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Reliable root detection with the qd-algorithm: When Bernoulli, Hadamard and Rutishauser cooperate

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ABSTRACT

When using Rutishauser's qd-algorithm for the determination of the roots of a polynomial (originally the poles of a meromorphic function), or for related problems, conditions have been formulated for the interpretation of the computed q- and e-values. For a correct interpretation, the so-called critical indices play a crucial role. They index a column of e-values that tends to zero because of a jump in modulus among the poles. For more than 50 years the qd-algorithm in exact arithmetic was considered to be fully understood. In this presentation we push the detailed theoretical investigation of the qd-algorithm even further and we present a new aspect that seems to have been overlooked. We indicate a new element that makes a column of e-values tend to zero, namely a jump in multiplicity among equidistant poles. This result is obtained by combining the qd-algorithm with a deflation technique, and hence mainly relying on Bernoulli's method and Hadamard's formally orthogonal polynomials. Our results round up the theoretical analysis of the qd-algorithm as formulated in its original form, and are of importance in a variety of practical applications as outlined in the introduction.

0. Introduction

The qd-algorithm in its original form [25,29,28] is used for the determination of poles of a meromorphic function or for related problems [26,13,18,7]. The proper understanding when a column index is critical or not, is crucial because making the wrong decision leads to erroneous results. We focus on the correct determination of the columns in the qd-table that contain important information about the location of the poles of a meromorphic function. In this analysis a new element that makes a column of e-values tend to zero, is discovered: important e-columns do not only occur at a jump in modulus among the poles but also at a jump in multiplicity among equimodular poles. The distinction between the two cases can be made from the speed of convergence. The convergence speed of the e-columns at jumps in modulus is extensively discussed in [14,23]. The new result which concerns jumps in multiplicity among poles, is of importance in a variety of practical applications, through its connection with eigenvalue computation [22,10,11] and sparse polynomial interpolation [7] and polynomial systems. Sparse techniques are currently at the core of some robot kinematics [31], sparse signal processing [20], drug design [12] and organism evolution [21]. The issue of numeric stability of the qd-algorithm is not being discussed in the current presentation which focuses on new theoretical results.

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A new and detailed analysis of the convergence speed of the e-values (see the proofs of Theorems 2.2 and 3.3) shows how one can distinguish between a correct and an incorrect interpretation of the values in the qd-table. The new technique prevents erroneous conclusions from the inspection of the columns in the qd-algorithm, particularly in the case of poles of equal modulus and different multiplicity. In the latter case the new results, and their combination with deflation, significantly improve the accuracy of the extracted poles. The scheme is illustrated with two detailed numerical examples and the influence of cancellation is discussed which is typical for deflation techniques.

The convergence of so-called inbetween or additional columns has also been observed in other tables, namely in [30] for the Padé table and in [19] for the Walsh array. In the former the column number was not explicited: it resulted as the solution of an integer programming problem. In the latter explicit column numbers were given but the explicit extraction of the poles was not discussed. In this presentation we show how to obtain all the poles and their multiplicities from the mere computation of the qd-table. Neither the algorithm nor the proofs necessitate the computation of entries in the Padé table or the Walsh array. We add to the convergence result of [19] what Rutishauser has added in [25] to the convergence theorem of de Montessus de Ballore [9].

The paper is structured as follows. In Section 1 we summarize the classical results on the qd-algorithm which can be found in [17] or the original publications [25,27,24]. Sections 2 and 3 guide the reader through the new results which are then summarized and illustrated in Sections 4 and 5. Some particular cases are discussed in Section 6. The proofs of all new results can be found in Section 7.

1. The classical qd-algorithm

Let the function $f(z)$ be known by its formal power series expansion (FPS)

$$f(z)=\sum_{i=0}^{\infty} c_i z^i. \tag{1}$$

The series expansion is taken around the origin only to simplify the notation. We set $c_i=0$ for $i<0$. For arbitrary integers n and for integers $m\geqslant 0$ we define the Hankel determinants

$$H_m^{(n)}=\begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+m-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+m} \\ \vdots & & & \vdots \\ c_{n+m-1} & c_{n+m} & \cdots & c_{n+2m-2} \end{vmatrix}, \qquad H_0^{(n)}=1,$$

and the Hadamard polynomials

$$P_m^{(n)}(z)=\frac{H_m^{(n)}(z)}{H_m^{(n)}}, \quad m\geqslant 0,\ n\geqslant 0,$$

with

$$H_m^{(n)}(z)=\begin{vmatrix} c_n & \cdots & c_{n+m-1} & c_{n+m} \\ \vdots & \ddots & & \\ & & \vdots & \vdots \\ c_{n+m-1} & \cdots & & c_{n+2m-1} \\ 1 & \cdots & z^{m-1} & z^m \end{vmatrix}, \qquad H_0^{(n)}(z)=1.$$

The series (1) is termed *k-normal* if $H_m^{(n)}\neq 0$ for $m=0,1,\ldots,k$ and $n\geqslant 0$. It is called *ultimately k-normal* if for every $0\leqslant m\leqslant k$ there exists an $n(m)$ such that $H_m^{(n)}\neq 0$ for $n>n(m)$. With (1) as input we can also define the qd-scheme [25]:

1. the initial columns are given by

$$e_0^{(n)}=0, \quad n=1,2,\ldots,$$
$$q_1^{(n)}=\frac{c_{n+1}}{c_n}, \quad n=1,2,\ldots,$$

2. and the rhombus rules for continuation of the scheme by

$$e_m^{(n)}=q_m^{(n+1)}-q_m^{(n)}+e_{m-1}^{(n+1)}, \quad m=1,2\ldots,\ n=1,2,\ldots,$$
$$q_{m+1}^{(n)}=\frac{e_m^{(n+1)}}{e_m^{(n)}}q_m^{(n+1)}, \quad m=1,2\ldots,\ n=1,2,\ldots. \tag{2}$$

