Constr. Approx. (1992) 8: 69–86

CONSTRUCTIVE APPROXIMATION

On the Structure of a Table of Multivariate Rational Interpolants

H. Allouche and A. Cuyt

Abstract. In the table of multivariate rational interpolants the entries are arranged such that the row index indicates the number of numerator coefficients and the column index the number of denominator coefficients. If the homogeneous system of linear equations defining the denominator coefficients has maximal rank, then the rational interpolant can be represented as a quotient of determinants. If this system has a rank deficiency, then we identify the rational interpolant with another element from the table using less interpolation conditions for its computation and we describe the effect this dependence of interpolation conditions has on the structure of the table of multivariate rational interpolants. In the univariate case the table of solutions to the rational interpolation problem is composed of triangles of so-called minimal solutions, having minimal degree in numerator and denominator and using a minimal number of interpolation conditions to determine the solution.

1. The Structure of the Univariate Rational Interpolation table

Let the univariate function $f(x)$ be known in the points x_i with $i \in \mathbf{N}$, and let the functions

$$B_i(x) = \prod_{k=0}^{i-1} (x - x_k)$$

span the space of univariate polynomials. The rational interpolation problem of order (n, m) for f consists in finding polynomials

$$\text{(1a)} \qquad p_{n,m}(x) = \sum_{i=0}^{n} a_i B_i(x), \qquad q_{n,m}(x) = \sum_{i=0}^{m} b_i B_i(x),$$

such that

$$\text{(1b)} \qquad (f q_{n,m} - p_{n,m})(x_k) = 0, \qquad k = 0, \ldots, n + m.$$

Date received: May 1, 1989. Date revised: March 5, 1990. Communicated by Dietrich Braess.
AMS classification: 41A20, 41A63.
Key words and phrases: Multivariate, Newton–Padé approximant, Nonnormal, Block structure.

Condition (1b) can be reformulated as

$$(2) \qquad (fq_{n,m} - p_{n,m})(x) = \sum_{i \geq n+m+1} d_i B_i(x),$$

where the Newton series development is formal, and which can also be used if some or all of the interpolation points coincide [1]. Let us denote the divided difference $f[x_i, \ldots, x_j]$ by c_{ij} with $c_{ij} = 0$ if $i > j$. Then condition (2), stating that $d_i = (fq_{n,m} - p_{n,m})[x_0, \ldots, x_i] = 0$ for $i = 0, \ldots, n+m$, is equivalent with

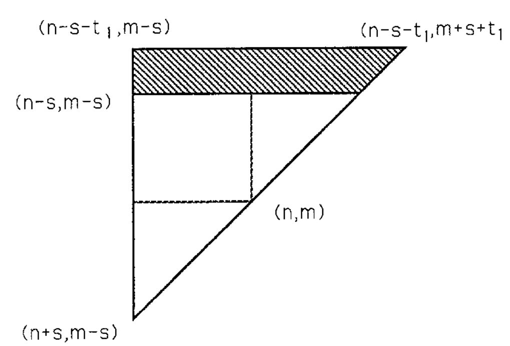
$$(3a) \qquad \begin{pmatrix} c_{00} & 0 & & \cdots & 0 \\ c_{01} & c_{11} & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ c_{0n} & c_{1n} & c_{2n} & \cdots & c_{mn} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix},$$

$$(3b) \qquad \begin{pmatrix} c_{0,n+1} & \cdots & c_{m,n+1} \\ \vdots & & \vdots \\ c_{0,n+m} & \vdots & c_{m,n+m} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If the rank of (3b) is $m - s$ [2] then (up to a multiplicative constant factor) a unique solution $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ of (3) exists with $\partial \bar{p}_{n,m} \leq n - s$ and $\partial \bar{q}_{n,m} \leq m - s$, where at least one of the upper bounds is attained. Since the polynomials $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ have the property that their degrees cannot be lowered simultaneously anymore, unless some interpolation conditions are lost, they are called a "minimal solution." This means that they solve (1)–(3) with a minimal number of parameters a_i and b_i, namely, at most, $n + m - 2s$. All other solutions $p_{n,m}(x)$ and $q_{n,m}(x)$ are a polynomial multiple of the minimal solution $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$. The minimal solutions can be arranged in a table where the numerator degree n is the row index and the denominator degree m is the column index. For this table, the following theorem due to Claessens holds [2]. We denote the coefficient matrix of the linear system (3b) by $C_{n+1,n+m}$.

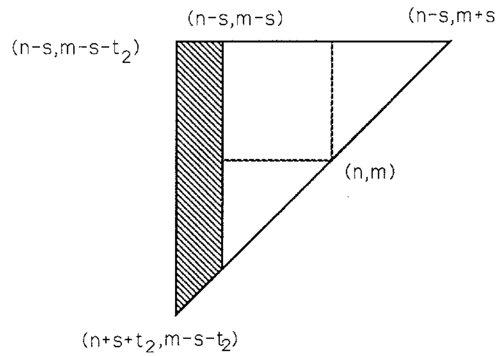
Theorem 1. *Let the rank of $C_{n+1,n+m}$ and the rank of $C_{n-s+1,n+m-2s}$ be $m - s$, and let $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ be the minimal solution of the rational interpolation problem of order (n, m) for f.*

(a) *If $\partial \bar{p}_{n,m} = n - s - t_1$ and the rank of $C_{n-s-t_1+1,n+m-2s-t_1} = m - s$, then all*



the rational interpolation problems of order (i, j) with (i, j) lying in the triangle with corner elements $(n-s-t_1, m-s)$, $(n-s-t_1, m+s+t_1)$, and $(n+s, m-s)$ have $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ as minimal solution.

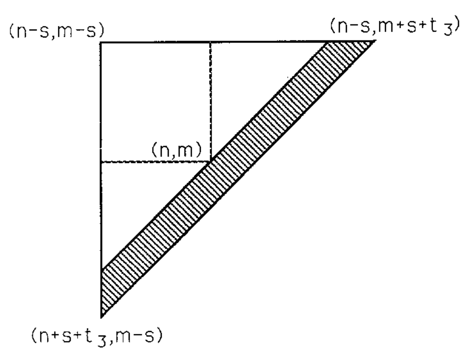
(b) *If $\partial\bar{q}_{n,m} = m-s-t_2$ and the rank of $C_{n-s+1,n+m-2s-t_2} = m-s-t_2$, then all the rational interpolation problems of order (i, j) with (i, j) lying in the triangle with corner elements $(n-s, m-s-t_2)$, $(n-s, m+s)$, and $(n+s+t_2, m-s-t_2)$ have $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ as minimal solution.*



(c) *If*

$$(f\bar{q}_{n,m} - \bar{p}_{n,m})(x) = \sum_{i \geq n+m+t_3+1} d_i B_i(x)$$

with $d_{n+m+t_3+1} \neq 0$, then all the rational interpolation problems of order (i, j) with (i, j) lying in the triangle with corner elements $(n-s, m-s)$, $(n-s, m+s+t_3)$, and $(n+s+t_3, m-s)$ have $\bar{p}_{n,m}(x)$ and $\bar{q}_{n,m}(x)$ as minimal solution.



