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Well-defined determinant representations for non-normal multivariate rational interpolants

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If the system of linear equations defining a multivariate rational interpolant is singular, then the table of multivariate rational interpolants displays a structure where the basic building block is a hexagon. Remember that for univariate rational interpolation the structure is built by joining squares. In this paper we associate with every entry of the table of rational interpolants a well-defined determinant representation, also when this entry has a nonunique solution. These determinant formulas are crucial if one wants to develop a recursive computation scheme.

In section 2 we repeat the determinant representation for nondegenerate solutions (nonsingular systems of interpolation conditions). In theorem 1 this is generalized to an isolated hexagon in the table. In theorem 2 the existence of such a determinant formula is proven for each entry in the table. We conclude with an example in section 5.

1. Multivariate rational Hermite interpolants

For the sake of notational simplicity we will restrict our description to the bivariate case. Let a bivariate function $f(x,y)$ be known in the data points (x_i, y_j) with $(i,j) \in \mathbb{N}^2$ and let I be a finite subset of $\mathbb{N}^2$ indexing those data points which will be used as interpolation points. With the data points we construct the polynomial basis functions

$$B_{ij}(x,y) = \prod_{k=0}^{i-1}(x - x_k)\prod_{l=0}^{j-1}(y - y_l).$$

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The problem of interpolating these data by a bivariate rational function was formulated in [5] as follows. Choose finite subsets N (from "Numerator") and D (from "Denominator") of $\mathbb{N}^2$ with $N \subset I$ and compute bivariate polynomials

$$p(x,y) = \sum_{(i,j)\in N} a_{ij}B_{ij}(x,y), \quad \#N = n+1,$$

$$q(x,y) = \sum_{(i,j)\in D} b_{ij}B_{ij}(x,y), \quad \#D = m+1, \tag{1a}$$

where we denote $\partial p = N$ and $\partial q = D$, such that

$$(fq-p)(x_i,y_j) = 0, \quad (i,j)\in I, \#I = n+m+1. \tag{1b}$$

If $q(x_i,y_j) \neq 0$ then this last condition implies that

$$f(x_i,y_j) = \frac{p}{q}(x_i,y_j), \quad (i,j)\in I. \tag{1c}$$

We say that I satisfies the inclusion property if whenever a point (i,j) belongs to I, all the points in the rectangle emanating from the origin with (i,j) as its furthermost corner belong to I. Condition (1b), for instance, is met if the following two conditions are satisfied by the polynomials given in (1a) [5]:

$$(fq-p)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus I} d_{ij}B_{ij}(x,y), \tag{2a}$$

$$I \text{ satisfies the inclusion property}, \tag{2b}$$

where the series development (2a) is formal. Condition (2a), with restriction (2b), can also be used if some or all of the interpolation points or their coordinates coincide since it can be replaced by conditions in terms of bivariate divided differences [5]:

$$(fq)[x_0,\ldots,x_i][y_0,\ldots,y_j] = p[x_0,\ldots,x_i][y_0,\ldots,y_j], \quad (i,j)\in N, \tag{3a}$$

$$(fq)[x_0,\ldots,x_i][y_0,\ldots,y_j] = 0, \quad (i,j)\in I\setminus N. \tag{3b}$$

Using a generalization of Leibniz' theorem [5] we can substitute $(fq)[x_0,\ldots,x_i][y_0,\ldots,y_j]$ in (3a–b), with the notation $c_{\mu i,\nu j} = f[x_\mu,\ldots,x_i][y_\nu,\ldots,y_j]$, by:

$$\begin{aligned}(fq)[x_0,\ldots,x_i][y_0,\ldots,y_j] &= \sum_{\mu=0}^{i}\sum_{\nu=0}^{j} q[x_0,\ldots,x_\mu][y_0,\ldots,y_\nu]c_{\mu i,\nu j}\\ &= \sum_{\mu=0}^{i}\sum_{\nu=0}^{j} b_{\mu\nu}c_{\mu i,\nu j}\\ &= \sum_{(\mu,\nu)\in D} b_{\mu\nu}c_{\mu i,\nu j}.\end{aligned}$$

Also

$$p[x_0,\dots,x_i][y_0,\dots,y_j] = a_{ij}, \quad (i,j)\in N .$$

From now on we denote a rational function satisfying (3) by $[N/D]_I$. Numbering the points in the sets N, D and I as:

$$N = \{(i_0,j_0),\dots,(i_n,j_n)\}, \tag{4a}$$

$$D = \{(d_0,e_0),\dots,(d_m,e_m)\}, \tag{4b}$$

$$I = N\cup\{(i_{n+1},j_{n+1}),\dots,(i_{n+m},j_{n+m})\}, \tag{4c}$$

condition (3) becomes [5]:

$$\begin{pmatrix} c_{d_0i_0,e_0j_0} & \cdots & c_{d_mi_0,e_mj_0}\\ \vdots & & \vdots\\ c_{d_0i_n,e_0j_n} & \cdots & c_{d_mi_n,e_mj_n}\end{pmatrix}\begin{pmatrix} b_{d_0e_0}\\ \vdots\\ b_{d_me_m}\end{pmatrix} = \begin{pmatrix} a_{i_0j_0}\\ \vdots\\ a_{i_nj_n}\end{pmatrix}, \tag{5a}$$

$$\begin{pmatrix} c_{d_0i_{n+1},e_0j_{n+1}} & \cdots & c_{d_mi_{n+1},e_mj_{n+1}}\\ \vdots & & \vdots\\ c_{d_0i_{n+m},e_0j_{n+m}} & \cdots & c_{d_mi_{n+m},e_mj_{n+m}}\end{pmatrix}\begin{pmatrix} b_{d_0e_0}\\ \vdots\\ b_{d_me_m}\end{pmatrix} = \begin{pmatrix} 0\\ \vdots\\ 0\end{pmatrix}. \tag{5b}$$

If the rank of (5b) is not maximal, we look at $[N/D]_I$ as being a set of rational functions of which the numerator and denominator are given by (1a) and are satisfying (5a) and (5b). Such a solution $[N/D]_I$ is called degenerate.