Usually the values $q_m^{(n)}$ and $e_m^{(n)}$ are arranged in a table where subscripts indicate columns and superscripts downward sloping diagonals and the continuation rules link elements in a rhombus:

$$\begin{array}{ccccccc}
q_1^{(1)} & & & & & & \\
 & e_1^{(1)} & & & & & \\
q_1^{(2)} & & q_2^{(1)} & & & & \\
 & e_1^{(2)} & & e_2^{(1)} & & & \\
q_1^{(3)} & & q_2^{(2)} & & \ddots & & \\
 & e_1^{(3)} & & e_2^{(2)} & & & \\
q_1^{(4)} & & q_2^{(3)} & & \ddots & & \\
\vdots & e_1^{(4)} & \vdots & e_2^{(3)} & & & \\
 & \vdots & & \vdots & & &
\end{array}$$

The proof of the following lemma and theorem can be found in [17, pp. 610–613]. As indicated in Lemma 1.1, the condition of (ultimate) k-normality guarantees the computation of the q- and e-columns. How the periodic occurrence of zeroes can be avoided is indicated in Section 6.

Lemma 1.1. *Let* (1) *be the FPS at* $z = 0$ *of a function* f *which is* k*-normal for some integer* $k > 0$*. Then the values* $q_m^{(n)}$ *and* $e_m^{(n)}$ *exist for* $m = 1, \ldots, k$ *and*

$$q_m^{(n)} = \frac{H_m^{(n+1)} H_{m-1}^{(n)}}{H_m^{(n)} H_{m-1}^{(n+1)}},$$

$$e_m^{(n)} = \frac{H_{m+1}^{(n)} H_{m-1}^{(n+1)}}{H_m^{(n)} H_m^{(n+1)}}.$$

Theorem 1.2. *Let* (1) *be the FPS at* $z = 0$ *of a function* f *meromorphic in the disk* $B(0, R) = \{z\colon |z| < R\}$*, and let the poles* z_i *of* f *in* $B(0, R)$ *be numbered such that*

$$z_0 = 0 < |z_1| \leqslant |z_2| \leqslant \cdots < R,$$

with each pole occurring as many times in the sequence $\{z_i\}_i$ *as indicated by its order. If* f *is ultimately* k*-normal for some integer* $k > 0$*, then the qd-scheme associated with* f *has the following properties.*

(a) *For each* m *with* $0 < m \leqslant k$ *and* $|z_{m-1}| < |z_m| < |z_{m+1}|$,

$$\lim_{n\to\infty} q_m^{(n)} = z_m^{-1}.$$

(b) *For each* m *with* $0 < m \leqslant k$ *and* $|z_m| < |z_{m+1}|$,

$$\lim_{n\to\infty} e_m^{(n)} = 0.$$

Any index m such that the strict inequality

$$|z_m| < |z_{m+1}|$$

holds, is called a *critical index*. It is clear that the critical indices of a function do not depend on the order in which the poles of equal modulus are numbered. The theorem above states that if m is a critical index and f is ultimately m-normal, then

$$\lim_{n\to\infty} e_m^{(n)} = 0.$$

The e-columns with critical index column number do complicate the computation of the values in the qd-table when using (2). The fact that they tend to zero and are used in divisions, destabilizes the computation. In order to overcome this problem a progressive form of the qd-algorithm can be used as in [17, pp. 614–615] or a breakdown-free formulation as in [16,15].

Thus the qd-table of a meromorphic function is divided into subtables by the e-columns that tend to zero. Any q-column corresponding to a simple pole of isolated modulus is flanked by such e-columns and converges to the reciprocal of the corresponding pole. If a subtable contains $j > 1$ columns of q-values, the presence of j poles of equal modulus is indicated. In Theorem 1.3 [17, p. 642] it is also explained how to determine these poles if $j > 1$.

Theorem 1.3. *Let m and $m + j$ with $j > 1$ be two consecutive critical indices and let f be $(m + j)$-normal. Let the polynomials $p_i^{(n)}(z)$ be defined by*

$$p_0^{(n)}(z) = 1,$$
$$p_{i+1}^{(n)}(z) = zp_i^{(n+1)}(z) - q_{m+i+1}^{(n)} p_i^{(n)}(z), \quad n \geqslant 0,\ i = 0, 1, \ldots, j-1.$$

Then there exists a subsequence $\{p_j^{n(\ell)}\}_\ell$ such that

$$\lim_{\ell\to\infty} p_j^{(n(\ell))}(z) = \left(z - z_{m+1}^{-1}\right)\cdots\left(z - z_{m+j}^{-1}\right).$$

According to the above two theorems the qd-scheme seems to be an ingenious tool for determining, under certain conditions, the poles of a meromorphic function f directly from its FPS at the origin. If f is rational, the last e-column is even identically equal to zero [17, pp. 610–613].

If $m = 0$ in Theorem 1.3, then the polynomials $p_i^{(n)}(z)$ coincide with the Hadamard polynomials $P_i^{(n)}(z)$ for $i = 0, \ldots, j$ and the entire sequence converges, as we can see from Theorem 1.4 [17, p. 626].

Theorem 1.4. *Let (1) be the FPS at $z = 0$ of a function f meromorphic in the disk $B(0, R) = \{z\colon |z| < R\}$, and let the poles z_i of f in $B(0, R)$ be numbered such that*

$$z_0 = 0 < |z_1| \leqslant |z_2| \leqslant \cdots < R,$$

with each pole occurring as many times in the sequence $\{z_i\}_i$ as indicated by its order. If f is ultimately m-normal and if $|z_m| < |z_{m+1}|$, then

$$\lim_{n\to\infty} P_m^{(n)}(z) = \left(z - z_1^{-1}\right)\left(z - z_2^{-1}\right)\cdots\left(z - z_m^{-1}\right)$$

uniformly on compact subsets of $\mathbb{C}$.

The Hadamard polynomials are usually computed as follows.