(d) *If $\partial\bar{p}_{n,m} = n-r_1$, $\partial\bar{q}_{n,m} = m-r_2$, the rank of $C_{n-r_1+1,n+m-r_1-r_2}$ is $m-r_2$ and*

$$(f\bar{q}_{n,m} - \bar{p}_{n,m})(x) = \sum_{i \geq n+m+1} d_i B_i(x)$$

with $d_{n+m+1} \neq 0$, then $\bar{p}_{i,j} = \bar{p}_{n,m}$ and $\bar{q}_{i,j} = \bar{q}_{n,m}$ if and only if (i, j) belongs to the triangle with corner elements $(n-r_1, m-r_2)$, $(n-r_1, m+r_1)$, and $(n+r_2, m-r_2)$.

In Section 3 a multivariate analogon of Theorem 1 on minimal solutions will be proved and its differences with the univariate theorem will be discussed. The minimal solutions defined above may still be reducible by a common polynomial factor. Lemma 1 on these common factors is proved in [1]. We shall also make clear in Section 3 why Corollary 1 is harder to generalize.

Lemma 1. *If $(x-\alpha)^\beta$ represents a common factor of the minimal solution $\bar{p}_{n,m}$ and $\bar{q}_{n,m}$, then $\alpha \in \{x_k\}_{k=0}^{n+m}$ and $\beta \le m_\alpha$ where m_α denotes the multiplicity of the interpolation point α in $\{x_k\}_{k=0}^{n+m}$.*

In other words, when minimal solutions are reducible, the true rational interpolation problem

$$\left(f - \frac{p_{n,m}}{q_{n,m}}\right)(x_k) = 0, \qquad k = 0, \ldots, n+m,$$

has a number of "unattainable" interpolation points, namely, precisely those $\alpha = x_k$ for which $q_{n,m}(x_k) = p_{n,m}(x_k) = 0$. Although a number of interpolation conditions are then lost by considering the irreducible form of solutions of the rational interpolation problem, we still call this irreducible form the "rational interpolant."

Corollary 1. *Let $\bar{p}(x)$ and $\bar{q}(x)$ be the minimal solution of the (n, m) rational interpolation problem with $\partial\bar{p} = n - r_1$, $\partial\bar{q} = m - r_2$. Let the rank of $C_{n-r_1+1,n+m-r_1-r_2}$ be $m - r_2$ and $(f\bar{q} - \bar{p})(x) = \sum_{i \ge n+m+1} d_i B_i(x)$ with $d_{n+m+1} \ne 0$. Then for all (i, j) in the square determined by $(n - r_1, m - r_2)$ and $(n + r_2 . m + r_1)$ the solutions of the rational interpolation problem of order (i, j) have the same irreducible form.*

The proof is given in [2]. From Corollary 1 the reader could get the impression that, once the order of the interpolation points is fixed, the table of rational interpolants (which is different from the table of minimal solutions) has, under certain conditions, a square block structure. This is not true. The block mentioned in Corollary 1 is square and hence symmetric with respect to its own block diagonal. In [2] it is proved that further down the table one may encounter more "trailing" blocks filled with the same irreducible rational interpolant and built symmetrically above and below the extended diagonal of the first block.

2. The Multivariate Rational Interpolation Problem

For the sake of notational simplicity we will restrict our description to the bivariate case. Let a bivariate function $f(x, y)$ be known in the data points (x_i, y_j) with $(i, j) \in \mathbb{N}^2$, and let I be a finite subset of $\mathbb{N}^2$ indexing those data points which will be used as interpolation points. With the data points we construct the polynomial basis functions

$$B_{ij}(x, y) = \prod_{k=0}^{i-1} (x - x_k) \prod_{l=0}^{j-1} (y - y_l).$$

The problem of interpolating these data by a bivariate rational function was formulated in [3] as follows. Choose finite subsets N (from "Numerator") and D (from "Denominator") of $\mathbf{N}^2$ with $N \subset I$ and compute bivariate polynomials

$$\text{(4a)} \qquad \begin{aligned} p(x, y) &= \sum_{(i,j)\in N} a_{ij}B_{ij}(x, y), \qquad \#N = n + 1, \\ q(x, y) &= \sum_{(i,j)\in D} b_{ij}B_{ij}(x, y), \qquad \#D = m + 1, \end{aligned}$$

such that

$$\text{(4b)} \qquad (fq - p)(x_i, y_j) = 0, \qquad (i, j) \in I, \qquad \#I = n + m + 1.$$

If $q(x_i, y_j) \neq 0$, then this last condition implies that

$$\text{(4c)} \qquad f(x_i, y_j) = \frac{p}{q}(x_i, y_j), \qquad (i, j) \in I.$$

We say that I satisfies the inclusion property if whenever a point (i, j) belongs to I, all the points in the rectangle emanating from the origin with (i, j) as its furthermost corner belong to I. Condition (4b) is, for instance, met if the following two conditions are satisfied by the polynomials given in (4a) [3]:

$$\text{(5a)} \qquad (fq - p)(x, y) = \sum_{(i,j)\in \mathbf{N}^2\setminus I} d_{ij}B_{ij}(x, y),$$

(5b) I satisfies the inclusion property,

where the series development (5a) is formal. Condition (5a) can also be used if some or all of the interpolation points or their coordinates coincide since it can be replaced by conditions in terms of bivariate divided differences [3]

$$\text{(6a)} \qquad (fq)[x_0, \ldots, x_i][y_0, \ldots, y_j] = p[x_0, \ldots, x_i][y_0, \ldots, y_j], \qquad (i, j) \in N,$$

$$\text{(6b)} \qquad (fq)[x_0, \ldots, x_i][y_0, \ldots, y_j] = 0, \qquad (i, j) \in I\setminus N.$$