2. A determinant representation for nondegenerate solutions

With the numberings (4a), (4b) and (4c) of the respective indices in N, D and I we can set up descending chains of index sets, defining bivariate polynomials of "lower degree" and bivariate rational interpolation problems of "lower order":

$$N = N_n \supset \dots \supset N_k = \{(i_0,j_0),\dots,(i_k,j_k)\} \supset \dots \supset N_0 = \{(i_0,j_0)\},$$
$$k = 0,\dots,n, \tag{6a}$$

$$D = D_m \supset \dots \supset D_l = \{(d_0,e_0),\dots,(d_l,e_l)\} \supset \dots \supset D_0 = \{(d_0,e_0)\},$$
$$l = 0,\dots,m, \tag{6b}$$

$$I = I_{n+m} \supset \dots \supset I_{k+l} = \{(i_0,j_0),\dots,(i_{k+l},j_{k+l})\} \supset \dots \supset I_0 = \{(i_0,j_0)\},$$
$$k+l = 0,\dots,n+m, \tag{6c}$$

$$I_{k+1,k+l} = \{(i_{k+1}, j_{k+1}), \ldots, (i_{k+l}, j_{k+l})\}, \quad I \backslash N = I_{n+1,n+m}\,.$$

If we assume that each set $I_{k+l} \subset I$ satisfies the inclusion property in its turn and also that each $I_k = N_k$ for $0 \leqslant k \leqslant n$, then we can compute with these subsets the following entries in a "table" of multivariate rational interpolants:

$$\begin{array}{ccc} [N_0/D_0]_{I_0} & \cdots & [N_0/D_m]_{I_m} \\ \vdots & & \vdots \\ [N_n/D_0]_{I_n} & \cdots & [N_n/D_m]_{I_{n+m}} \end{array} \tag{7}$$

If we let n and m increase, infinite chains of index sets as in (6) can be constructed and an infinite table of multivariate rational interpolants results. Of course, in practice, only a finite number of entries will be computed. For a nondegenerate entry in this table the following determinant representation can be given [4]:

$$[N_k/D_l]_{I_{k+l}} = \frac{\begin{vmatrix} \sum_{(i,j)\in N_k} c_{d_0 i, e_0 j} B_{ij}(x,y) & \cdots & \sum_{(i,j)\in N_k} c_{d_l i, e_l j} B_{ij}(x,y) \\ c_{d_0 i_{k+1}, e_0 j_{k+1}} & \cdots & c_{d_l i_{k+1}, e_l j_{k+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{k+l}, e_0 j_{k+l}} & \cdots & c_{d_l i_{k+l}, e_l j_{k+l}} \end{vmatrix}}{\begin{vmatrix} B_{d_0 e_0}(x,y) & \cdots & B_{d_l e_l}(x,y) \\ c_{d_0 i_{k+1}, e_0 j_{k+1}} & \cdots & c_{d_l i_{k+1}, e_l j_{k+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{k+l}, e_0 j_{k+l}} & \cdots & c_{d_l i_{k+l}, e_l j_{k+l}} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} t_0(k) & \cdots & t_0(k+l) \\ \delta t_0(k) & \cdots & \delta t_0(k+l) \\ \vdots & & \\ \delta t_{l-1}(k) & \cdots & \delta t_{l-1}(k+l) \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ \delta t_0(k) & \cdots & \delta t_0(k+l) \\ \vdots & & \\ \delta t_{l-1}(k) & \cdots & \delta t_{l-1}(k+l) \end{vmatrix}}\,,$$

with

$$t_k(l) = \sum_{(i,j)\in N_l} c_{d_k i, e_k j} B_{d_k, i, e_k j}(x,y), \quad k = 0, \ldots, m, l = 0, \ldots, n+m\,, \tag{8a}$$

$$\begin{aligned} t_j(i) &= 0, \quad i<0\,, \\ \delta t_j(i) &= t_{j+1}(i) - t_j(i), \quad j \geqslant 0\,. \end{aligned} \tag{8b}$$

It was shown in [4] that this last ratio of determinants can be computed recursively via the E-algorithm of Brezinski [2], at least if one is dealing with a nondegenerate solution and if the computation scheme does not break down. Then $[N_k/D_l]_{I_{k+l}} = E_l^{(k)}$ and $[N_n/D_m]_{I_{n+m}} = E_m^{(n)}$ with

$$B_{ik,jl}(x,y) = B_{kl}(x,y)/B_{ij}(x,y), \quad k \geqslant i, l \geqslant j,$$

$$E_0^{(k)} = \sum_{(i,j)\in N_k} c_{d_{0i},e_{0j}} B_{d_0 i, e_0 j}(x,y), \quad k = 0,\dots,n+m, \tag{9a}$$

$$E_l^{(k)} = \frac{E_{l-1}^{(k)} g_{l-1,l}^{(k+1)} - E_{l-1}^{(k+1)} g_{l-1,l}^{(k)}}{g_{l-1,l}^{(k+1)} - g_{l-1,l}^{(k)}}, \quad k = 0,1,\dots,n, l = 1,2,\dots,m, \tag{9b}$$

$$g_{0,l}^{(k)} = t_l(k) - t_{l-1}(k), \quad l = 1,\dots,m, k = 0,\dots,n+m, \tag{10a}$$

$$g_{h,l}^{(k)} = \frac{g_{h-1,l}^{(k)} g_{h-1,h}^{(k+1)} - g_{h-1,l}^{(k+1)} g_{h-1,h}^{(k)}}{g_{h-1,h}^{(k+1)} - g_{h-1,h}^{(k)}}, \quad l = h+1, h+2, \dots. \tag{10b}$$

We have seen in [1] that degeneracy has consequences for neighbouring elements in the table. If the rank of the homogeneous system is not maximal, but has a deficiency of s, then we proved in [1] that the table of rational interpolants (7) is, under certain conditions, composed of hexagonal blocks built around each degenerate $[N_n/D_m]_{I_{n+m}}$ (see fig. 1).

