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Unattainable Points in Multivariate Rational Interpolation

H. ALLOUCHE

UFR-IEEA, Groupe ANO, Université de Lille I, Bâtiment M3, F-59655 Villeneuve d'Ascq Cedex, France

AND

A. CUYT*

Department of Mathematics and Computer Science, University of Antwerp (UIA), Universiteitsplein 1, B-2610 Wilrijk, Belgium

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The problem of unattainable points is typical for the case of rational interpolation. Having computed the rational interpolant p/q from "linearized" interpolation conditions, in other words, conditions expressed for $fq - p$ instead of for $f - (p/q)$, it may occur that an interpolation point is also a common zero of p and q and hence that the rational function p/q is undefined in that interpolation point. Consequently the "nonlinear" interpolation condition cannot be satisfied in that interpolation point anymore, not even by the irreducible form of p/q. The interpolation point has become "unattainable."

1. UNATTAINABLE POINTS IN UNIVARIATE RATIONAL INTERPOLATION

The univariate problem of unattainable points was extensively discussed by Claessens in [3]. We list his results that will serve as a starting point for our discussion of the multivariate situation.

Let f be a univariate complex function known in the complex interpolation points $(x_i)_{i \in \mathbb{N}}$. We construct the polynomial basis functions

$$B_i(x) = \prod_{k=0}^{i-1} (x - x_k).$$

The problem of interpolating f by a rational function with numerator of

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degree n and denominator of degree m is formulated as follows. Find polynomials

$$p(x)=\sum_{i=0}^{n} a_i B_i(x)$$

$$q(x)=\sum_{i=0}^{m} b_i B_i(x)$$

satisfying

$$(fq-p)(x)=\sum_{i\geqslant n+m+1} d_i B_i(x) \tag{1}$$

for the formal Newton interpolating series development of $fq-p$. Condition (1) means that

$$d_i=(fq-p)[x_0, ..., x_i]=0, \qquad i=0, ..., n+m \tag{2}$$

and implies

$$(fq-p)(x_i)=0, \qquad i=0, ..., n+m.$$

The advantage of expressing the rational interpolation problem as a Newton–Padé approximation problem in (1) is that this formulation can also be used in case some of the interpolation points coincide, because the divided differences in (2) are defined for coalescent points. Let us denote

$$c_{ij}=f[x_i, ..., x_j], \qquad i\leqslant j$$

with $c_{ij}=0$ if $i>j$. Using a kind of Leibniz rule for divided differences the conditions in (2) for the coefficients a_i and b_i can be rewritten as

$$\sum_{i=0}^{m} c_{ij} b_i = a_j, \qquad j=0, ..., n$$

$$\begin{pmatrix} c_{0,n+1} & \cdots & c_{m,n+1} \\ \vdots & & \vdots \\ c_{0,n+m} & \cdots & c_{m,n+m} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}. \tag{3}$$

If the rank of the coefficient matrix in (3) is $m-s$, then it was proved in [2] that a unique solution $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ of (1) exists with $\partial\bar{p}_{n,m}\leqslant n-s$ and $\partial\bar{q}_{n,m}\leqslant m-s$ where at least one of the upper bounds for the degrees is attained. Every other solution $p(x)$ and $q(x)$ is of the form $p(x)=t(x)\,\bar{p}_{n,m}(x)$ and $q(x)=t(x)\,\bar{q}_{n,m}(x)$. Since no solution exists where both the degrees can be lowered simultaneously, these unique polynomials $\bar{p}_{n,m}$ and $\bar{q}_{n,m}$ are called the "minimal solution." This minimal solution in fact solves a whole triangle of interpolation problems in the rational interpolation table, namely all those lying in the triangle with corners $(n-s, m-s)$, $(n+s, m-s)$, and $(n-s, m+s)$ depicted in Fig. 1.

Since the degrees in the minimal solution are minimal, the amount of troubling common factors in the numerator and denominator of the minimal solution is kept minimal. We call an interpolation point x_i an "unattainable point" of order l if

$$\bar{p}_{n,m}^{(k)}(x_i) = 0 = \bar{q}_{n,m}^{(k)}(x_i), \qquad k = 0, ..., l$$

with l as large as possible. The following theorem, proved in [3], says that common factors in minimal solutions only involve unattainable interpolation points.