Using a generalization of Leibniz' theorem [3] we can substitute $(fq)[x_0, \ldots, x_i][y_0, \ldots, y_j]$ in (6a–b), with the notation $c_{\mu i, \nu j} = f[x_\mu, \ldots, x_i][y_\nu, \ldots, y_j]$, by

$$\begin{aligned} (fq)[x_0, \ldots, x_i][y_0, \ldots, y_j] &= \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} q[x_0, \ldots, x_\mu][y_0, \ldots, y_\nu]c_{\mu i, \nu j} \\ &= \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} b_{\mu\nu}c_{\mu i, \nu j}. \\ &= \sum_{(\mu,\nu)\in D} b_{\mu\nu}c_{\mu i, \nu j}. \end{aligned}$$

Also

$$p[x_0, \ldots, x_i][y_0, \ldots, y_j] = a_{ij}, \qquad (i, j) \in N.$$

From now on we denote a rational function satisfying (6) by $[N/D]_I$. Numbering the points in the sets N, D, and I as

$$N = \{(i_0, j_0), \ldots, (i_n, j_n)\}, \tag{7a}$$

$$D = \{(d_0, e_0), \ldots, (d_m, e_m)\}, \tag{7b}$$

$$I = N \cup \{(i_{n+1}, j_{n+1}), \ldots, (i_{n+m}, j_{n+m})\}, \tag{7c}$$

condition (6) becomes

$$\begin{pmatrix} c_{d_0 i_0, e_0 j_0} & \cdots & c_{d_m i_0, e_m j_0} \\ \vdots & & \vdots \\ c_{d_0 i_n, e_0 j_n} & \cdots & c_{d_m i_n, e_m j_n} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} a_{i_0 j_0} \\ \vdots \\ a_{i_n j_n} \end{pmatrix}, \tag{8a}$$

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{8b}$$

It is obvious that at least one nontrivial solution of (8) exists, but it is not so (unlike the univariate case) that different solutions p_1, q_1 and p_2, q_2 of (8) are necessarily equivalent, meaning that $(p_1 q_2)(x, y) = (p_2 q_1)(x, y)$. Hence p_1/q_1 and p_2/q_2 may be different functions. Consider the following example:

$$f(x, y) = \frac{x + y}{x - y - xy},$$

$$x_i = i, \qquad i = 0, 1, 2, \ldots,$$

$$y_j = j + 1, \qquad j = 0, 1, 2, \ldots,$$

$$I_5 = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\},$$

$$f(I_5) = \{-1, -2, -1, -3, -1, -1\},$$

$$N_2 = \{(0, 0), (1, 0), (0, 1)\},$$

$$D_3 = \{(0, 0), (1, 0), (0, 1), (2, 0)\},$$

$$[N_2/D_3]_{I_5} = \frac{p_1}{q_1}(x, y) = \frac{3(x - 1)}{(x - 1)(x - 3)} = \frac{3}{x - 3},$$

$$[N_2/D_3]_{I_5} = \frac{p_2}{q_2}(x, y) = \frac{y - 1}{1 - y} = -1.$$

In the next section we shall discuss what is so typical about the multivariate case and causes this to happen. Note that in the above example both irreducible forms satisfy neither (4b) nor (4c). This phenomenon, which was also described in the previous section for the univariate case, can occur even if (4b) has an essentially unique solution. Consider the same example as above now with

$$\begin{aligned} I_4 &= \{(0,0),(1,0),(0,1),(2,0),(1,1)\},\\ f(I_4) &= \{-1,-2,-1,-3,-1\}\\ N_1 &= \{(0,0),(1,0)\},\\ D_3 &= \{(0,0),(1,0),(0,1),(2,0)\},\\ [N_1/D_3]_{I_4} &= \frac{p}{q}(x,y) = \frac{3(x-1)}{(x-1)(x-3)} = \frac{3}{x-3}. \end{aligned}$$

Up to a multiplicative constant factor in the numerator and denominator this solution is unique, but its irreducible form $r(x, y) = 3/(x - 3)$ does not satisfy the interpolation conditions in (x_1, y_0) and (x_1, y_1) anymore

$$\begin{aligned} f(x_1, y_0) &= -2 \neq r(x_1, y_0) = -3/2,\\ f(x_1, y_1) &= -1 \neq r(x_1, y_1) = -3/2. \end{aligned}$$

Also the polynomials $p(x, y) = 3$ and $q(x, y) = x - 3$ do not satisfy (8) anymore.

When this second phenomenon (loss of interpolation conditions) occurs, the interpolation problem is said to have "unattainable" interpolation points. We shall not discuss this here in more detail. The first phenomenon (loss of equivalence) can only occur if essentially different solutions of (8) exist and in that case the coefficient matrix in (8b) is rank-deficient. Hence, if the rank of (8b) is not maximal, we should look at $[N/D]_I$ as being a set of rational functions of which the numerator and denominator are given by (4a) and are satisfying (8a) and (8b). A solution $[N/D]_I$ containing numerators and denominators of different "degrees" is called "degenerate."

With the numberings (7a), (7b), and (7c) of the respective indices in N, D, and I we can set up descending chains of index sets, defining bivariate polynomials of "lower degree" and bivariate rational interpolation problems of "lower order"

(9a) $$N = N_n \supset \cdots \supset N_k = \{(i_0, j_0), \ldots, (i_k, j_k)\} \supset \cdots \supset N_0 = \{(i_0, j_0)\}, \quad k = 0, \ldots, n,$$

(9b) $$D = D_m \supset \cdots \supset D_l = \{(d_0, e_0), \ldots, (d_l, e_l)\} \supset \cdots \supset D_0 = \{(d_0, e_0)\}, \quad l = 0, \ldots, m,$$

(9c) $$I = I_{n+m} \supset \cdots \supset I_{k+l} = \{(i_0, j_0), \ldots, (i_{k+l}, j_{k+l})\} \supset \cdots \supset I_0 = \{(i_0, j_0)\}, \quad k + l = 0, \ldots, n + m,$$

$$I_{k+1,k+l} = \{(i_{k+1}, j_{k+1}), \ldots, (i_{k+l}, j_{k+l})\}, \qquad I \backslash N = I_{n+1,n+m}.$$

With these subsets we can compute the following entries in a "table" of multivariate rational interpolants:

(10) $$\begin{array}{ccc} [N_0/D_0]_{I_0} & \cdots & [N_0/D_m]_{I_m} \\ \vdots & & \vdots \\ [N_n/D_0]_{I_n} & \cdots & [N_n/D_m]_{I_{n+m}} \end{array}$$

If we let n and m increase, infinite chains of index sets as in (9) can be constructed, and an infinite table of multivariate rational interpolants results. Of course, in practice, only a finite number of entries will be computed. We shall see in the next section that degeneracy has consequences for neighboring elements in the table.