Lemma 1.5. *For all Hadamard polynomials that are well defined,*

$$P_m^{(n)}(z) = zP_{m-1}^{(n+1)}(z) - q_m^{(n)} P_{m-1}^{(n)}(z).$$

Theorems 1.2, 1.3 and 1.4 summarize what is known about the extraction of pole information from the FPS coefficients c_n of a meromorphic function. But several problems remain:

- Theorem 1.2(b) gives only sufficient conditions to locate a jump in modulus between the poles of f. If $\lim_{n\to\infty} e_m^{(n)} = 0$, there is no guarantee that $|z_m| < |z_{m+1}|$. We only know for sure that if $\lim_{n\to\infty} e_m^{(n)} \neq 0$, then $|z_m| = |z_{m+1}|$. In case the poles z_m and z_{m+1} have equal modulus, we do not know how the sequence $\{e_m^{(n)}\}_n$ behaves.
- When dealing with poles of equal modulus, Theorem 1.3 only ensures the existence of a converging subsequence, not the convergence of the entire sequence. The degree of the polynomials $p_j^{(n)}(z)$ is determined from the knowledge of consecutive critical indices. The convergence of the sequence of polynomials $P_m^{(n)}(z)$ in Theorem 1.4 is only guaranteed when m is the first critical index. But determining a critical index is easier said than done. One usually observes the convergence of the e-columns and from this decides which column numbers serve as critical indices. However, Theorem 1.2(b) does not guarantee the correctness of this procedure because it does not give a necessary condition to locate a jump in modulus.

Let us now investigate whether these problems can be overcome in some way, and at what price.

2. Computing simple or multiple poles isolated in modulus

In this section we generalize Bernoulli's method for the extraction of a simple pole of isolated modulus to the case of a multipole of isolated modulus.

Suppose the poles $z_1, \ldots, z_s$ of $f(z)$ have been determined. Then we can compute the coefficients $C_n^{(s)}$ given by

$$C_n^{(0)} = c_n,$$
$$C_0^{(i)} = z_i C_0^{(i-1)}, \quad i = 1, \dots, s,$$
$$C_n^{(i)} = z_i C_n^{(i-1)} - C_{n-1}^{(i-1)}, \quad n = 1, 2, \dots, \ i = 1, \dots, s,$$

which are the coefficients in the FPS of

$$f_s(z) = (z_1 - z) \cdots (z_s - z) f(z) = \sum_{n=0}^{\infty} C_n^{(s)} z^s.$$

For $s = 0$, $f_0(z) = f(z)$. The computation of the coefficients $C_n^{(s)}$ can also be done by solving the linear system (only back-substitution)

$$\sum_{j=0}^{n} z_s^{-j-1} C_{n-j}^{(s)} = C_n^{(s-1)}, \quad n = 0, 1, 2, \dots.$$

The following lemma, which was proved in [17, pp. 569–570], will be very useful.

Lemma 2.1. *Let f be a rational function with distinct poles $z_1, \dots, z_s$. Let the multiplicity of z_i be denoted by m_i. Then the FPS coefficients c_n of f are of the form*

$$c_n = \sum_{i=0}^{s} \gamma_i(n) z_i^{-(n+1)},$$

where $\gamma_i(n)$ is a polynomial of degree $m_i - 1$ in the index n.

Using Lemma 2.1 the following new result can be proved.

Theorem 2.2. *Let (1) be the FPS at $z = 0$ of a function f meromorphic in the disk $B(0, R) = \{z\colon |z| < R\}$ and let the poles z_i of f in $B(0, R)$ be numbered such that their modulus does not decrease, each pole occurring as many times in the sequence $\{z_i\}_i$ as indicated by its order. Let $f_{s+i}(z)$ be ultimately 1-normal for $i = 0, \dots, t-1$ and let z_{s+1} be a pole isolated in modulus, meaning that if for some $i \geqslant 1$, $|z_i| = |z_{s+1}|$ then $z_i = z_{s+1}$. Then z_{s+1} is a pole of multiplicity t, in other words*

$$z_0 = 0 < |z_1| \leqslant \cdots \leqslant |z_s| < |z_{s+1}| = \cdots = |z_{s+t}| < |z_{s+t+1}| \leqslant \cdots < R,$$
$$z_{s+1} = \cdots = z_{s+t},$$

if and only if

$$\begin{cases} \lim_{n\to\infty} \dfrac{C_n^{(s)}}{C_{n+1}^{(s)}} = \cdots = \lim_{n\to\infty} \dfrac{C_n^{(s+t-1)}}{C_{n+1}^{(s+t-1)}} = z_{s+1}, \\ \lim_{n\to\infty} \dfrac{C_n^{(s+t)}}{C_{n+1}^{(s+t)}} \neq z_{s+1}. \end{cases}$$

The limit of $C_n^{(s+t)}/C_{n+1}^{(s+t)}$ not being equal to z_{s+1}, means that either it does not exist or, if it exists, it is infinite or a different finite complex number. The sufficient conditions in Theorem 2.2 are a means to compute z_{s+1}, while the necessary conditions allow to detect the presence and the multiplicity of a multipole of isolated modulus.

When inspecting column $e_1^{(n)}$ computed from the coefficients $C_n^{(s)}$, then we find that it converges slowly but surely to zero. This phenomenon will be explained in the sequel. Also column $e_t^{(n)}$ computed from $C_n^{(s)}$ converges to zero, by Theorem 1.2 because t is a critical index. However, it converges much faster.

The technique even applies to functions like $f(z) = \log(1+z)$ having a pole of infinite multiplicity. Here for each $s \geqslant 1$ and $n \geqslant s$, we find

$$\lim_{n\to\infty} \frac{C_n^{(s)}}{C_{n+1}^{(s)}} = -1, \qquad C_n^{(s)} = \frac{(-1)^{n+1-s} s!}{n(n-1)\cdots(n-s)}.$$

The limit of $C_n^{(s)}/C_{n+1}^{(s)}$ can also be larger than R in modulus, in which case we can stop the search for poles in $B(0, R)$ because they have all been found. When it does not exist at all, we have to turn to the results of the next section.