THEOREM 1. *Let $(x-\alpha)^k$ be a common factor of the minimal solution $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ of (1). Then $\alpha \in \{x_i \mid i = 0, ..., n+m\}$ and $k \leqslant m(\alpha)$ where $m(\alpha)$ denotes the multiplicity of α in $\{x_i \mid i = 0, ..., n+m\}$.*

In other words, k of the $m(\alpha)$ interpolation conditions in α cannot be attained when computing the rational interpolant [3].

THEOREM 2. *Let $x_{i_1} = \cdots = x_{i_{m(\alpha)}} = \alpha$, with $i_1 \leqslant \cdots \leqslant i_{m(\alpha)}$, belong to $\{x_i \mid i = 0, ..., n+m\}$. Then α is an unattainable point of order l if and only if for the irreducible form $r_{n,m}$ of $\bar{p}_{n,m}/\bar{q}_{n,m}$*

$$r_{n,m}^{(i)}(\alpha) = f^{(i)}(\alpha), \qquad i = 0, ..., m(\alpha) - l - 1$$
$$r_{n,m}^{(m(\alpha)-l)}(\alpha) \neq f^{(m(\alpha)-l)}(\alpha).$$

For the irreducible form $r_{n,m}(x)$ even more can be proved. The following result in fact introduces unattainable points in further entries of the rational interpolation table [3].

THEOREM 3. *Let the minimal solution $\bar{p}_{n,m}$ and $\bar{q}_{n,m}$ of (1) be such that $\partial\bar{p}_{n,m} = n - s_1$ and $\partial\bar{q}_{n,m} = m - s_2$. Then all the rational interpolants lying in*

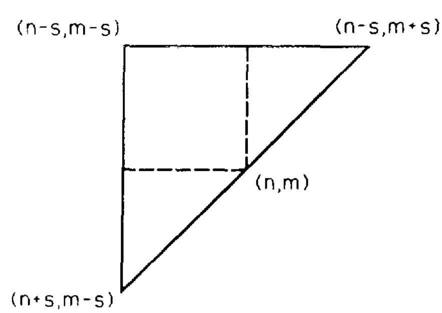

FIGURE 1

the square with corners $(n-s_1, m-s_2)$ *and* $(n+s_2, m+s_1)$ *have the same irreducible form* $r_{n,m}(x)$.

This result should be read as: since $r_{k,l} = r_{n,m}$ for $k+l > n+m$ we know that $\bar{p}_{k,l}(x_i) = 0 = \bar{q}_{k,l}(x_i)$ for $i = n+m+1, ..., k+l$ and hence that $x_{n+m+1}, ..., x_{k+l}$ are unattainable points. More details can be found in [3]. This square is in fact the union of the triangle emanating from the minimal solution and its mirror image. For each entry in this mirror image its minimal solution is constructed by multiplying $\bar{p}_{n,m}$ and $\bar{q}_{n,m}$ with a factor containing interpolation points and hence unattainable points. This technique ensures us that more coefficients in the Newton series for $fq - p$ vanish.

THEOREM 4. *Let the minimal solution* $\bar{p}_{n-n_1,m-m_1}$ *and* $\bar{q}_{n-n_1,m-m_1}$ *be such that*

(a) $\partial\bar{p}_{n-n_1,m-m_1} = n - n_1$

(b) $\partial\bar{q}_{n-n_1,m-m_1} = m - m_1$

(c) $(f\bar{q}_{n-n_1,m-m_1} - \bar{p}_{n-n_1,m-m_1})(x) = \sum_{i \geqslant n+m+1} d_i B_i(x)$

(d) $r_{n-n_1,m-m_1}(x)$ *also satisfies the interpolation conditions in the points* $x_{n+m+1+\beta_j}$ *for* $j = 1, ..., t$ *and* $0 \leqslant \beta_1 < \cdots < \beta_t$.

Then if $\beta_j < zj + n_1 + m_1$ *we have for* $l = \beta_j + 1, ..., 2j + n_1 + m_1$: $r_{n+m_1+j,m-m_1+l-j} = r_{n-n_1,m-m_1} = r_{n-n_1+l-j,m+n_1+j}$.

This theorem explains that the square block described in the previous theorem is only a starting point and that it can have a sort of tail concentrated along its main diagonal as illustrated in Fig. 2. Let us now investigate what remains valid if we turn to the case of more variables.