3. Some particular degenerate solutions

Let us introduce some new ratios of determinants. Let $E_{l,s}^{(k,u)}$ denote

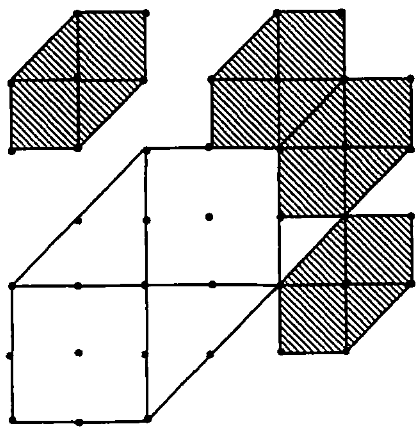

Fig. 1.

$$E_{l,s}^{(k,u)} = \frac{\begin{vmatrix} t_0(k) & \dots & t_0(u-s) & t_0(u+1) & \dots & t_0(k+l+s) \\ \delta t_0(k) & \dots & & & & \\ \vdots & & & & & \\ \delta t_{l-1}(k) & \dots & & & & \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ \delta t_0(k) & \dots & \\ \vdots & & \\ \delta t_{l-1}(k) & \dots & \end{vmatrix}}. \quad (11)$$

These values strongly resemble the values $E_l^{(k)}$ of the previous section: the nondegenerate values are obtained for $s = 0$ and also for $u \geqslant k + l + s$. We now prove the following lemma and theorem for these new ratios of determinants. In both proofs we shall only focus on well-defined values, with nonzero denominator.

LEMMA 1

Let $p(x, y)$ and $q(x, y)$ be defined by (1), (4) and (5) with I satisfying the inclusion property. Let the rank of the coefficient matrix $C_{n+1,n+m}$ in (5b) be given by $m - s$ and let the s linearly dependent rows in $C_{n+1,n+m}$ be consecutive. Let the singular block built around $[N_n/D_m]_{I_{n+m}}$ be an isolated hexagonal. Then for some u with $n + s \leqslant u \leqslant n + m + s - 1$, the rational function $E_{m,s}^{(n,u)}$ belongs to $[N_n/D_m]_{I_{n+m}} \cap [N_n/D_{m+s}]_{I_{n+m+s}}$.

Proof

Let us focus on the defining system of eqs. (5b) that determines the denominator coefficients. Suppose that the s consecutive linearly dependent rows of $C_{n+1,n+m}$ are

$$\begin{pmatrix} c_{d_0 i_{u-s+1}, e_0 j_{u-s+1}} & \dots & c_{d_m i_{u-s+1}, e_m j_{u-s+1}} \\ \vdots & & \vdots \\ c_{d_0 i_u, e_0 j_u} & \dots & c_{d_m i_u, e_m j_u} \end{pmatrix}.$$

Then the homogeneous system (5b) reduces to

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \dots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{u-s}, e_0 j_{u-s}} & \dots & c_{d_m i_{u-s}, e_m j_{u-s}} \\ c_{d_0 i_{u+1}, e_0 j_{u+1}} & \dots & c_{d_m i_{u+1}, e_m j_{u+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \dots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Because of the linear dependence, the coefficient row

$$\left(c_{d_0 i_{u+1}, e_0 j_{u+1}} \dots c_{d_m i_{u+1}, e_m j_{u+1}}\right)$$

can be replaced by

$$\left(\sum_{k=1}^{s+1} \frac{c_{d_0 i_{u-s+k}, e_0 j_{u-s+k}}}{B_{i_{u-s+k} i_{u+1}, j_{u-s+k} j_{u+1}}(x,y)} \dots \sum_{k=1}^{s+1} \frac{c_{d_m i_{u-s+k}, e_m j_{u-s+k}}}{B_{i_{u-s+k} i_{u+1}, j_{u-s+k} j_{u+1}}(x,y)}\right)$$

and s equations can be added. This gives us the following determinant representation for $p(x, y)$ and $q(x, y)$

$$p = \begin{vmatrix} \sum_{(i,j)\in N} c_{d_0 i, e_0 j} B_{ij}(x,y) & \dots & \sum_{(i,j)\in N} c_{d_m i, e_m j} B_{ij}(x,y) \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} & \dots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{u-s}, e_0 j_{u-s}} & \dots & c_{d_m i_{u-s}, e_m j_{u-s}} \\ \sum_{k=1}^{s+1} \frac{c_{d_0 i_{u-s+k}, e_0 j_{u-s+k}}}{B_{i_{u-s+k} i_{u+1}, j_{u-s+k} j_{u+1}}(x,y)} & \dots & \sum_{k=1}^{s+1} \frac{c_{d_m i_{u-s+k}, e_m j_{u-s+k}}}{B_{i_{u-s+k} i_{u+1}, j_{u-s+k} j_{u+1}}(x,y)} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \dots & c_{d_m i_{n+m}, e_m j_{n+m}} \\ \vdots & & \vdots \\ c_{d_0 i_{n_m+s}, e_0 j_{n+m+s}} & \dots & c_{d_m i_{n+m+s}, e_m j_{n+m+s}} \end{vmatrix},$$

$$q = \begin{vmatrix} B_{d_0 e_0}(x,y) & \dots & B_{d_m, e_m}(x,y) \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} & \dots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{u-s}, e_0 j_{u-s}} & \dots & c_{d_m i_{u-s}, e_m j_{u-s}} \\ \sum_{k=1}^{s+1} \frac{c_{d_0 i_{u-s+k}, e_0 j_{u-s+k}}}{B_{i_{u-s+k} i_{u+1}, j_{u-s+k} j_{u+1}}(x,y)} & \dots & \sum_{k=1}^{s+1} \frac{c_{d_m i_{u-s+k}, e_m j_{u-s+k}}}{B_{i_{u-s+k} i_{u+1}, j_{u-s+k} j_{u+1}}(x,y)} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \dots & c_{d_m i_{n+m}, e_m j_{n+m}} \\ \vdots & & \vdots \\ c_{d_0 i_{n_m+s}, e_0 j_{n+m+s}} & \dots & c_{d_m i_{n+m+s}, e_m j_{n+m+s}} \end{vmatrix},$$