In the sequel we drop the superscript $^{(s)}$ in the coefficients $C_n^{(s)}$ and the subscript $_s$ in the function $f_s(z)$. In order to simplify the notation we denote the deflated function $f_s(z)$ by $f(z)$ and its FPS coefficients $C_n^{(s)}$ by c_n.

3. Computing distinct poles of equal modulus

When dealing with several poles of equal modulus, we have to distinguish between the computation of several simple poles of equal modulus and the case where at least one pole has a multiplicity larger than one. As before we assume that the poles of lesser modulus have been computed and removed from the FPS expansion.

Lemma 3.1. *Let* (1) *be the FPS at* $z = 0$ *of a function* f *meromorphic in the disk* $B(0, R) = \{z\colon\ |z| < R\}$*, and let the poles* z_i *of* f *in* $B(0, R)$ *be such that the first* t *poles are all simple and of equal modulus, in other words*

$$\begin{aligned} &z_0 = 0 < |z_1| = \cdots = |z_t| < |z_{t+1}| \cdots < R, \\ &z_j = re^{i\theta_j}, \qquad \theta_k \neq \theta_j, \quad k \neq j,\ k, j = 1, \ldots, t. \end{aligned} \tag{3}$$

Then for $1 \leqslant k < t$ *the sequence of determinants* $\{\tilde{H}_k^{(n)}\}_n$ (*with* $\gamma_j(n)$ *abbreviated by* γ_j) *where*

$$\tilde{H}_k^{(n)} = \begin{vmatrix} \sum_{j=1}^t \gamma_j e^{-i(n+1)\theta_j} & \ldots & \frac{1}{r^{k-1}} \sum_{j=1}^t \gamma_j e^{-i(n+k)\theta_j} \\ \frac{1}{r} \sum_{j=1}^t \gamma_j e^{-i(n+2)\theta_j} & \ldots & \frac{1}{r^k} \sum_{j=1}^t \gamma_j e^{-i(n+k+1)\theta_j} \\ \vdots & & \vdots \\ \frac{1}{r^{k-1}} \sum_{j=1}^t \gamma_j e^{-i(n+k)\theta_j} & \ldots & \frac{1}{r^{2k-2}} \sum_{j=1}^t \gamma_j e^{-i(n+2k-1)\theta_j} \end{vmatrix}$$

does not converge. For $k = t$ *we have* $\tilde{H}_t^{(n)} \neq 0$.

Theorem 3.2. *Let* (1) *be the FPS at* $z = 0$ *of a function* f *meromorphic in the disk* $B(0, R) = \{z\colon\ |z| < R\}$*, and let the poles* z_i *of* f *in* $B(0, R)$ *be such that* (3) *holds. Then for* $1 \leqslant k < t$ *the sequence* $\{e_k^{(n)}\}_n$ *does not converge.*

In the situation of Theorem 3.2, the poles $z_1, \ldots, z_t$ are computed using Theorem 1.4 which is a special case of Theorem 1.3 with $m = 0$. So the Hadamard polynomials of degree t contain the information on the next t simple poles of equal modulus. From the proof of Lemma 3.1 it is obvious that an infinite number of entries in the sequences $\{e_k^{(n)}\}_n$ exists because an infinite number of the determinants $\tilde{H}_k^{(n)}$ is nonzero.

Next we consider the situation where at least one of the poles that we want to compute next has a multiplicity larger than one. We again distinguish between two cases, which are respectively dealt with by Theorems 3.3 and 3.6.

Theorem 3.3. *Let* (1) *be the FPS at* $z = 0$ *of a function* f *meromorphic in the disk* $B(0, R) = \{z\colon\ |z| < R\}$*, and let* z_i *denote the distinct poles of* f *in* $B(0, R)$ *with the first* t *poles* $z_1, \ldots, z_t$ *having equal modulus but unequal multiplicity. Let* f *be ultimately* 1*-normal. If only one pole's multiplicity among the first* t *poles is maximal, say that of* z_s*, then*

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_s, \quad 1 \leqslant s \leqslant t.$$

From the proof in Section 7 it is clear that in this case the convergence of c_n/c_{n+1} to z_s and that of $e_1^{(n)}$ to zero is slow, of the order of $1/n^{\mu}$ where μ is the gap in multiplicity between that of z_s and the largest multiplicity of the other poles of modulus $|z_s|$. This situation is precisely the one that is not detected in Theorem 1.2(b) and that may lead to an incorrect interpretation of the values in the qd-table. Instead of recommending a convergence acceleration technique, we introduce the new concept of critical multiplicity index. Rather than merely being a drawback, the slow convergence of this e-column plays a crucial role in the correct interpretation of the values in the qd-table.

Definition 3.4. Let z_i denote the distinct poles of f and let m_i denote their multiplicity. Let the z_i be numbered such that their modulus does not decrease and such that the multiplicity among poles of equal modulus does not increase. Let $|z_1| = \cdots = |z_t|, 1 \leqslant s \leqslant t$ and $m_s \geqslant m_{s+1}$. If the respective multiplicities m_s and m_{s+1} of z_s and z_{s+1}, being two poles of equal modulus, satisfy $m_s > m_{s+1}$, or if $s = t$ and $1 < m_1 = \cdots = m_t$, then s is called a critical multiplicity index.

This definition distinguishes, among the vanishing columns $e_m^{(n)}$, those that vanish because of a jump in modulus from those that vanish because of a jump in multiplicity among poles of equal modulus. Its importance was overlooked in the past. For both the critical multiplicity index and the critical index the e-column tends to zero. But the latter indicates a jump in modulus while the former does not.

After eliminating, by applying Theorem 3.3, among the poles of equal modulus, one by one those with strictly larger multiplicity, in the order of decreasing multiplicity, we are left with a number of poles of equal modulus and equal multiplicity. The following theorem deals with this case. Again the function $f(z)$ is the deflated function and the next poles in line are those of equal modulus and equal multiplicity.