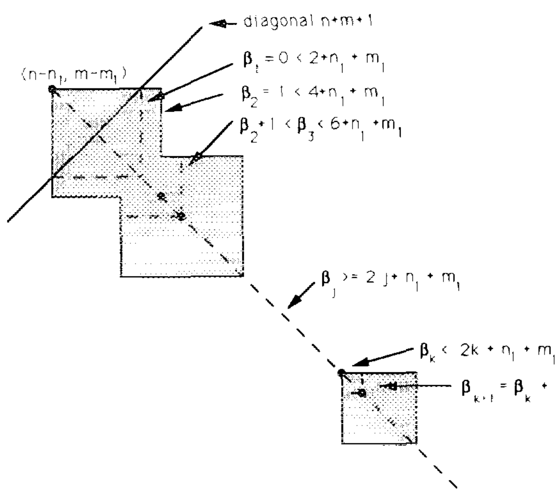

FIGURE 2

2. The Multivariate Situation

It is immediately clear that there will be one very important difference between the univariate and the multivariate case. In the univariate situation a common zero α of two polynomials gives rise to a common factor $(x-\alpha)$. In the multivariate case one can have $p(\alpha, \beta)=0=q(\alpha, \beta)$ without having a factor like $(x-\alpha)(y-\beta)$ common in p and q. However, unattainable interpolation points still do exist and a theorem analogous to Theorem 1 can be proved. We first resume the definition of multivariate rational interpolant and the notion of multivariate minimal solution from [4].

Let a bivariate function $f(x, y)$ be known in the complex data points $((x_i, y_j))_{(i,j)\in \mathbb{N}^2}$ and let I be a finite subset of $\mathbb{N}^2$ indexing those data points that will be used as interpolation points. With the data points we construct the polynomial basis functions

$$B_{ij}(x, y)=\prod_{k=0}^{i-1}(x-x_k)\prod_{l=0}^{j-1}(y-y_l).$$

The problem of interpolating these data by a bivariate rational function was formulated in [4] as follows. Choose finite subsets N and D of $\mathbb{N}^2$ with $N\subset I$ and compute bivariate polynomials

$$\begin{aligned} p(x, y)&=\sum_{(i,j)\in N} a_{ij}B_{ij}(x, y), \qquad \# N=n+1 \\ q(x, y)&=\sum_{(i,j)\in D} b_{ij}B_{ij}(x, y), \qquad \# D=m+1 \end{aligned} \tag{4a}$$

such that

$$(fq-p)(x_i, y_j)=0, \qquad (i, j)\in I, \quad \# I=n+m+1. \tag{4b}$$

If $q(x_i, y_j)\neq 0$ then this last condition implies that

$$f(x_i, y_j)=\frac{p}{q}(x_i, y_j), \qquad (i, j)\in I. \tag{4c}$$

We say that I satisfies the inclusion property if whenever a point (i, j) belongs to I, all the points in the rectangle emanating from the origin with (i, j) as its furthermost corner belong to I. Condition (4b) is for instance met if the following two conditions are satisfied by the polynomials given in (4a) [4],

$$(fq-p)(x, y)=\sum_{(i,j)\in \mathbb{N}^2\setminus I} d_{ij}B_{ij}(x, y) \tag{5a}$$

$$I \text{ satisfies the inclusion property,} \tag{5b}$$

where the Newton interpolating series development (5a) is formal. Condition (5a) can also be used if some or all of the interpolation points or their coordinates coincide since it can be replaced by conditions in terms of bivariate divided differences [4],

$$(fq)[x_0, ..., x_i][y_0, ..., y_j] = p[x_0, ..., x_i][y_0, ..., y_j], \qquad (i, j) \in N \quad (6a)$$

$$(fq)[x_0, ..., x_i][y_0, ..., y_j] = 0, \qquad (i, j) \in I \backslash N. \quad (6b)$$

Using a multivariate generalization of the Leibniz rule for divided differences [4] we can substitute $(fq)[x_0, ..., x_i][y_0, ..., y_j]$ in (6a), (6b) with the notation

$$c_{\mu i, \nu j} = f[x_\mu, ..., x_i][y_\nu, ..., y_j]$$

by

$$(fq)[x_0, ..., x_i][y_0, ..., y_j] = \sum_{(\mu, \nu) \in D} b_{\mu\nu} c_{\mu i, \nu j}.$$