where $q(x, y)$ is nonzero because, as we shall see, it reduces to the denominator of $E_{m,s}^{(n,u)}$, which is well-defined. It cannot be singular because otherwise one more equation could be added to (5b) and $E_{m,s}^{(n,u)}$ would solve $[N_n/D_{m+s+1}]_{I_{n+m+s+1}}$ too, which contradicts the fact that the singular hexagonal block is isolated. By multiplying the consecutive rows in p and q, from the second on, respectively by $B_{i_{n+1}j_{n+1}}(x,y),\dots,B_{i_{u-s}j_{u-s}}(x,y),B_{i_{u+1}j_{u+1}}(x,y),\dots,B_{i_{n+m+s}j_{n+m+s}}(x,y)$, in the two determinants above, and after dividing the $(l+1)$th column for $l=0,\dots,m$ by $B_{d_le_l}(x,y)$ in the two determinants above, we get a representation for p and q using partial sums and terms of the series $t_0,\dots,t_m$ defined in (8). Some manipulation with the rows and columns (linear combinations) finally gives us, in the same way as in [4],

$$\frac{p(x,y)}{q(x,y)} = E_{m,s}^{(n,u)},$$

with numerator of degree $\partial p = N_n$, denominator of degree $\partial q = D_m \subset D_{m+s}$ and satisfying the interpolation conditions imposed by $I_{n+m+s} \supset I_{n+m}$. □

THEOREM 1

Let $p(x,y)$ and $q(x,y)$ be defined by (1), (4) and (5) with I satisfying the inclusion property. Let the rank of the coefficient matrix $C_{n+1,n+m}$ in (5b) be given by $m-s$. Let for each pair (k,l) with $0\leqslant k\leqslant s, 0\leqslant l\leqslant s, k+l=s$, the rank of $C_{n-k+1,n+m-s}$ equal its maximal rank $m-l$. Let the hexagonal block of degenerate solutions be isolated, which means that for $0\leqslant k\leqslant s$ the coefficient matrices $C_{n-s+1,n+m-s+k}$ (top row), $C_{n-k+1,n+m-s}$ (leftmost diagonal), $C_{n+k+1,n+m-s+k}$ (leftmost column), $C_{n+s+1,n+m+k}$ (bottom row), $C_{n+s-k+1,n+m+s}$ (rightmost antidiagonal) and finally $C_{n-s+k+1,n+m+k}$ (rightmost column) all have maximal rank. Then for $i=0,\dots,s$ the following can be proved.

(a) $E_{m,s}^{(n-s+i,n+m)}$ is well-defined and solves $[N_{n-s+i}/D_{m+s}]_{I_{n+m+i}}$. It also belongs to $[N_{n-s+i+k}/D_{m+s-k}]_{I_{n+m+i}}$ with $k=0,\dots,s$, meaning that $E_{m,s}^{(n-s+i,n+m)}$ solving $[N_{n-s+i}/D_{m+s}]_{I_{n+m+i}}$ can be shifted downwards in the hexagonal block in the direction of the antidiagonal because it also solves the interpolation problems posed in $[N_{n-s+i+k}/D_{m+s-k}]_{I_{n+m+i}}$.

(b) $E_{m-s+i,s}^{(n,n+m)}$ is well-defined and solves $[N_{n+s}/D_{m-s+i}]_{I_{n+m+i}}$. It also belongs to $[N_{n+s-k}/D_{m-s+i+k}]_{I_{n+m+i}}$ with $k=0,\dots,s$, meaning that $E_{m-s+i,s}^{(n,n+m)}$ solving $[N_{n+s}/D_{m-s+i}]_{I_{n+m+i}}$ can be shifted upwards in the hexagonal block in the direction of the antidiagonal because it also solves the interpolation problems posed in $[N_{n+s-k}/D_{m-s+i+k}]_{I_{n+m+i}}$.

(c) On the rightmost upward sloping diagonal we have for $i=0,\dots,s$: $[N_{n+s-i}/D_{m+i}]_{I_{n+m+s}} = E_{m,s}^{(n,n+m)}$.

Proof

(a) Let us take a look at the system of equations defining $[N_{n-s+i}/D_{m+s}]_{I_{n+m+i}}$:

$$\begin{pmatrix} c_{d_0 i_0, e_0 j_0} & \cdots & c_{d_{m+s} i_0, e_{m+s} j_0} \\ \vdots & & \vdots \\ c_{d_0 i_{n-s+i}, e_0 j_{n-s+i}} & \cdots & c_{d_{m+s} i_{n-s+i}, e_{m+s} j_{n-s+i}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_{m+s} e_{m+s}} \end{pmatrix} = \begin{pmatrix} a_{i_0 j_0} \\ \vdots \\ a_{i_{n-s+i} j_{n-s+i}} \end{pmatrix},$$

$$\begin{pmatrix} c_{d_0 i_{n-s+i+1}, e_0 j_{n-s+i+1}} & \cdots & c_{d_{m+s} i_{n-s+i+1}, e_{m+s} j_{n-s+i+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m+i}, e_0 j_{n+m+i}} & \cdots & c_{d_{m+s} i_{n+m+i}, e_{m+s} j_{n+m+i}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_{m+s} e_{m+s}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We know that the rank of $C_{n-s+i+1,n+m-s}$ is maximal, namely $m-i$ and this for $i = 0, \ldots, s$. So for $i = s$ the matrix