Lemma 3.5. *Let* (1) *be the FPS at $z = 0$ of a function f meromorphic in the disk $B(0, R) = \{z\colon |z| < R\}$, and let z_i denote the distinct poles of f in $B(0, R)$. Let the first t poles have equal modulus and let the equimodular poles be numbered according to decreasing multiplicity. In other words,*

$$\begin{aligned} &z_0 = 0 < |z_1| = \cdots = |z_t| < |z_{t+1}| < \cdots < R, \\ &z_j = re^{i\theta_j}, \qquad \theta_i \neq \theta_j, \quad i, j = 1, \ldots, t, \\ &m_1 = \cdots = m_s > m_{s+1} \geqslant \cdots, \quad t \geqslant s > 1, \end{aligned} \tag{4}$$

with s being the first critical multiplicity index. Then

(a) *there exist constants $\alpha_1, \ldots, \alpha_s$ and an integer $\tau \geqslant 1$ such that*

$$c_n = \frac{n^{m_s-1}}{r^{n+1}}\left(\alpha_1 e^{-i(n+1)\theta_1} + \cdots + \alpha_s e^{-i(n+1)\theta_s} + O\left(\frac{1}{n^\tau}\right)\right);$$

(b) *the Hankel determinant $H_{s+1}^{(n)}$ equals*

$$H_{s+1}^{(n)} = \frac{n^{(s+1)(m_s-1)}}{r^{(s+1)(n+1)}}\left(\frac{A}{n^\mu}e^{-i(n+1)\theta} + O\left(\frac{1}{n}\right)\right),$$

where A is a constant and $\mu = m_s - m_{s+1} \geqslant 1$ and $\theta = \theta(\theta_1, \ldots, \theta_t)$.

Theorem 3.6. *Let* (1) *be the FPS at $z = 0$ of a function f meromorphic in the disk $B(0, R) = \{z\colon |z| < R\}$, and let z_i denote the distinct poles of f in $B(0, R)$. Let the first t poles have equal modulus, and let the equimodular poles be numbered according to decreasing multiplicity, as in* (4)*. Let s be the first critical multiplicity index and let f be ultimately s-normal. Then*

$$\begin{cases} \text{the sequence } \{e_k^{(n)}\}_n \text{ does not converge}, \quad k = 1, \ldots, s-1, \\ \lim_{n\to\infty} e_s^{(n)} = 0, \end{cases}$$

and the sequence $\{P_s^{(n)}(z)\}_n$ of Hadamard polynomials converges to

$$\lim_{n\to\infty} P_s^{(n)}(z) = (z - z_1^{-1})\cdots(z - z_s^{-1})$$

uniformly on compact subsets of $\mathbb{C}$.

In the case of Theorem 3.6, again Hadamard's polynomials can be used for the extraction of the poles $z_1, \ldots, z_s$ from the FPS representation of f. Their computation is guaranteed by the ultimate s-normality. Theorem 3.6 generalizes Theorem 3.2 to the case of equimodular poles of different multiplicity or of equal but higher multiplicity.

When $s = t$ in Theorem 3.6, then all poles $z_1, \ldots, z_t$ have the same multiplicity. The difference with the result found in Theorem 3.2 is that here the multiplicity m_t of these equimodular poles is allowed to be larger than one. In that case, Theorem 3.6 says that t is still a critical multiplicity index, because $P_t^{(n)}$ will only extract each pole once. Actually Theorem 3.6 will be applied $m_t - 1$ times. In the last step the poles $z_1, \ldots, z_t$ remain with multiplicity 1 and then Theorem 3.2 can be applied.

When m is a critical index and not a critical multiplicity index, then the convergence speed of $e_m^{(n)}$ to zero is of the order of $(|z_m/z_{m+1}|)^n$ because m signals a jump in modulus. These e-columns are more clearly visible in the qd-table than the ones with critical multiplicity index number. But the latter often jeopardize the correct interpretation of the qd-table.

When there is an infinite number of poles of equal modulus, then no e-column will tend to zero because the next critical index is $+\infty$.

4. Illustrative example

In order to explain the impact of the above results on a pole extraction algorithm based on the qd-table, let us discuss the application of Theorems 2.2, 3.2, 3.3 and 3.6 to the symbolic function

$$f(z) = \frac{g(z)}{(z - z_1)^4(z - z_2)(z - z_3)(z - z_4)(z - z_5)^4(z - z_6)^3(z - z_7)^2}, \tag{5}$$

where $g(z)$ is holomorphic in $B(0, R)$, and

$$\begin{aligned} &|z_1| = r_1, && m_1 = 4, \\ &|z_2| = |z_3| = |z_4| = r_2, && m_2 = m_3 = m_4 = 1, \\ &|z_5| = |z_6| = |z_7| = r_3, && m_5 = 4, \qquad m_6 = 3, \qquad m_7 = 2, \end{aligned}$$

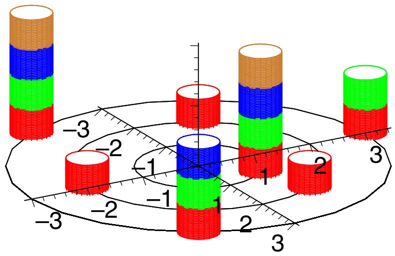

Fig. 1. Poles of $f(z)$ and their multiplicities.

with $r_1 < r_2 < r_3 < R$. We turn to a truly numeric example in Section 5. Using the deflation technique described above in combination with the qd-algorithm, the poles are extracted in order of increasing magnitude, and among the equimodular poles in order of decreasing multiplicity. In case a pole, be it isolated in modulus or not, has higher multiplicity, then it will be extracted, in different steps, as many times as indicated by its multiplicity. Let us describe what happens in the above example. At the same time we comment, from our extensive numerical experience, on the numerical quality of the results.