Also

$$p[x_0, ..., x_i][y_0, ..., y_j] = a_{ij}, \qquad (i, j) \in N.$$

From now on we denote a rational function satisfying (6) by $[N/D]_I$. Numbering the points in the sets N, D, and I as

$$N = \{(i_0, j_0), ..., (i_n, j_n)\} \quad (7a)$$

$$D = \{(d_0, e_0), ..., (d_m, e_m)\} \quad (7b)$$

$$I = N \cup \{(i_{n+1}, j_{n+1}), ..., (i_{n+m}, j_{n+m})\} \quad (7c)$$

condition (6) becomes

$$\sum_{\mu=0}^{m} c_{d_\mu i_\nu, e_\mu j_\nu} b_{d_\mu e_\mu} = a_{i_\nu j_\nu}, \qquad \nu = 0, ..., n \quad (8a)$$

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (8b)$$

It is obvious that at least one nontrivial solution of (8b) exists, but it is not so (unlike the univariate case) that different solutions p_1, q_1 and p_2, q_2 of (8) are necessarily equivalent, meaning that $(p_1 q_2)(x, y) = (p_2 q_1)(x, y)$. With the numberings (7a), (7b), and (7c) of the respective indices in N, D, and I we can set up descending chains of index sets, defining bivariate polynomials of "lower degree" and bivariate rational interpolation problems of "lower order":

$$N = N_n \supset \cdots \supset N_k = \{(i_0, j_0), ..., (i_k, j_k)\} \supset \cdots$$
$$\supset N_0 = \{(i_0, j_0)\}, \qquad k = 0, ..., n \tag{9a}$$
$$D = D_m \supset \cdots \supset D_l = \{(d_0, e_0), ..., (d_l, e_l)\} \supset \cdots$$
$$\supset D_0 = \{(d_0, e_0)\}, \qquad l = 0, ..., m \tag{9b}$$
$$I = I_{n+m} \supset \cdots \supset I_{k+l} = \{(i_0, j_0), ..., (i_{k+l}, j_{k+l})\} \supset \cdots$$
$$\supset I_0 = \{(i_0, j_0)\}, \qquad k + l = 0, ..., n + m \tag{9c}$$
$$I_{k+1,k+l} = \{(i_{k+1}, j_{k+1}), ..., (i_{k+l}, j_{k+l})\}, \qquad I \backslash N = I_{n+1,n+m}.$$

From now on we denote by ∂p the exact "degree set" of the polynomial $p(x, y)$. Hence

$$\partial p = N_n \Leftrightarrow p(x, y) \qquad \text{given by (4a) with} \quad a_{i_n j_n} \neq 0.$$

If all the sets $I_{k+l} \subset I$ also satisfy the inclusion property (this can easily be achieved by an appropriate enumeration), then we can compute the following entries in a "table" of multivariate rational interpolants:

$$\begin{matrix} [N_0/D_0]_{I_0} & \cdots & [N_0/D_m]_{I_m} \\ \vdots & & \vdots \\ [N_n/D_0]_{I_n} & \cdots & [N_n/D_m]_{I_{n+m}}. \end{matrix} \tag{10}$$

If we let n and m increase, infinite chains of index sets as in (9) can be constructed and an infinite table of multivariate rational interpolants results. Of course, in practice, only a finite number of entries will be computed. It was proved in [1] that if the rank of the coefficient matrix in (8b) equals $m - s$ then some rational interpolants of "minimal" degree can be computed for $[N/D]_I = [N_n/D_m]_{I_{n+m}}$.

THEOREM 5. *Let $p(x, y)$ and $q(x, y)$ be defined by* (4a) *and* (8a), (8b). *Let the rank of $C_{n+1,n+m}$ in* (8b) *be given by $m - s$. Then for $0 \leqslant k \leqslant s$ and the rank of $C_{n-k+1,n+m-s}$ equal to its maximal rank $m - s + k$, the unique rational function*

$$[N_{n-k}/D_{m-s+k}]_{I_{n+m-s}}$$

also solves $[N/D]_I = [N_n/D_m]_{I_{n+m}}$.