3. The Degenerate Case

Let us denote the coefficient matrix of (8b) by $C_{n+1,n+m}$. We remark that the rows are indexed by $I_{n+1,n+m}$. Let the set $H_{n+1,n+m} \subseteq I_{n+1,n+m}$ denote the indices in $I_{n+1,n+m}$ of the rows in $C_{n+1,n+m}$ that are linearly independent. If the rank of $C_{n+1,n+m}$ is maximal, then $H_{n+1,n+m} = I_{n+1,n+m}$ and we have a representation of $[N/D]_I$ as a quotient of determinants [3]:

$$[N/D]_I = \frac{\begin{vmatrix} \sum_{(i,j)\in N} c_{d_0 i, e_0 j} B_{ij}(x,y) & \cdots & \sum_{(i,j)\in N} c_{d_m i, e_m j} B_{ij}(x,y) \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} & & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & & c_{d_m i_{n+m}, e_m j_{n+m}} \end{vmatrix}}{\begin{vmatrix} B_{d_0 e_0}(x,y) & \cdots & B_{d_m e_m}(x,y) \\ c_{d_0 i_{n+1}, e_0 j_{n+1}} & & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & & c_{d_m i_{n+m}, e_m j_{n+m}} \end{vmatrix}}$$

As pointed out above, there are very good reasons not to continue our discussion by studying irreducible forms of solutions:

(a) in a degenerate case different solutions may not be equivalent and hence (unlike the univariate case) not generate a unique irreducible form;
(b) an irreducible form may not satisfy all the interpolation conditions anymore and its numerator and denominator polynomials may not satisfy (8) anymore;
(c) in the univariate case the structure of the rational interpolation table is deduced from that of the minimal solution table and we shall see that this is also the case for the multivariate rational interpolation table.

Hence, in the sequel of the text, we shall always work with solutions of (8).

In the univariate case and under certain conditions, the table of minimal solutions of the rational interpolation problem consists of triangles, once the numbering of the interpolation points is fixed [2]. The size of the triangles, as pointed out in Theorem 1, is related to the rank deficiency of the interpolation problem. We shall now prove a similar multivariate theorem and point out the differences with the univariate theorem. From this discussion it will also become clear why different solutions of the same rational interpolation problem are not necessarily equivalent, as was shown in the example of the previous section.

Given data, indexed by the set I as in (7c), satisfying the inclusion property, and given index sets N and D as in (7a) and (7b), a table of multivariate rational interpolants as in (10) can be set up. We assume in the following theorem that the involved $[N/D]_I$ belong to the part of the table that can be computed with the given data. In the sequel of the text we shall use the notation

$$\partial p = N_{n-t} \quad \Leftrightarrow \quad p(x, y) = \sum_{(i,j) \in N_{n-t}} a_{ij} B_{ij}(x, y) \qquad \text{with} \quad a_{i_{n-t} j_{n-t}} \neq 0.$$

Theorem 2. *Let $p(x, y)$ and $q(x, y)$ be defined by (4), (7), and (8). Let the rank of $C_{n+1,n+m}$ in (8b) be given by $m - s$. Then for each pair (k, l), with $0 \leq k \leq s$, $0 \leq l \leq s$, $k + l = s$, and the rank of $C_{n-k+1,n+m-s}$ equal to $m - l$, the following statements hold:*

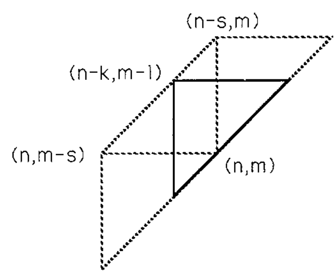
(a) *For $0 \leq s_1$, $0 \leq s_2$, and $s_1 + s_2 \leq s$, $[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$ belongs to the solution set $[N_{n-k+s_1}/D_{m-l+s_2}]_{I_{n+m-s+s_1+s_2}}$, meaning that the (up to a multiplicative constant factor) unique rational function*

$$[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$$

also solves the interpolation problems posed in

$$[N_{n-k+s_1}/D_{m-l+s_2}]_{I_{n+m-s+s_1+s_2}},$$

where $[N_{n-k+s_1}/D_{m-l+s_2}]_{I_{n+m-s+s_1+s_2}}$ lies in the triangle of the table of rational interpolants with corner elements $[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$, $[N_{n-k}/D_{m+k}]_{I_{n+m}}$, and $[N_{n+l}/D_{m-l}]_{I_{n+m}}$.



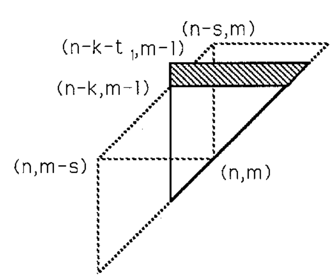
(b) *If the solution $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} = (p/q)(x, y)$ is such that $\partial p = N_{n-k-t_1}$ with $t_1 > 0$, then under the condition that rank $C_{n-k-t_1+1,n+m-s-t_1} = m - l$,*

$$[N_{n-k-t_1}/D_{m-l}]_{I_{n+m-s-t_1}}$$

also solves

$$[N_{n-k-t_1+s_1}/D_{m-l+s_2}]_{I_{n+m-s-t_1+s_1+s_2}}$$

for $0 \leq s_1$, $0 \leq s_2$, and $s_1 + s_2 \leq s + t_1$.



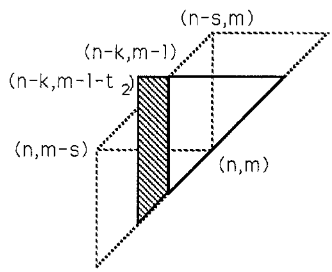
(c) *If the solution* $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} = (p/q)(x, y)$ *is such that* $\partial q = D_{m-l-t_2}$ *with* $t_2 > 0$, *then under the condition that rank* $C_{n-k+1,n+m-s-t_2} = m - l - t_2$,

$$[N_{n-k}/D_{m-l-t_2}]_{I_{n+m-s-t_2}}$$

also solves

$$[N_{n-k+s_1}/D_{m-l-t_2+s_2}]_{I_{n+m-s-t_2+s_1+s_2}}$$

for $0 \leq s_1$, $0 \leq s_2$, *and* $s_1 + s_2 \leq s + t_2$.