Fig. 1 is a graphical representation in $\mathbb{C} \times \mathbb{N}$ of the situation: the poles of f given in (5) are depicted in the complex plane and their multiplicity is added as a third dimension.

At the start $C_n^{(0)} = c_n$, and for $i = 0, \ldots, 3$

$$\lim_{n \to \infty} \frac{C_n^{(i)}}{C_{n+1}^{(i)}} = z_1,$$

while this does not happen for $C_n^{(4)}/C_{n+1}^{(4)}$ because of the specifics on the z_i and m_i. Hence Theorem 2.2 applies with $t = 4$. The pole of smallest radius is z_1, apparently of multiplicity 4, but isolated in modulus. A crude approximation of z_1 can be obtained from $\lim_{n \to \infty} C_n^{(i)}/C_{n+1}^{(i)}$. Knowing from the application of Theorem 2.2 that the multiplicity of the first pole isolated in modulus equals 4 (also $e_4^{(n)}$ converges to zero fast), it is better to compute the Hadamard polynomials $P_4^{(n)}(z)$ from the $C_n^{(0)}$ (this is from the q- and e-values computed from the $C_n^{(0)}$) and write it as

$$\lim_{n \to \infty} P_4^{(n)}(z) = \left(z - z_1^{-1}\right)^4.$$

So we have discovered the first critical index, namely 4. All poles of modulus $|z_1| = r_1$ have been extracted. All remaining poles are of modulus larger than r_1.

To move on, we recompute the FPS coefficients $C_n^{(4)}$ of

$$f_4(z) = (z_1 - z)^4 f(z),$$

and restart with checking $\lim_{n \to \infty} C_n^{(4)}/C_{n+1}^{(4)}$. Since it does not converge, we are certain to encounter equimodular poles of equal multiplicity. Otherwise either Theorem 2.2 or Theorem 3.3 would apply. So in this case, either Theorem 3.2 (simple poles) or Theorem 3.6 (with $1 < s$ here) should be applied to extract $(z - z_2)(z - z_3)(z - z_4)$. To this end new Hadamard polynomials are computed from the qd-table constructed with the $C_n^{(4)}$. When we compute this qd-table, of which we denote the entries by $q_m^{(n,4)}$ and $e_m^{(n,4)}$, then we see that sufficiently fast

$$\lim_{n \to \infty} e_3^{(n,4)} = 0,$$

and that no preceding e-column tends to zero slowly. Therefore all multiplicities are equal. Hence 3 is a critical index and we are in the situation of Theorem 3.2. The Hadamard polynomials of degree 3 deliver the information on these poles. We are sure that all poles of modulus r_2 have been extracted because we found the critical index. All remaining poles are of modulus larger than r_2.

To continue our search, we now compute the FPS coefficients for

$$f_7(z) = (z_2 - z)(z_3 - z)(z_4 - z) f_4(z).$$

As usual we start by investigating $\lim_{n \to \infty} C_n^{(7)}/C_{n+1}^{(7)}$. It now converges to z_5 because m_5 is strictly greater than the other multiplicities. We are either facing the case covered by Theorem 2.2 or that covered by Theorems 3.3 and 3.6 (case $1 < s$). By looking at $C_n^{(8)}/C_{n+1}^{(8)}$ the case of Theorem 2.2 can be excluded. Both Theorems 3.3 and 3.6 deal with the situation of a critical multiplicity index. The difference is that in case of Theorem 3.3 column $e_1^{(n,7)}$ slowly tends to zero, while in case of Theorem 3.6 with $s > 1$ column $e_s^{(n,7)}$ slowly converges to zero. In our case

$$\lim_{n \to \infty} e_1^{(n,7)} = 0,$$

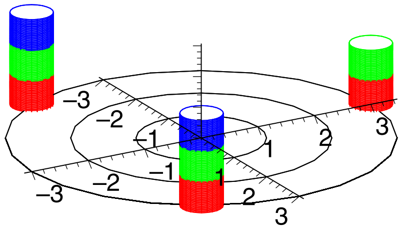

Fig. 2. Poles of $f_8(z)$ with multiplicities.

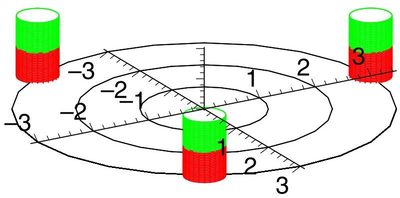

Fig. 3. Poles of $f_{10}(z)$ with multiplicities.

and hence Theorem 3.3 applies. Since 1 is not a true critical index, more poles of the same modulus are present (see Fig. 2). From the $C_n^{(8)}$ we obtain

$$\lim_{n\to\infty} \frac{C_n^{(8)}}{C_{n+1}^{(8)}} = z_6.$$

Is this again a case of Theorem 3.3 or of Theorem 3.6? It cannot be a case of Theorem 2.2 since $|z_6| = |z_5|$ with $z_6 \neq z_5$. Apparently, slowly but surely,

$$\lim_{n\to\infty} e_2^{(n,8)} = 0.$$

Hence Theorem 3.6 can be applied. After finding $(z - z_5)(z - z_6)$, we can compute the coefficients $C_n^{(10)}$.

From the fact that $\lim_{n\to\infty} C_n^{(10)}/C_{n+1}^{(10)}$ does not converge we know that we are in the situation of Theorem 3.2 or Theorem 3.6 (case $s = t$), namely the case of equimodular poles having the same multiplicity (see Fig. 3). We investigate the columns $e_m^{(n,10)}$ and find that $\lim_{n\to\infty} e_3^{(n,10)} = 0$ slowly, while $\lim_{n\to\infty} e_6^{(n,10)}$ quickly. Hence 3 is a critical multiplicity index, while 6 is a true critical index. All information on the poles has been found. One can check that

$$\lim_{n\to\infty} \frac{C_n^{(16)}}{C_{n+1}^{(16)}} = \infty,$$

indicating that all poles have been discovered, because our function is meromorphic in the entire complex plane. The last 9 poles, all of modulus r_3 are best computed by composing the Hadamard polynomial of degree 9 using the $q_m^{(n,7)}$ because the convergence speed of the Hadamard polynomials is always very high when their degree equals a true critical index.