Clearly minimal solutions aren't uniquely determined anymore. In Theorem 5 all solutions $\bar{p}_{n,m,k}/\bar{q}_{n,m,k} = [N_{n-k}/D_{m-s+k}]_{I_{n+m-s}}$ are "minimal" in the sense that they use a minimal number of parameters and data to solve the (n, m) rational interpolation problem, in other words, each of the minimal solutions on the $(n+m-s)$th diagonal (with numerator and denominator "degree" respectively less than or equal to n and m). Again each minimal solution in fact solves a whole triangle of interpolation problems, namely those in the triangle with corners $(n-k, m-s+k)$, $(n+s-k, m-s+k)$, and $(n-k, m+k)$ [1]. This triangle is smaller than in the univariate case. From Fig. 3 it is clear that the union of all the triangles emanating from the minimal solutions $\bar{p}_{n,m,k}$ and $\bar{q}_{n,m,k}$ is a trapezoidal structure contained in the circumscribing triangle with corners $(n-s, m-s)$, $(n+s, m-s)$, and $(n-s, m+s)$. Why this trapezium cannot be enlarged to cover the large triangle completely is explained in [1]. If the rank of $C_{n-k+1,n+m-s}$ in Theorem 5 is not yet maximal then one can further retreat in the table of rational interpolants as will be shown in the following lemma.

In the univariate case the minimal solution is unique and is either a true irreducible solution or a reducible solution with unattainable interpolation points. We have just seen that in the multivariate case a minimal solution is not unique anymore. What's more, in the multivariate case a rational interpolation problem can have both a true irreducible minimal solution and a reducible minimal solution at the same time. Consider the following example and the solution sets $[N_n/D_m]_{I_{n+m}}$ for a number of n and m.

$$x_i = i, \qquad i = 0, 1, 2, \ldots$$

$$y_j = j+1, \qquad j = 0, 1, 2, \ldots$$

$$I_5 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$$

$$f(I_5) = \{-1, -2, -1, -3, -1, -1\}$$

$$N_2 = \{(0,0), (1,0), (0,1)\}$$

$$D_4 = \{(0,0), (1,0), (0,1), (2,0), (1,1)\}.$$

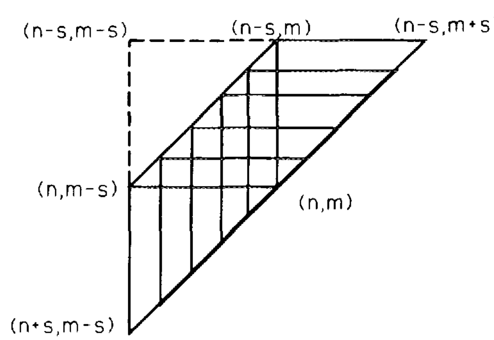

FIGURE 3

Note that

$$[N_0/D_4]_{I_4} = \frac{-6}{6-3x+x(x-1)+3x(y-1)}$$

and

$$[N_1/D_3]_{I_4} = \frac{3-3x}{-3+3x-x(x-1)}$$

are two minimal solutions for

$$[N_1/D_4]_{I_5} = \frac{-\alpha+3\beta x}{\alpha-\alpha x+\beta x(x-1)+(\alpha-3\beta)\,x(y-1)},$$

where $[N_0/D_4]_{I_4}$ is a true irreducible solution satisfying (4c) and $[N_1/D_3]_{I_4}$ is a reducible solution containing the common factor $(x-1)$ with (x_1, y_0) and (x_1, y_1) as unattainable points. In the multivariate case a solution must not be reducible in order to have unattainable interpolation points. Take a look at $[N_2/D_3]_{I_5}$ and you see that the general solution $(p/q)(x, y)$ with $\alpha\beta \neq 0$ is irreducible while (x_1, y_0) is an unattainable interpolation point since $p(x_1, y_0) = 0 = q(x_1, y_0)$. This is a situation which is essentially different from the univariate one. However, common factors of minimal solutions still always involve unattainable points as will be shown in the following theorem. We first formalize the definition of unattainable point. An interpolation point (x_i, y_j) is called "unattainable" of order (l_1, l_2) if for some $0 \leq k \leq s$ the solution $\bar{p}_{n,m,k}/\bar{q}_{n,m,k} = [N_{n-k}/D_{m-s+k}]_{I_{n+m-s}}$ of $[N_n/D_m]_{I_{n+m}}$ satisfies

$$\frac{\partial^{\mu+\nu}\bar{p}_{n,m,k}}{\partial x^\mu\,\partial y^\nu}(x_i, y_j) = 0 = \frac{\partial^{\mu+\nu}\bar{q}_{n,m,k}}{\partial x^\mu\,\partial y^\nu}(x_i, y_j), \qquad 0 \leq \mu \leq l_1,\ 0 \leq \nu \leq l_2.$$