(d) *If the solution* $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} = (p/q)(x, y)$ *is such that*

$$(fq - p)(x, y) = \sum_{(i,j)\in\mathbf{N}^2\setminus I_{n+m+t_3}} d_{ij}B_{ij}(x, y)$$

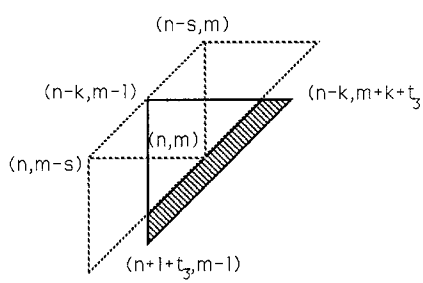
with $t_3 > 0$, *then*

$$[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$$

also solves

$$[N_{n-k+s_1}/D_{m-l+s_2}]_{I_{n+m-s+s_1+s_2}},$$

where $0 \leq s_1$, $0 \leq s_2$, *and* $s_1 + s_2 \leq s + t_3$.



(e) *If the solution* $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} = (p/q)(x, y)$ *is such that* $\partial p = N_{n-k}$, $\partial q = D_{m-l}$, *and*

$$(fq - p)(x, y) = \sum_{(i,j)\in\mathbf{N}^2\setminus I_{n+m}} d_{ij}B_{ij}(x, y)$$

with $d_{i_{n+m+1}j_{n+m+1}} \neq 0$, *then* $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} \in [N_i/D_j]_{I_{i+j}}$ *if and only if* (i, j) *belongs to the triangle with corner elements* $(n-k, m-l)$, $(n+l, m-l)$, *and* $(n-k, m+k)$.

Proof. (a) Take k and l with $0 \le k \le s$, $0 \le l \le s$, $k + l = s$ and write down (8) for $[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$:

$$\begin{pmatrix} c_{d_0 i_0, e_0 j_0} & \cdots & c_{d_{m-l} i_0, e_{m-l} j_0} \\ \vdots & & \vdots \\ c_{d_0 i_{n-k}, e_0 j_{n-k}} & \cdots & c_{d_{m-l} i_{n-k}, e_{m-l} j_{n-k}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_{m-l} e_{m-l}} \end{pmatrix} = \begin{pmatrix} a_{i_0 j_0} \\ \vdots \\ a_{i_{n-k} j_{n-k}} \end{pmatrix},$$

$$\begin{pmatrix} c_{d_0 i_{n-k+1}, e_0 j_{n-k+1}} & \cdots & c_{d_{m-l} i_{n-k+1}, e_{m-l} j_{n-k+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m-s}, e_0 j_{n+m-s}} & \cdots & c_{d_{m-l} i_{n+m-s}, e_{m-l} j_{n+m-s}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_{m-l} e_{m-l}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We know from the fact that $C_{n-k+1,n+m-s}$ has maximal rank, that (8) has a unique solution up to a multiplicative factor in the numerator and denominator of $[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$. We shall now prove that this solution also belongs to $[N_{n-k+s_1}/D_{m-l+s_2}]_{I_{n+m-s+s_1+s_2}}$. We know from the rank deficiency of the homogeneous system (8b) for $[N_n/D_m]_{I_{n+m}}$ that at most an extra s parameters among its unknowns can be choosen freely or an extra s linearly independent conditions can be imposed. By choosing

$$b_{d_{m-l+1} e_{m-l+1}} = \cdots = b_{d_m e_m} = 0$$

and imposing

$$a_{i_{n-k+1} j_{n-k+1}} = \cdots = a_{i_n j_n} = 0,$$

one constructs a solution $(\tilde{p}/\tilde{q})(x, y)$ of $[N_n/D_m]_{I_{n+m}}$ that belongs to the solution set $[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$ with a number of extra terms in $(f\tilde{q} - \tilde{p})(x, y)$ canceled out, namely those indexed by $I_{n+m-s+1,n+m}$. Hence $(\tilde{p}/\tilde{q})(x, y)$ also belongs to $[N_{n-k+s_1}/D_{m-l+s_2}]_{I_{n+m-s+s_1+s_2}}$ as long as $0 \le s_1$, $0 \le s_2$, and $s_1 + s_2 \le s$. Since the solution of (8) is unique, $(\tilde{p}/\tilde{q})(x, y) = [N_{n-k}/D_{m-l}]_{I_{n+m-s}}$.

(b) As in (a), the unique solution $(p/q)(x, y)$ of $[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$ can be proved to cancel all those terms in the formal series development of $(fq - p)(x, y)$ indexed by I_{n+m}. Since now $\partial p = N_{n-k-t_1}$ with $t_1 > 0$, $(p/q)(x, y)$ also belongs to $[N_{n-k-t_1+s_1}/D_{m-l+s_2}]_{I_{n+m-s-t_1+s_1+s_2}}$ for $0 \le s_1$, $0 \le s_2$, and $s_1 + s_2 \le s + t_1$.

(c) The proof is similar to the one for (b).

(d) The proof is trivial.

(e) Let us prove the sufficient condition. If (i, j) belongs to the given triangle, then $i \ge n - k$, $j \ge m - l$, and $i + j \le n + m$. This implies that $[N_{n-k}/D_{m-l}]_{I_{n+m-s}} = (p/q)(x, y)$ solves the rational interpolation problem of order (i, j) because the numerator and denominator polynomials have the desired degree and the rest series $(fq - p)(x, y)$ has the desired order. The proof of the necessary condition is obvious. ■

Let us now point out some differences between this theorem and its univariate counterpart in [2]. First of all, it is important to note that both the univariate and the multivariate theorem are proved under the same conditions. With the rank of $C_{n+1,n+m}$ equal to $m-s$, we are able in both cases to construct solutions p_1, q_1 of $[N_{n-s}/D_m]_{I_{n+m-s}}$ and p_2, q_2 of $[N_n/D_{m-s}]_{I_{n+m-s}}$ that are also contained in $[N_m/D_m]_{I_{n+m}}$. We have

$$\begin{aligned}(p_1q_2 - p_2q_1)(x, y) &= [q_1(fq_2 - p_2) - q_2(fq_1 - p_1)](x, y)\\ &= q_1(x, y) \sum_{(i,j)\in \mathbf{N}^2\backslash n+m} d_{ij}^{(2)} B_{ij}(x, y)\\ &\quad - q_2(x, y) \sum_{(i,j)\in \mathbf{N}^2\backslash I_{n+m}} d_{ij}^{(1)} B_{ij}(x, y),\end{aligned}$$

from which we can conclude that $(p_1q_2 - q_1p_2)(x_i, y_j) = 0$ for all $(i, j) \in I_{n+m}$ with I_{n+m} satisfying the inclusion property. We also have

$$\partial(p_1q_2 - q_1p_2) = \{(i, j) = (r, s) + (t, u) \mid (r, s) \in N_n, (t, u) \in D_m\}.$$