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_{m-s} i_{n+1}, e_{m-s} j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m-s}, e_0 j_{n+m-s}} & \cdots & c_{d_{m-s} i_{n+m-s}, e_{m-s} j_{n+m-s}} \end{pmatrix}$$

has maximal rank $m-s$. Hence also the matrix

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m-s}, e_0 j_{n+m-s}} & \cdots & c_{d_m i_{n+m-s}, e_m j_{n+m-s}} \end{pmatrix} \tag{12}$$

with only longer rows has maximal rank $m-s$. Since $C_{n+1,n+m}$ has a rank deficiency of s and since (12) consists of the top $m-s$ rows of $C_{n+1,n+m}$, the s remaining rows of $C_{n+1,n+m}$ must all be linearly dependent on the rows of (12). Let us discard from (5b) the s linearly dependent equations and replace them by the equations

$$\begin{pmatrix} c_{d_0 i_{n-s+i+1}, e_0 j_{n-s+i+1}} & \cdots & c_{d_m i_{n-s+i+1}, e_m j_{n-s+i+1}} \\ \vdots & & \vdots \\ c_{d_0 i_n, e_0 j_n} & \cdots & c_{d_m i_n, e_m j_n} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

on top and the equations

$$\begin{pmatrix} c_{d_0 i_{n+m+1}, e_0 j_{n+m+1}} & \cdots & c_{d_m i_{n+m+1}, e_m j_{n+m+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m+i}, e_0 j_{n+m+i}} & \cdots & c_{d_m i_{n+m+i}, e_m j_{n+m+i}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

at the end, in total s of them. This completed homogeneous system still has at least one nontrivial solution. A determinant representation for the rational Hermite interpolant p/q given by (1a), (5a), (5b) constructed with this particular solution is precisely the ratio $E_{m,s}^{(n-s+i,n+m)}$. From lemma 1 we know that $E_{m,s}^{(n-s+i,n+m)}$ also solves $[N_{n-s+i}/D_{m+s}]_{I_{n+m+i}}$. Moreover, the denominator of $E_{m,s}^{(n-s+i,n+m)}$ is nonsingular because otherwise one could add one more equation to (5b) and prove that

$E_{m,s}^{(n-s+i,n+m)}$ also solves $[N_{n-s+i}/D_{m+s+1}]_{I_{n+m+i+1}}$, which contradicts the fact that we are dealing with an isolated hexagonal block. Since $E_{m,s}^{(n-s+i,n+m)}$ has a numerator $p(x,y)$ and a denominator $q(x,y)$ of respective degrees $\partial p = N_{n-s+i}$ and $\partial q = D_m$, this solution also solves the interpolation problems $[N_{n-s+i+k}/D_{m+s-k}]_{I_{n+m+i}}$ for $k = 0, \dots, s$.

(b) The proof is constructed in a completely similar way.

(c) Since the block does not stretch beyond diagonal $n+m+s$, we know that for each $i = 0, \dots, s$ the rank of $C_{n+s-i,n+m+s}$ is maximal and given by $m+i$. Now $E_{m,s}^{(n,n+m)}$ solves $[N_n/D_{m+s}]_{I_{n+m+s}}$ and $E_{m,s}^{(n,n+m)}$ also solves $[N_{n+s}/D_m]_{I_{n+m+s}}$ according to (a) and (b). Hence we have a rational Hermite interpolant with numerator and denominator respectively indexed by N_n and D_m and solving the interpolation problem imposed by I_{n+m+s}. Because for each $i = 0, \dots, s$ the solution $[N_{n+s-i}/D_{m+i}]_{I_{n+m+s}}$ is nondegenerate and hence unique, we necessarily have for $i = 0, \dots, s$ that $[N_{n+s-i}/D_{m+i}]_{I_{n+m+s}} = E_{m,s}^{(n,n+m)}$. □

Note that the theorem provides us with a solution in the rightmost column of the isolated hexagonal block, column $m+s$, in the form of a ratio of determinants of size $m+1$, while the coefficient matrix $C_{n-s+i,n+m+i}$ is regular because the block is isolated, implying that its unique solution (up to a multiplicative constant) can also be represented as a ratio of determinants of size $m+s+1$. From this we can conclude that $E_{m,s}^{(n-s+i,n+m)}$ and $E_{m+s}^{(n-s+i)}$ differ only in a common multiplicative factor in numerator and denominator.

4. Special rules for an isolated hexagonal block

When, we run across such an isolated singular hexagonal block as in the previous section, we want to know the values on the edges of the block, because from there on we can take up the nonsingular rules again and proceed with our recursive scheme. Let us walk around the block and try to identify the rational interpolants on all the edges. Remember that $[N_l/D_k]_{I_{l+k}}$ denotes the complete set of solutions while $E_k^{(l)}$ or $E_{k,s}^{(l,u)}$ denote a particular solution from that set.

First there is the upward sloping diagonal with regular entries $[N_{n-s+i}/D_{m-i}]_{I_{n+m-s}}$ because $C_{n-s+i+1,n+m-s}$ has maximal rank for all $i = 0, \dots, s$. Then we proved in [1] that for $i = 0, \dots, s$ the value $E_m^{(n-s)}$ also solves the rational interpolation problems posed in $[N_{n-s}/D_{m+i}]_{I_{n+m-s+i}}$ and analogously for $E_{m-s}^{(n)}$ and $[N_{n+i}/D_{m-s}]_{I_{n+m-s+i}}$. So this deals with the top row and leftmost column of our isolated block. The values in the rightmost column and on the bottom line of the hexagonal block were just respectively identified as $E_{m,s}^{(n-s+i,n+m)}$ and $E_{m-s+i,s}^{(n,n+m)}$ with $i = 1, \dots, s-1$. The closing rightmost upward sloping diagonal is filled with $E_{m,s}^{(n,n+m)}$.