LEMMA. *Let $p(x, y)$ and $q(x, y)$ be defined by* (4a) *and* (8a), (8b) *for the $(n-n_1, m-m_1)$ rational interpolation problem and let*

(a) $\partial p = N_{n-n_1}$

(b) $\partial q = D_{m-m_1}$

(c) $(fq-p)(x, y) = \sum_{(i,j)\in \mathbb{N}^2\setminus I_{n+m}} d_{ij} B_{ij}(x, y)$ *with* $d_{i_{n+m+1} j_{n+m+1}} \neq 0$.

Then $p(x, y)$ and $q(x, y)$ solve the multivariate rational interpolation problem at entry (k, l) if and only if (k, l) belongs to the triangle with corners $(n-n_1, m-m_1)$, $(n+m_1, m-m_1)$, and $(n-n_1, m+n_1)$.

Proof. We first prove the necessary condition. If p/q solves the (k, l) rational interpolation problem, then k and l must be such that

$$n - n_1 \leqslant k \qquad \text{implying} \quad N_{n-n_1} \subset N_k$$
$$m - m_1 \leqslant l \qquad \text{implying} \quad D_{m-m_1} \subset D_l$$
$$n + m \geqslant k + l \qquad \text{implying} \quad I_{k+l} \subset I_{n+m}.$$

Now we concentrate on the sufficient condition. Consider the (k, l) rational interpolation problem with (k, l) in the triangle in question. Then clearly $p(x, y)$ and $q(x, y)$ solving the $(n - n_1, m - m_1)$ rational interpolation problem also satisfy

$$\partial p \subset N_k$$
$$\partial q \subset D_l$$
$$(fq - p)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I_{k+l}} d_{ij} B_{ij}(x, y)$$

which completes the proof. ▌

The importance of this lemma lies in the fact that it describes a structure of the table of multivariate rational interpolants "emanating" from a "minimal solution." Each minimal solution solves a whole triangle of interpolation problems and the maximal triangle is given in this lemma. We now generalize Theorem 1 to the multivariate case.

THEOREM 6. *Let $p(x, y)$ and $q(x, y)$ be defined by* (4a) *and* (8a), (8b) *with $\partial p = N_{n-n_1}$, $\partial q = D_{m-m_1}$, and $C_{n-n_1+1, n+m-n_1-m_1}$ of maximal rank. If p and q have a common factor*

$$t(x, y) = \sum_{(i,j) \in T} t_{ij} B_{ij}(x, y)$$

with $\{(0, 0)\} \neq T$ then $t(x, y)$ passes through at least one interpolation point of $I_{n+m-n_1-m_1}$, meaning that $t(x_i, y_j) = 0$ for some $(i, j) \in I_{n+m-n_1-m_1}$.

Proof. We know that $p = tp^*$, $q = tq^*$, and that

$$(fq - p)(x, y) = (fq^* - p^*)(x, y)\, t(x, y)$$

with $\partial p^* \subset \partial p$ and $\partial q^* \subset \partial q$. Let us suppose that $t(x_i, y_j) \neq 0$ for $(i, j) \in I_{n+m-n_1-m_1}$. Then

$$(fq^* - p^*)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I_{n+m-n_1-m_1}} d_{ij} B_{ij}(x, y).$$

Since $T \neq \{(0,0)\}$ we can renumber N_{n-n_1} and D_{m-m_1} in such a way that

$$\partial p^* = N_{n-n_1-u}$$
$$\partial q^* = D_{m-m_1-v}$$

with u and v strictly positive. This contradicts the maximal triangle described in the lemma for the $(n-n_1, m-m_1)$ rational interpolation problem. ▮

This Theorem 6 is a multivariate counterpart for the univariate Theorem 1. However, a multivariate counterpart for Theorems 3 and 4 does not hold as will be shown in the following examples.

If a generalization of Theorem 3 would hold, then from a minimal solution on the leftmost upward sloping diagonal of the trapezium in Fig. 3, another solution could be constructed by adding some unattainable points such that more terms in the Newton series would disappear. The property of having unattainable points should be investigated for each entry in the hexagon depicted in Fig. 4. This hexagon consists of the trapezium and its mirror image just as in the univariate case the square comes from the triangle and its mirror image.