Before we continue our reasoning we prove the following lemma:

Lemma 2. *Let $I \subset \mathbf{N}^2$ satisfy the inclusion property and let the bivariate polynomial*

$$p(x,y) = \sum_{(i,j)\in \partial p} a_{ik} x^i y^i$$

be such that $\partial p \subset I$ and $p(x_i, y_j) = 0$ for $(i, j) \in I$, then $p \equiv 0$.

Proof. Given

$$p(x_i, y_j) = 0, \qquad (i, j) \in I,$$

for

$$\begin{aligned}p(x, y) &= \sum_{(i,j)\in \partial p} a_{ij} x^i y^j\\ &= \sum_{(i,j)\in \partial p} \tilde{a}_{ij} B_{ij}(x, y),\end{aligned}$$

one can prove that

$$p[x_0, \ldots, x_i][y_0, \ldots, y_j] = 0, \qquad (i, j) \in I.$$

Since $\tilde{a}_{ij} = p[x_0, \ldots, x_i][y_0, \ldots, y_j]$ the proof is completed. ■

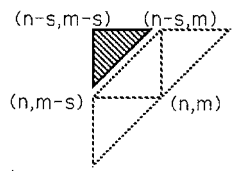
Of course, we should want to apply this lemma to $(p_1q_2 - q_1p_2)$, but since we do not always have that

$$\partial(p_1q_2 - q_1p_2) \subset I_{n+m},$$

we cannot conclude that $(p_1q_2 - q_1p_2)(x\ y) = 0$, and hence we cannot prove as in [2] that it is also possible to construct a solution p_3, q_3 of $[N_n/D_m]_{I_{n+m}}$ with $\partial p_3 \subset N_{n-s}$ and $\partial q_3 \subset D_{m-s}$. In the univariate case, however,

$$N_n = \{(i, 0) \mid 0 \le i \le n\},$$
$$D_m = \{(j, 0) \mid 0 \le j \le m\},$$
$$\{(i + j, 0) \mid i \in N_n, j \in D_m\} \subseteq I_{n+m} = \{(k, 0) \mid 0 \le k \le n + m\},$$

and hence $p_1q_2 = p_2q_1$. Hence in the univariate case the configuration described in Theorem 2 can also be enlarged with the triangle with corner elements $[N_{n-s}/D_{m-s}]_{I_{n+m-2s}}$, $[N_{n-1}/D_{m-s}]_{I_{n+m-s-1}}$, and $[N_{n-s}/D_{m-1}]_{I_{n+m-s-1}}$ resulting in the configuration described in Theorem 1.



To illustrate this important remark, consider again our first example,

$$f(x, y) = \frac{x + y}{x - y - xy},$$
$$x_i = i, \qquad i = 0, 1, 2, \ldots,$$
$$y_j = j + 1, \qquad j = 0, 1, 2, \ldots,$$
$$N_2 = \{(0, 0), (1, 0), (0, 1)\},$$
$$D_3 = \{(0, 0), (1, 0), (0, 1), (2, 0)\},$$
$$I_5 = N_2 \cup \{(2, 0), (1, 1), (0, 2)\},$$
$$f(I_5) = \{-1, -2, -1, -3, -1, -1\}.$$

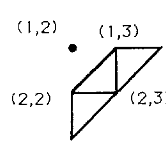
The homogeneous system to be solved for the computation of $[N_2/D_3]_{I_5}$ is

$$\begin{pmatrix} 0 & -1 & 0 & -3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{vmatrix} b_{00} \\ b_{10} \\ b_{01} \\ b_{20} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

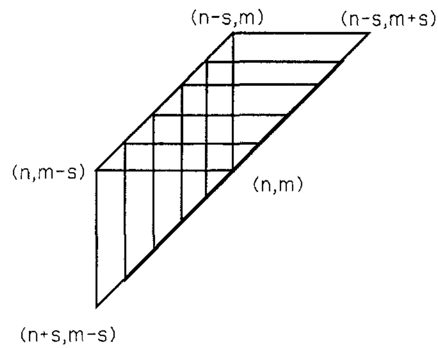
with rank $C_{3,5} = 2$. The general solution of $[N_2/D_3]_{I_5}$ is given by

$$p(x, y) = 3\alpha(x - 1) - \beta(y - 1),$$
$$q(x, y) = -3\alpha(x - 1) + \beta(y - 1) + \alpha x(x - 1).$$

With $\alpha = 1$ and $\beta = 0$ we find $[N_1/D_3]_{I_4}$. With $\alpha = 0$ and $\beta = 1$ we find $[N_2/D_2]_{I_4}$. But clearly $[N_1/D_2]_{I_3} = -1 - x$ is not contained in $[N_2/D_3]_{I_5}$.



How is Theorem 2 then to be understood as a generalization of Theorem 1? Clearly, minimal solutions are not uniquely determined anymore. In Theorem 2 all solutions $[N_{n-k}/D_{m-l}]$ with $k + l = s$ are "minimal" in the sense that they use a minimal number of parameters and data to solve the (n, m) rational interpolation problem. Now each of the minimal solutions on the $(n + m - s)^{\text{th}}$ diagonal (with numerator and denominator "degree," respectively, less than or equal to n and m) give rise to a triangular structure in the table. There is a whole triangle of rational interpolation problems that is solved by each minimal solution from the $(n + m - s)^{\text{th}}$ diagonal.



