

$$\begin{aligned}\tilde{V}_2^{(0)}(t, s) &= u^4 \tilde{\mathscr{V}}_2^{(0)}(u^{-1}) \\ &= -\tfrac{1}{12}(t^4 + 2t^3 s + 5t^2 s^2 + 2ts^3 + s^4) \\ &\quad + \tfrac{1}{6}(2t^3 + 5t^2 s + 5ts^2 + 2s^3) - \tfrac{1}{2}(t^2 + 3ts + s^2)\end{aligned}$$

and

$$\begin{aligned}W_2^{(0)}(t, s) &= \Gamma\left(\frac{\mathscr{V}_2^{(0)}(z) - \mathscr{V}_2^{(0)}(u)}{z - u}\right) \\ \tilde{W}_2^{(0)}(t, s) &= u^3 \mathscr{W}_2^{(0)}(u^{-1}) \\ &= -\frac{1}{2}(t^2 + 3ts + s^2) - \frac{1}{6}(t^3 + 7t^2 s + 7ts^2 + s^3)\end{aligned}$$

to obtain $[1/2]_H^f$.

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