The following example is a counterexample. Consider

$$x_i = i, \qquad i = 0, 1, 2, ...$$
$$y_j = j, \qquad j = 0, 1, 2, ...$$
$$I_5 = \{(0,0), (1,0), (0,1), (1,1), (2,0), (2,1)\}$$
$$f(I_5) = \{1, -1, -1, 0, 1, -1\}$$
$$N_2 = \{(0,0), (1,0), (0,1)\}$$
$$D_3 = \{(0,0), (1,0), (0,1), (1,1)\}.$$

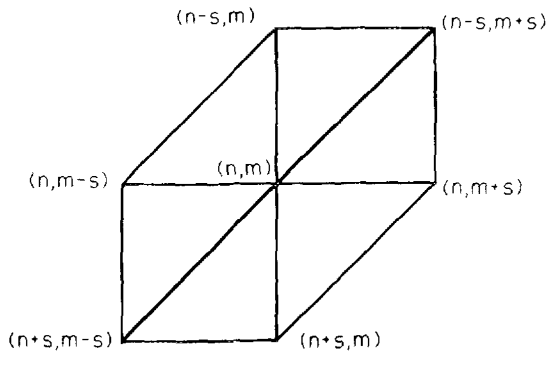

FIGURE 4

Then $[N_1/D_3]_{I_4}$ is singular with $[N_1/D_2]_{I_3}$ as one of its minimal solutions on the leftmost upward sloping diagonal from the hexagonal block concentrated around the entry (1, 3). The maximal triangle emanating from this minimal solution spans $[N_1/D_2]_{I_3}$, $[N_2/D_2]_{I_4}$, and $[N_1/D_3]_{I_4}$. So we investigate

$$[N_2/D_3]_{I_5} = \frac{2-2x}{2-2x-4y+4xy}$$

in the mirror image of this triangle. We would expect to find that (x_2, y_1) is an unattainable interpolation point but this is not the case.

If a generalization of Theorem 4 would hold, then under similar conditions as in Theorem 4 one would find unattainable points further down the table of multivariate rational interpolants outside the initial hexagonal block. We give some counterexamples to discourage anyone from believing that similarities with the univariate case can be proved. Consider

$$x_i = i, \qquad i = 0, 1, 2, \ldots$$
$$y_j = j, \qquad j = 0, 1, 2, \ldots$$

$$I_7 = \{(0,0), (1,0), (0,1), (1,1), (2,0), (2,1), (0,2), (1,2)\}$$
$$f(I_7) = \{1, -1, -1, 0, 1, -1, -1/3, 0\}$$

$$N_3 = \{(0,0), (1,0), (0,1), (1,1)\}$$
$$D_4 = \{(0,0), (1,0), (0,1), (1,1), (2,0)\}.$$

The rational interpolation problem $[N_2/D_3]_{I_5}$ is singular with a minimal solution in $[N_1/D_2]_{I_3}$. This minimal solution satisfies in addition the interpolation conditions in (x_{i_6}, y_{j_6}) and (x_{i_7}, y_{j_7}). Since the interpolation condition in (x_{i_5}, y_{j_5}) is not satisfied, the hexagonal block built around entry (1, 3) does not stretch beyond the diagonal $n+m=5$. We take a look at

$$[N_3/D_4]_{I_7} = \frac{1-x}{1-x-2y+2xy}$$

and see whether it has (x_{i_5}, y_{j_5}) as an unattainable point. It does not.

The following example makes us even more pessimistic.

$$x_i = i, \qquad i = 0, 1, 2, \ldots$$
$$y_j = j, \qquad j = 0, 1, 2, \ldots$$

$$I_{12} = \{(0,0), (1,0), (0,1), (1,1), (2,0), (2,1), (0,2), (1,2), (2,2), (3,0), (3,1), (3,2), (4,0)\}$$

$$f(I_{12}) = \{1, -1, -1, 0, 10, 1/3, 10, 0, 1/5, 1, 1/2, 1/4, 13\}$$

$$N_5 = \{(0,0), (1,0), (0,1), (1,1), (2,0), (2,1)\}$$

$$D_7 = \{(0,0), (1,0), (0,1), (1,1), (2,0), (2,1), (0,2), (1,2)\}.$$

Here we find that the same irreducible form repeats itself in the multivariate table of rational interpolants without any structural explanation. It solves two multivariate rational interpolation problems at different entries which cannot be linked by a singular structure built around one of them. There are no singularities involved, not even in any of their neighbouring entries. Also the second entry is not located on a so-called tail of the first one. In particular we have