In the univariate case the minimal solution is unique and is either a true irreducible solution or a deficient solution with unattainable interpolation points. We have just seen that in the multivariate case a minimal solution is not unique anymore. What is more, in the multivariate case, a rational interpolation problem can have both a true irreducible minimal solution and a deficient minimal solution at the same time. For the same function as used in the examples above, Table 1 lists the solution sets $[N_n/D_m]_{I_{n+m}}$ for a number of (n, m) rational interpolation problems. Note that $[N_0/D_4]_{I_4}$ and $[N_1/D_3]_{I_4}$ are two minimal solutions for $[N_1/D_4]_{I_5}$, where $[N_0/D_4]_{I_4}$ is a true irreducible solution satisfying (4c) and $[N_1/D_3]_{I_4}$ is a deficient solution reducible by $(x - 1)$, with (x_1, y_0) and (x_1, y_1) as unattainable points. In the multivariate case a solution must not be reducible in order to have unattainable interpolation points. Take a look at $[N_2/D_3]_{I_5}$ and you see that the general solution $(p/q)(x, y)$ with $\alpha\beta \neq 0$ is irreducible while (x_1, y_0) is an unattainable interpolation point since $p(x_1, y_0) = 0 = q(x_1, y_0)$. This is a situation which is essentially different from the univariate one.

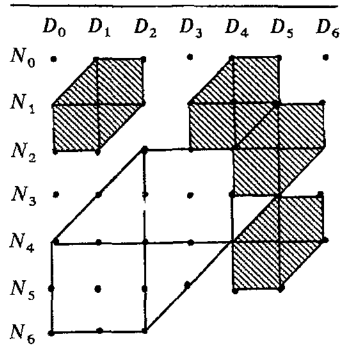
Table 2 gives the configuration proved in Theorem 2(a) for the table of minimal solutions. It does not give the "maximal" triangles of equal "minimal" solutions as described in Theorem 2(b)-(e). One can see that the rational interpolation problem $[N_4/D_2]_{I_6}$ has a rank deficiency of $s = 2$ and that each of the functions $[N_4/D_0]_{I_4}$, $[N_3/D_1]_{I_4}$, and $[N_2/D_2]_{I_4}$ solves a triangle of interpolation problems emanating from itself. Theorem 2(d) applies to $[N_2/D_2]_{I_4}$ with $t_3 = 1$. From Table 1 one can also see that $[N_1/D_1]_{I_2}$, $[N_1/D_4]_{I_5}$, and $[N_2/D_5]_{I_7}$ all have a rank deficiency of $s = 1$. Theorem 2(c) applies to $[N_1/D_5]_{I_6}$ with $t_2 = 1$. In the univariate case Theorem 1(a) and 1(b) never apply simultaneously [2] while this can be true in the

Table 1

	D_0	D_1	D_2
N_0	-1	$\frac{-2}{2-x}$	$\frac{-2}{2-x}$
N_1	$-1-x$	$\frac{-1-(2\alpha+1)x}{1+\alpha x}$	$-1-x$
N_2	$-1-x$	$-1-x$	$\frac{-(y-1)}{(y-1)}$
N_3	$-1-x$	$\frac{-1+x+x(x-1)}{1-x}$	$\frac{\alpha(-1+x+x(x-1))-\beta(y-1)}{\alpha(1-x)+\beta(y-1)}$
N_4	$-1-x+x(y-1)$	$\frac{-\alpha-(\alpha+2\beta)x-\beta x(x-1)+(\alpha+\beta)x(y-1)}{\alpha+\beta x}$	$\frac{-\alpha-(\alpha+2\beta)x-\gamma(y-1)-\beta x(x-1)+(\alpha+\beta)x(y-1)}{\alpha+\beta x+\gamma(y-1)}$
N_5	$-1-x+x(y-1)$	$\frac{-\alpha-(\alpha+2\beta)x-\beta x(x-1)+(\alpha+\beta)x(y-1)}{\alpha+\beta x}$	$\frac{-\alpha(1+x)-\beta(y-1)+\alpha x(y-1)}{\alpha+\beta(y-1)}$
N_6	$-1-x+x(y-1)$	$-1-x+x(y-1)$	$-1-x+x(y-1)$

	D_3	D_4	D_5	D_6
N_0	$\frac{6}{-6+3x-x(x-1)}$	$\frac{-6}{6-3x+x(x-1)+3x(y-1)}$	$\frac{-6}{6-3x+x(x-1)+3x(y-1)}$	$\frac{-24}{24-12x+4x(x-1)+12x(y-1)-x(x-1)(x-2)}$
N_1	$\frac{3-3x}{-3+3x-x(x-1)}$	$\frac{-\alpha+3\beta x}{\alpha-\alpha x+\beta x(x-1)+(\alpha-3\beta)x(y-1)}$	$\frac{-1-x}{1+x(y-1)}$	$\frac{-1-x}{1+x(y-1)}$
N_2	$\frac{-3\alpha+3\alpha x-\beta(y-1)}{-3\alpha x+\beta(y-1)+\alpha x(x-1)}$	$\frac{-\alpha-\alpha x-\beta(y-1)}{\alpha+\beta(y-1)+\alpha x(y-1)}$	$\frac{-\alpha-\alpha x-\beta(y-1)}{\alpha+\beta(y-1)+\alpha x(y-1)}$	$\frac{-1-x+(y-1)}{1-(y-1)+x(y-1)}$
N_3	$\frac{-1+x+x(x-1)-\beta(y-1)}{\alpha(1-x)+\beta(y-1)}$	$\frac{-\alpha(1+x)-\beta(y-1)}{\alpha+\beta(y-1)+\alpha x(y-1)}$	$\frac{-1-x+(y-1)}{1-(y-1)+x(y-1)}$	
N_4	$\frac{(1+x)-\beta(y-1)+\alpha x(y-1)}{\alpha+\beta(y-1)}$	$\frac{-\alpha(1+x)-\beta(y-1)+(\alpha+\beta)x(y-1)}{\alpha+\beta(y-1)-\beta x(y-1)}$		
N_5	$-1-x+x(y-1)$			
N_6				

Table 2



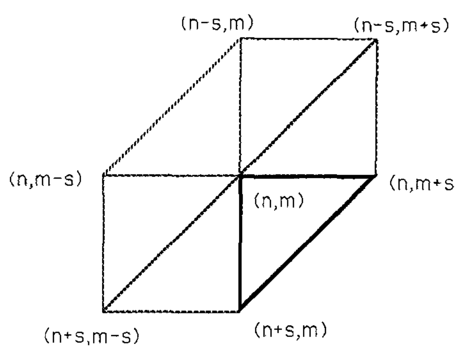
multivariate case. Look at the rational interpolation problem $[N_4/D_5]_{I_9}$ which has a rank deficiency of $s = 1$. For $[N_3/D_5]_{I_8}$ Theorem 2(b) and 2(c) apply at the same time with $t_1 = 1 = t_2$, such that the triangular configuration emanates from the rational interpolation problem of order $(n - k - t_1, m - l - t_2)$.