Let us now discuss some particular solutions at the interior of the hexagon. It is essential when identifying certain rational interpolants that we present solutions which are well-defined, in other words, which can be represented as E-values with nonzero denominator determinants. In [1] we mentioned how to fill the left upper half of the hexagonal block with nonsingular E-values, namely by copying the non-degenerate solutions from the upward sloping diagonal over the small triangles that fill this half of the hexagon, as shown in the figure below. Using theorem 1c the triangle emanating from (n,m) in the hexagon can be filled with the regular values from the rightmost upward sloping diagonal, which are all equal because of theorem 1c and which have the correct degrees N_n and D_m (see fig. 2). We shall now see how the rest of the right lower half of the hexagonal structure can be filled with regular E-values. From theorem 1a and 1b we learn that well-defined solutions for the rational Hermite interpolation problems posed in this half of the hexagon come from copies of the rightmost column or copies of the bottom line (see fig. 3).

Essentially this leaves us with the problem of computing these new values $E_{m,s}^{(n-s+i,n+m)}$ and $E_{m-s+i,s}^{(n,n+m)}$. When trying to provide a coherent computation scheme we must be careful not to involve intermediate singular values. A first singular rule for the E-algorithm was proved by Brezinski [3], but not in the context of multivariate rational interpolation. So this rule does not exploit the special hexagonal structure of the table of interpolants. When we would like to apply it, as it stands, to our problem, we jump from the regular E-values in column $m-s$ to the regular E-values in column $m+s$ in one step:

$$E_{m+s}^{(n-s+k)} = E_{m-s}^{(n-s+k)} - \left(g_{m-s,m-s+1}^{(n-s+k)} \cdots g_{m-s,m+s}^{(n-s+k)}\right)$$
$$\times \begin{pmatrix} \Delta g_{m-s,m-s+1}^{(n-s+k)} & \cdots & \Delta g_{m-s,m+s}^{(n-s+k)} \\ \vdots & & \vdots \\ \Delta g_{m-s,m-s+1}^{(n+s+k-1)} & \cdots & \Delta g_{m-s,m+s}^{(n+s+k-1)} \end{pmatrix}^{-1} \begin{pmatrix} \Delta E_{m-s}^{(n-s+k)} \\ \vdots \\ \Delta E_{m-s}^{(n+s+k)} \end{pmatrix}.$$

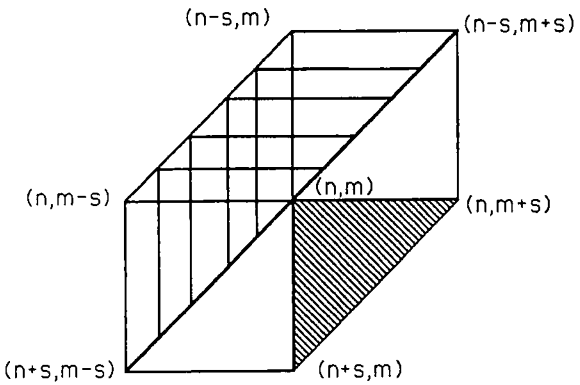

Fig. 2.

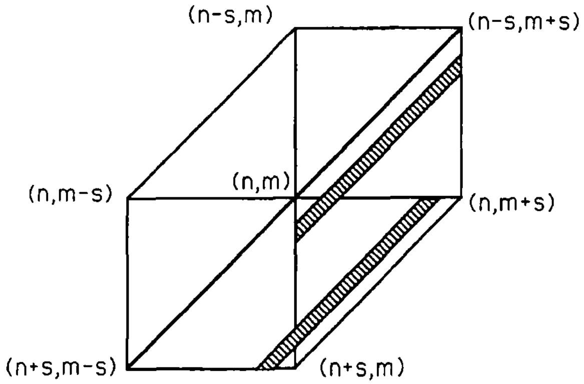

Fig. 3.

On the other hand, an algorithm that is more tailored to this problem in the sense that it exploits the hexagonal structure, is the following application of the E-algorithm to the newly defined values $E_{k,s}^{(l,u)}$. This application uses help-entries $g_{h,k,s}^{(l,u)}$ where the linearly dependent rows in the matrices are discarded. The fact that the E-algorithm remains valid is due to the special form of the determinant ratio (11) which is very similar to the form of the original $E_l^{(k)}$. Using the initialisation

$$E_{m-i,s}^{(n-s+i,n+m)} = E_{m-i}^{(n-s+i)}, \quad i = 1, \ldots, s-1,$$

$$E_{m-i,s}^{(n,n+m)} = E_{m-i}^{(n+s)}, \quad i = 1, \ldots, s-1,$$

$$g_{m-i,r,s}^{(n-s+i,n+m)} = g_{m-i,r}^{(n-s+i)}, \quad i = 0, \ldots, s,$$

$$g_{m-s+i,r,s}^{(n,n+m)} = g_{m-s+i,r}^{(n+s)}, \quad i = 1, \ldots, s, \tag{13a}$$

and the rules

$$E_{l,s}^{(k,n+m)} = \frac{E_{l-1,s}^{(k,n+m)} g_{l-1,l,s}^{(k+1,n+m)} - E_{l-1,s}^{(k+1,n+m)} g_{l-1,l,s}^{(k,n+m)}}{g_{l-1,l,s}^{(k+1,n+m)} - g_{l-1,l,s}^{(k,n+m)}}, \quad k = 0, 1, \ldots, n, l = 1, 2, \ldots, m,$$

$$g_{h,l,s}^{(k,n+m)} = \frac{g_{h-1,l,s}^{(k,n+m)} g_{h-1,h,s}^{(k+1,n+m)} - g_{h-1,l,s}^{(k+1,n+m)} g_{h-1,h,s}^{(k,n+m)}}{g_{h-1,h,s}^{(k+1,n+m)} - g_{h-1,h,s}^{(k,n+m)}}, \quad l = h+1, h+2, \ldots,$$

we can fill the following quasi-triangular table of values:

$$
\begin{array}{ccccc}
 & & & E^{(n-s+1,n+m)}_{m-1,s} & E^{(n-s+1,n+m)}_{m,s} \\
 & & \cdot^{\cdot^{\cdot}} & E^{(n-s+1,n+m)}_{m-1,s} & E^{(n-s+1,n+m)}_{m,s} \\
 & E^{(n-2,n+m)}_{m-s+2,s} & & \vdots & \vdots \\
E^{(n-1,n+m)}_{m-s+1,s} & E^{(n-1,n+m)}_{m-s+2,s} & & & \\
E^{(n,n+m)}_{m-s+1,s} & E^{(n,n+m)}_{m-s+2,s} & \cdots & E^{(n,n+m)}_{m-1,s} & E^{(n,n+m)}_{m,s}
\end{array}
\qquad (13b)
$$

where the bottom row and rightmost column of (13b) respectively solve the interpolation problems in the bottom row and rightmost column of our hexagon. A proof for these rules can be constructed as in [4] for the case $s = 0$. The quasi-triangular table (13b) can in its turn only be filled completely if we do not encounter indefinite values at the interior of (13b).

The initialisations in (13a) are easy to understand. The first is merely by notation: from the determinant representations for $E^{(n-s+i)}_{m-i}$ and $E^{(n-s+i,n+m)}_{m-i,s}$ one can see that these expressions are equal. The second initialisation follows from theorem 1b. The initialisations for the g-values are analogous. The first is by notation, the other by theorem 1b. How do we now get the starting E- and g-values on the bottom row of the hexagon? The bottom row of (13b) is computed from the nondegenerate rules with input values $E^{(n+s)}_{m-s}$ and $E^{(n+s+1)}_{m-s}$. The newly described extension together with its initialisations can then best be understood from fig. 4.

COROLLARY 1

From (13) it can be seen that the new rule, especially designed for the multivariate rational Hermite interpolation problem and summarized in fig. 4 computes $E^{(n-s+k,n+m)}_{m,s} = [N_{n-s+k}/D_{m+s}]_{I_{n+m+k}}$ making use of only $E^{(n-1)}_{m-s+1}, \ldots, E^{(n-s+k)}_{m-k}$ on the leftmost upward sloping diagonal and of $E^{(n+s)}_{m-s+1}, \ldots, E^{(n+s)}_{m-s+k}$ on the bottom row of the hexagonal structure.

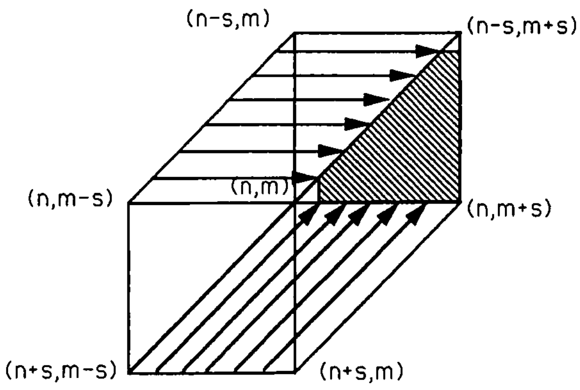

Fig. 4.

5. A general degenerate table

Up to now we have only considered isolated hexagonal blocks for which the linearly dependent rows in $C_{n+1,n+m}$ are consecutive. In general one may have several nonconsecutive groups of linearly dependent rows in the coefficient matrix of the homogeneous system of defining equations. However, it is always possible to give a determinant representation with consecutive column numbers as will be shown in the following theorem. In (11) we already introduced $E_{m,s}^{(n,u)}$ involving the coefficient rows of $C_{n+1,n+m}$ except for the linearly dependent ones

$$\begin{pmatrix} c_{d_0 i_{u-s+1}} & \cdots & c_{d_m i_{u-s+1}} \\ \vdots & & \vdots \\ c_{d_0 i_u} & \cdots & c_{d_m i_u} \end{pmatrix}.$$

In general we will denote by $E_{m,s_1+\ldots+s_t}^{(n,k_1,\ldots,k_t)}$ a ratio of determinants similar to the one given in (11) but now with t groups of linearly dependent rows lacking, namely those indexed in $C_{n+1,n+m}$ by

$$\begin{pmatrix} c_{d_0 i_{k_1-s_1+1}} & \cdots & c_{d_m i_{k_1-s_1+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{k_1}} & \cdots & c_{d_m i_{k_1}} \\ & \vdots & \\ c_{d_0 i_{k_t-s_t+1}} & \cdots & c_{d_m i_{k_t-s_t+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{k_t}} & \cdots & c_{d_m i_{k_t}} \end{pmatrix}.$$

Explicitly $E_{m,s_1+\ldots+s_t}^{(n,k_1,\ldots,k_t)}$ is given by

$$E_{m,s_1+\ldots+s_t}^{(n,k_1,\ldots,k_t)} = \frac{\begin{vmatrix} t_0(n) & \ldots t_0(k_1-s_1)t_0(k_1+1)\ldots t_0(k_t-s_t)t_0(k_t+1)\ldots t_0(n+m+s_1+\ldots+s_t) \\ \delta t_0(n) & \ldots \\ \vdots & \\ \delta t_{m-1}(n)\ldots & \end{vmatrix}}{\begin{vmatrix} 1 & \ldots & 1 \\ \delta t_0(n) & \ldots & \\ \vdots & & \\ \delta t_{m-1}(n) & \ldots & \end{vmatrix}}. \tag{14}$$

THEOREM 2

For each entry $[N_n/D_m]_{I_{n+m}}$ in the table of rational Hermite interpolants a well-defined determinant representation of the form $E^{(n,k_1,\ldots,k_t)}_{l,s_1+\ldots+s_t}$ exists which belongs to $[N_n/D_m]_{I_{n+m}}$ with $l \leqslant m$ and $s_i > 0$ for $i = 1, \ldots, t$. Numerator p and denominator q of $E^{(n,k_1,\ldots,k_t)}_{l,s_1+\ldots+s_t}$ have respective degrees $\partial p = N_n$ and $\partial q = D_l$.