$$[N_1/D_2]_{I_3} = \frac{1-x}{1-x-2y} = [N_4/D_6]_{I_{10}}$$

with the entries (1, 3), (2, 2), (2, 3), (4, 5), (5, 5), (5, 6), (4, 7), and (5, 7) all nondegenerate. Besides all these negative results, we can prove the following theorem for the particular case $D = D_m = I_m$. We always have $N = N_n = I_n$ but now we also extend this to the denominator.

THEOREM 7. *Let the rank of the submatrix consisting of the last $m-l+1$ columns of the coefficient matrix $C_{n+1,n+m}$ be at most $m-l$ with $l>0$. Then the interpolation points indexed by I_{l-1} are unattainable points for $[N_n/I_m]_{I_{n+m}}$.*

Proof. Since it is given that the matrix

$$\begin{pmatrix} c_{d_l i_{n+1}, e_l j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_l i_{n+m}, e_l j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix}$$

has rank at most $m-l$, we know that the homogeneous system of equations

$$\begin{pmatrix} c_{d_l i_{n+1}, e_l j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_l i_{n+m}, e_l j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix} \begin{pmatrix} b_{d_l e_l} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has at least one nontrivial solution. From this solution we can construct the solution

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots 0 \\ b_{d_l e_l} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$a_{i_0 j_0} = \cdots = a_{i_{l-1} j_{l-1}} = 0, \qquad \sum_{\mu=0}^{m} c_{d_\mu i_\nu, e_\mu j_\nu} b_{d_\mu e_\mu} = a_{i_\nu j_\nu}, \quad \nu = l, ..., n$$

for the rational interpolation problem $[N_n/D_m]_{I_{n+m}}$. If $D_m = I_m$ then $d_k = i_k$, $e_k = j_k$ for $k = 0, ..., m$, and D_m satisfies the inclusion property. Since N_n already equals I_n and hence also satisfies the inclusion property, we have that

$$p(x, y) = \sum_{(i,j) \in I_{l,n}} a_{ij} B_{ij}(x, y)$$

$$q(x, y) = \sum_{(i,j) \in I_{l,m}} b_{ij} B_{ij}(x, y)$$

satisfy

$$p(x_i, y_j) = 0 = q(x_i, y_j), \qquad (i, j) \in I_{l-1}$$

which completes the proof. ▮

COROLLARY. *Under the same conditions as in the preceding theorem we put* $d_k = \min\{d_l, ..., d_m\}$ *and* $e_k = \min\{e_l, ..., e_m\}$. *For* $d_k = 0$ *we define* $J_1 = \varnothing$ *and for* $d_k > 0$ *we define* $J_1 = \{0 \leqslant i < d_k \mid (i, j) \in I_{l-1}\}$. *The same is done for* e_k *producing a set* J_2. *Then*

$$\prod_{i \in J_1} (x - x_i) \prod_{j \in J_2} (y - y_j)$$

is a common factor of p and q solving $[N_n/D_m]_{I_{n+m}}$.

In the univariate case conditions such as $D_m = I_m$ are always satisfied. An enumeration of the interpolation points and numerator and denominator coefficients is quite obvious when one is working with only one variable. In the multivariate case familiar facts such as an enumeration satisfying

$$i_{n+k} - d_{m+k} = i_n - d_m,$$

which is always true in the univariate case, are not valid anymore and these facts disturb a lot of the univariate results when one tries to generalize them. This is the price paid for the generality of the framework. In future research we intend to concentrate on some particular enumerations or degree sets in order to preserve more of the univariate results.

References

1. H. Allouche and A. Cuyt, On the structure of a table of multivariate rational interpolants, *Constr. Approx.* **8** (1992), 69–86.
2. G. Claessens, On the Newton–Padé approximation problem, *J. Approx. Theory* **22** (1978), 150–160.
3. G. Claessens, On the structure of the Newton–Padé table, *J. Approx. Theory* **22** (1978), 304–319.
4. A. Cuyt and B. Verdonk, General order Newton–Padé approximants for multivariate functions, *Numer. Math.* **43** (1984), 293–307.