It must be clear by this time that Corollary 1 cannot easily be generalized to the multivariate case, since the concept of "rational interpolant" as a unique irreducible form is not well defined. However, the following multivariate counterpart holds.

Corollary 2. *Let $p(x, y)$ and $q(x, y)$ be defined by* (4), (7), *and* (8). *Let the rank of $C_{n+1,n+m}$ in* (8b) *be given by $m - s$. Then the following holds*:

(a) *For $s_1 \geq 0$, $s_2 \geq 0$, and $s_1 + s_2 \leq s$*:

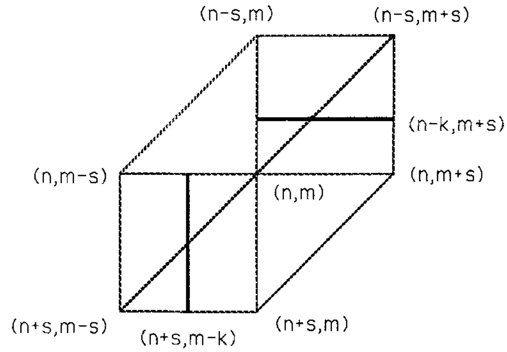
$$\bigcap_{(s_1, s_2)} [N_{n+s_1}/D_{m+s_2}]_{I_{n+m+s_1+s_2}} \neq \emptyset.$$



(b) *Let $0 \leq k \leq s$ and the rank of $C_{n-k+1,n+m-s}$ be equal to $m - s + k$*:

$$\bigcap_{j=0}^{s} [N_{n-k}/D_{m+j}]_{I_{n-k+m+j}} \neq \emptyset,$$

$$\bigcap_{i=0}^{s} [N_{n+i}/D_{m-k}]_{I_{n+i+m-k}} \neq \emptyset.$$



Proof. (a) Since the coefficient matrix $C_{n+1,m+1}$ of (8b) has a rank deficiency of s, we can add at least s conditions to the homogeneous system and still find a nontrivial solution for the coefficients b_{ij}. The rational function $(p/q)(x, y)$ with numerator coefficients a_{ij} and denominator coefficients b_{ij} indexed, respectively, by N and D as given in (7a) and (7b) and satisfying

$$\begin{pmatrix} c_{d_0 i_0, e_0 j_0} & \cdots & c_{d_m i_0, e_m j_0} \\ \vdots & & \vdots \\ c_{d_0 i_n, e_0 j_n} & \cdots & c_{d_m i_n, e_m j_n} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} a_{i_0 j_0} \\ \vdots \\ a_{i_n j_n} \end{pmatrix},$$

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m}, e_0 j_{n+m}} & \cdots & c_{d_m i_{n+m}, e_m j_{n+m}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m+s}, e_0 j_{n+m+s}} & \cdots & c_{d_m i_{n+m+s}, e_m j_{n+m+s}} \end{pmatrix} \begin{pmatrix} b_{d_0 e_0} \\ \vdots \\ b_{d_m e_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

belongs to the solution set $[N_n/D_m]_{I_{n+m}}$ and also to all the solution sets $[N_{n+s_1}/D_{m+s_2}]_{I_{n+m+s_1+s_2}}$ for all $s_1 \geq 0$ and $s_2 \geq 0$ with $s_1 + s_2 \leq s$.

(b) We shall give the proof only for the first intersection. The second intersection is proved analogously. Knowing that the rank of $C_{n-k+1,n+m-s}$ equals $m - s + k$, we can conclude that the rank of

$$\begin{pmatrix} c_{d_0 i_{n+1}, e_0 j_{n+1}} & \cdots & c_{d_m i_{n+1}, e_m j_{n+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m-s}, e_0 j_{n+m-s}} & \cdots & c_{d_m i_{n+m-s}, e_m j_{n+m-s}} \end{pmatrix},$$

equals $m - s$. Knowing that the rank of $C_{n+1,n+m}$ also equals $m - s$, we can conclude that in $C_{n+1,n+m}$ the last s rows are dependent from the first $m - s$ rows. Hence the rank of

$$\begin{pmatrix} c_{d_0 i_{n-k+1}, e_0 j_{n-k+1}} & \cdots & c_{d_m i_{n-k+1}, e_m j_{n-k+1}} \\ \vdots & & \vdots \\ c_{d_0 i_{n+m+s-k}, e_0 j_{n+m+s-k}} & \cdots & c_{d_m i_{n+m+s-k}, e_m j_{n+m+s-k}} \end{pmatrix}$$

is at most m, and consequently the homogeneous system with coefficient matrix $C_{n-k+1,n+m+s-k}$ and unknowns $b_{i_0j_0}, \ldots, b_{i_{m+s}j_{m+s}}$ has a nontrivial solution. This solution belongs to $[N_{n-k}/D_m]_{I_{n+m-k}} \cap [N_{n-k}/D_{m+s}]_{I_{n+m+s-k}}$. ■

The solution of $[N_n/D_m]_{I_{n+m}}$ common to all solution sets $[N_{n+s_1}/D_{m+s_2}]_{I_{n+m+s_1+s_2}}$ as described in Corollary 2(a) could be called the "optimal solution" in the sense that it satisfies as many conditions as possible. If the rank of $C_{n+1,n+m+s}$ is still not maximal even more conditions can be added. So in the rational interpolation table a triangle emanating from $[N_n/D_m]_{I_{n+m}}$ can be filled with the optimal solution, while triangles emanating from $[N_{n-k}/D_{m-l}]_{I_{n+m-s}}$ with $k + l = s$ can be filled with minimal solutions. The rest of the hexagon is filled with the solutions constructed in the proof of Corollary 2(b).

References

1. G. Claessens (1978): *On the Newton-Padé approximation problem,* J. Approx. Theory, **22**:150–160.
2. G. Claessens (1978): *On the structure of the Newton–Padé table.* J. Approx. Theory, **22**:304–319.
3. A. Cuyt, B. Verdonk (1985): *General order Newton–Padé approximants for multivariate functions.* Numer. Math., **43**:293–307.

H. Allouche
UFR-IEEA
Groupe ANO
Université de Lille I
Bâtiment M3
F-59655 Villeneuve d'Ascq Cedex
France

A. Cuyt
Research Associate NFWO
Department of Mathematics and Computer Science
University of Antwerp (UIA)
Universiteitsplein 1
B-2610 Wilrijk
Belgium