Proof

Let us first construct l, which will determine the size of the matrices in the determinant representation for $[N_n/D_m]_{I_{n+m}}$. Consider $C_{n+1,n+m}$ and permute its rows until you have an $l \times (l+1)$ submatrix of maximal rank l in the upper left corner. Do not permute any columns. Now consider the $m \times (l+1)$ matrix

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_l i_{n+1}, e_l j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_l i_{n+m}, e_l j_{n+m}} \end{pmatrix}.$$

It is clear that this matrix contains $m - l$ linearly dependent rows and so we have already determined the total size of the gap that will occur in the final determinant representation, namely $s_1 + \ldots + s_t = m - l$. However, the linearly dependent lines may occur in different groups (say t in total) at different places. By computing $b_{d_0 e_0}, \ldots, b_{d_l e_l}$ from

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_l i_{n+1}, e_l j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_l i_{n+m}, e_l j_{n+m}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_l e_l} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{15}$$

and putting $b_{d_{l+1} e_{l+1}} = \ldots = b_{d_m e_m} = 0$ we have constructed a nontrivial solution $q(x,y) = \sum_{i=0}^{l} b_{d_i e_i} B_{d_i e_i}(x,y)$ to (5b). It is clear that by discarding the linearly dependent rows in (15) and using the same technique as in lemma 1, a determinant representation for this solution is given by $E^{(n,k_1,\ldots,k_t)}_{l,s_1+\ldots+s_t}$. □

In a nondegenerate table every solution $[N_n/D_m]_{I_{n+m}}$ is given by a determinant representation $E^{(n)}_m$ where the numerator and denominator determinants are of size $m+1$. In a degenerate table one jumps over certain singularities and this implies that the size of a nonsingular determinant representation is smaller than the cardinality of the denominator index set D_m. Simply consider $E^{(n-s+i,n+m)}_{m,s}$ with determinants of size $m+1$ solving $[N_{n-s+i}/D_{m+s}]_{I_{n+m+i}}$, which in the nonsingular case only has a determinant formula of size $m+s+1$. Or recall the ideas from the previous proof to understand this phenomenon. From the last paragraph of section 3 it must also be clear that the size $l+1$ used in theorem 2 is not unique. In the rightmost column of an isolated hexagon one has different determinant representations, respectively with $l = m$ and $l = m + s$.

In a general table of multivariate rational interpolants, we do not only deal with isolated hexagons, but several hexagonal blocks can be adjacent or sometimes partially enlarged at one of the edges as described in [1]. In order to treat the following example we introduce the flags

$$\begin{aligned} sing(n,m) &= 0 \Leftrightarrow C_{n+1,n+m} \text{ has maximal rank}, \\ sing(n,m) &= 1 \Leftrightarrow C_{n+1,n+m} \text{ has a nonzero rank deficiency}. \end{aligned}$$

EXAMPLE 1

Consider

$$f(x,y) = \frac{x+y}{x-y-xy},$$

with interpolation points (x_i, y_j) given by $x_i = i$ for $i = 0, 1, \ldots$ and $y_j = j+1$ for $j = 0, 1, \ldots$. We want to obtain a determinant representation for $[N_k/D_l]_{I_{k+l}}$ with $1 \leqslant k, l \leqslant 5$ and $1 \leqslant k+l \leqslant 6$ where

$$N_6 = D_6 = I_6 = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(3,0)\}.$$

We display the table of values $sing(n,m)$ indicating where the singularities are located and we give the numerator determinants of the representation obtained in theorem 2, evaluated at $(x,y) = (5,2)$. The denominator determinants can be found by replacing the first row in the numerator determinants by $(1, \ldots, 1)$.

$sing(n,m)$

	D_0	D_1	D_2	D_3	D_4	D_5
N_1	1	0	0	1	0	0
N_2	1	1	1	0	0	
N_3	1	1	0	0		
N_4	1	0	0			
N_5	1	0				
N_6	1					

	D_0	D_1	D_2	D_3	D_4	D_5
N_1	-6	-6	$\begin{vmatrix} -6 & -6 \\ 4 & 0 \end{vmatrix}$	$\begin{vmatrix} -6 & -6 & -6 & -1 \\ 4 & 4 & 0 & -4 \\ 2 & 1 & 5 & 4 \\ 0 & 1 & -2 & -2 \end{vmatrix}$	$\begin{vmatrix} -6 & -6 & -6 & -1 \\ 4 & 4 & 0 & -4 \\ 2 & 1 & 5 & 4 \\ 0 & 1 & -2 & -2 \end{vmatrix}$	$\begin{vmatrix} -6 & -6 & -6 & -1 & -1 \\ 4 & 4 & 0 & -4 & -4 \\ 2 & 1 & 5 & 4 & 4 \\ 0 & 1 & -2 & -2 & -5 \\ 0 & 0 & 3 & 2 & 5 \end{vmatrix}$
N_2	-6	$\begin{vmatrix} -6 & -6 \\ 4 & 0 \end{vmatrix}$	$\begin{vmatrix} -6 & -6 & -6 \\ 4 & 0 & -4 \\ 1 & 5 & 4 \end{vmatrix}$	$\begin{vmatrix} -6 & -6 & -6 \\ 4 & 0 & -4 \\ 1 & 5 & 4 \end{vmatrix}$	$\begin{vmatrix} -6 & -6 & -6 \\ 4 & 0 & -4 \\ 1 & 5 & 4 \end{vmatrix}$	
N_3	-6	$\begin{vmatrix} -6 & -1 \\ 0 & -4 \end{vmatrix}$	$\begin{vmatrix} -6 & -1 \\ 0 & -4 \end{vmatrix}$	$\begin{vmatrix} -6 & -1 \\ 0 & -4 \end{vmatrix}$		
N_4	-1	-1	-1			
N_5	-1	-1				
N_6	-1					

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