The new results can also be used to obtain information on the multiplicity of the poles. For instance, the fact that 1 is a critical multiplicity index for $f_7(z)$ entails that $m_5 \geqslant m_6 + 1$ because 1 pole is of higher multiplicity. The fact that 2 is a critical multiplicity index for $f_8(z)$ entails that $m_5 \geqslant m_7 + 1$ and $m_6 \geqslant m_7 + 1$ and hence that $m_5 \geqslant m_7 + 2$. The fact that 3 is a critical multiplicity index for $f_{10}(z)$ and a true critical index for $f_{13}(z)$ leads us to conclude that

$$m_7 = 2, \qquad m_6 = m_7 + 1, \qquad m_5 = m_7 + 2.$$

Hence one knows that one is dealing with 3 distinct poles of respective multiplicities 2, 3 and 4 and one can compute the last 9 poles (we have also found that 9 is a true critical index for $f_7(z)$) from the above-mentioned Hadamard polynomial rewritten in the form

$$(z - z_5)^4 (z - z_6)^3 (z - z_7)^2.$$

Depending on the precision used in the computations, some nearby poles may of course appear as multipoles. It suffices to increase the precision to be able to separate those and write the Hadamard polynomial in the proper form (this aspect is outside the scope of this paper where we want to focus on the theoretical results). Otherwise a center of gravity is computed with the multiplicity equaling the proper number of poles it represents.

5. Numerical illustration

We now turn to a numeric illustration which is very representative for the multitude of experiments that we have carried out. We restrict ourselves to the case where the radii can be clearly distinguished. Let us consider the function

$$f(z) = \left(\frac{1}{1 - z^3} + \frac{1}{(2e^{i\pi/4} - z)^2} + \frac{1}{2i - z}\right)\exp(z),$$

which is a meromorphic function with 3 simple poles of modulus 1 and 3 poles of modulus 2, one simple and one double. We examine the classical qd-algorithm and its deflated generalization presented here. The detailed discussion of this numerical example leads to some recommendations for a practical implementation.

Table 1
No critical index.

$e_0^{(n)}$	$q_1^{(n)}$	$e_1^{(n)}$
	$0.81141-0.31799i$	
0		$-0.5338+0.022792i$
	$0.27791-0.29519i$	
0		$-0.71107+1.91288i$
	$-0.43316+1.61768i$	
0		$1.5832-1.43245i$
	$\vdots$	
		$\vdots$
	$0.8828+0.00431i$	
0		$-0.39795-0.00213i$
	$0.48484+0.00218i$	
0		$1.82941+0.01022i$
	$2.31426+0.01240i$	
0		$-1.42088-0.01192i$
	$\vdots$	
		$\vdots$

Table 2
True critical index.

$q_3^{(n)}$	$e_3^{(n)}$
$-0.37227-0.06994i$	
	$0.32911+1.12215i$
$-0.29196+1.48619i$	
	$-0.58098-1.29901i$
$\vdots$	
	$\vdots$
$1.69434-0.00425i$	
	$-0.00253-0.00200i$
$-0.51452-0.00076i$	
	$0.00336-0.00027i$
$\vdots$	
	$\vdots$
$-0.51296+1.58128\times 10^{-6}i$	
	$(7.98733-8.47731i)\times 10^{-6}$
$\vdots$	
	$\vdots$

Since column $q_1^{(n)} = C_{n+1}^{(0)}/C_n^{(0)}$, which can be found in Table 1, does not converge, one can conclude from Theorem 2.2 that f does not have a single smallest (in modulus) pole, be it simple or multiple. Hence the smallest (in modulus) poles of f present themselves as a collection of equimodular poles. One can also conclude that one is not dealing with the case described by Theorem 3.3 because then again $q_1^{(n)}$ would converge due to the larger multiplicity of one single pole.

It remains to distinguish the problem between the case of Theorem 3.2 and that described by Theorem 3.6. The first e-column to converge to zero is $e_3^{(n)}$ and its convergence is fast as we can see from Table 2. So we are in the case of Theorem 3.2 and column number 3 is a true critical index. If the convergence of the e-column had been slow, then we would have been in the situation described by Theorem 3.6 and column number 3 would have been a critical multiplicity index. The poles $|z_1| = |z_2| = |z_3|$ can be determined from Hadamard's polynomial sequence $P_3^{(n)}(z)$ given in Table 3, which converges to $-1+z^3$.

When advancing further in the qd-table, column $e_4^{(n)}$ seems to converge to zero (slowly), while column $e_6^{(n)}$ definitely converges to zero (fast), as can be seen from Table 4.

From the new results obtained in the previous section we can conclude that 4 is a critical multiplicity index while 6 is a true critical index in the traditional sense. Assume that we erroneously conclude from Theorem 1.2 that $\lim_{n\to\infty} q_4^{(n)} =$

Table 3
Hadamard polynomials.

$P_3^{(0)}(z) = (-0.29070 - 0.42071i) + (-0.04250 - 0.04693i)z + (-0.37732 - 0.25518i)z^2 + z^3$
$P_3^{(5)}(z) = (-0.89823 - 0.03704i) + (-0.08602 + 0.12290i)z + (-0.01319 + 0.00460i)z^2 + z^3$
$P_3^{(10)}(z) = (-0.99905 - 0.00179i) + (-0.00320 + 0.00804i)z + (-0.00168 - 0.01115i)z^2 + z^3$
$P_3^{(15)}(z) = (-0.99960 + 0.00035i) + (8.5 \times 10^{-6} + 0.00009i)z + (-0.00011 - 0.00039i)z^2 + z^3$
$P_3^{(20)}(z) = (-0.99998 + 7.2 \times 10^{-6}i) + (-0.00002 + 1.2 \times 10^{-6}i)z + (-3.2 \times 10^{-6} - 2.0 \times 10^{-6}i)z^2 + z^3$
$P_3^{(25)}(z) = (-1. - 2.9 \times 10^{-8}i) + (-6.3 \times 10^{-7} + 2.2 \times 10^{-7}i)z + (5.7 \times 10^{-7} - 6.4 \times 10^{-7}i)z^2 + z^3$

Table 4
Critical multiplicity index versus true critical index.

$e_4^{(n)}$	$e_5^{(n)}$	$e_6^{(n)}$
$-0.22709+0.12968i$		
$-0.10148+0.00752i$	$-0.25962+0.13164i$	
$-0.01123-0.00578i$	$1.07010+0.52425i$	$-0.00101-0.39214i$
$\vdots$	$\vdots$	$\vdots$
$-0.00211+0.00833i$	$-0.03414+0.03273i$	$(8.82-7.96i)\times 10^{-8}$
$0.00298+0.00651i$	$-0.05320+0.01246i$	$(1.47+0.36i)\times 10^{-8}$
$\vdots$	$\vdots$	$\vdots$
$0.00007-0.00400i$	$0.04118-0.03248i$	$0.+0.i$ (underflow)
$\vdots$	$\vdots$	$\vdots$

$1/z_4$ and consequently use the columns $q_5^{(n)}$, $e_5^{(n)}$ and $q_6^{(n)}$ to compose the quadratic polynomials $p_2^{(n)}(z)$ as indicated in Theorem 1.3. A subsequence of these polynomials converges to

$$p_2(z) \approx -(0.172966167422277 + 0.173275520875674i) \\ + (-0.350050652936770 + 0.846801223076168i)z + z^2.$$

From $\lim_{n\to\infty} q_4^{(n)}$ and the estimate for $p_2(z)$ one obtains the very bad estimates for the poles given by

$$z_4 = 1.363856230213032 + 1.345582135771676i,$$
$$z_5 = 0.011720961979471 + 2.015241354517116i,$$
$$z_6 = 1.426059438450287 + 1.440168938686209i. \tag{6}$$

Instead, let us compute the coefficients $C_n^{(3)}$ of the FPS of

$$f_3(z) = (z_1 - z)(z_2 - z)(z_3 - z)f(z).$$

The sequence $C_n^{(3)}/C_{n+1}^{(3)}$ converges very slowly to

$$z_4 = 1.40491 + 1.39518i. \tag{7}$$

Compared to the estimates for z_4, z_5 and z_6 obtained from the erroneous conclusion, the value

$$C_{30}^{(3)}/C_{31}^{(3)} = 1.39032 + 1.36117i$$

is already a much better estimate of z_4 than the approximation (6) extracted from $q_4^{(30)}$, which uses exactly the same data. Since the sequence $C_n^{(3)}/C_{n+1}^{(3)}$ converges, we are either in the situation described in Theorem 2.2 or in that covered by Theorem 3.3. By inspecting $\lim_{n\to\infty} C_n^{(4)}/C_{n+1}^{(4)}$, which does not converge to the same value as $C_n^{(3)}/C_{n+1}^{(3)}$, we can exclude the case of Theorem 2.2. When looking at the next columns in the qd-table for $f_3(z)$ (we denote its entries by $e_m^{(n,3)}$ and $q_m^{(n,3)}$), one can see that column $e_1^{(n,3)}$ converges slowly to zero, while $e_3^{(n,3)}$ tends to zero rather fast. Hence 1 is a critical multiplicity index for f_3, while 3 is a true critical index and we are in the case of Theorem 6. When composing the Hadamard polynomials $P_3^{(n,3)}(z)$ for f_3 (which is similar to composing the polynomials $p_3^{(n)}$ for f from $q_4^{(n)}$, $q_5^{(n)}$ and $q_6^{(n)}$) we find the poles

$$z_4 = 1.414212000843331 + 1.41421251989044i,$$
$$z_5 = 1.414215123904284 + 1.414214604854061i,$$
$$z_6 = -1.526112569647409 \times 10^{-12} + 1.999999999998472i.$$

Using Theorem 3.3, we could also have eliminated z_4 found in (7) and computed the $C_4^{(n)}, q_m^{(n,4)}, e_m^{(n,4)}$. It remains to find the same pole once again, because its multiplicity is 2, and the last pole which is of the same modulus but only of multiplicity 1. These two remaining poles can be extracted using Theorem 3.2. In doing so however, one obtains numerically less reliable results. The reason for this only partly lies in the repeated use of the deflation technique which causes a certain error buildup. The main reason for the loss in significance when eliminating z_4, is the fact that the deflation technique is not only being used when a critical index is encountered, but also at a critical multiplicity index. Although the previous theorems prove the existence of and explain the concept of critical multiplicity index, it is recommended not to apply deflation in case of Theorems 3.3 and 3.6.

A wiser use of the new results consists in identifying the critical multiplicity indices and continuing the computation of q- and e-columns up to the point where a true critical index has been reached. The poles can then be extracted by means of Theorem 1.4, in which the Hadamard polynomials are computed that contain information on all z_i of equal modulus. The application of Theorems 3.3 and 3.6 however is useful if the condition of m-normality in Theorem 1.4 is not fulfilled while the weaker 1- or s-normality is, with s being a critical multiplicity index.

The deflation technique allows to extract information on the multiplicity of the poles as explained in Section 4. This kind of information is new and could not be detected using the theorems given in the existing literature on the qd-algorithm. When applied to this example, the Hadamard polynomial $P_3^{(n,3)}(z)$ can be written in the form

$$P_3^{(n,3)}(z) = (z - z_4)^2 (z - z_5)$$

before extracting the three poles.

6. Particular cases

The remaining condition on the function $f(z)$ in Theorems 2.2 and 3.3 is the 1-normality. If this condition is not satisfied, the following technique can solve the problem. When for a fixed period k the FPS coefficients of $f(z)$ are such that

$$c_{ki+j} = 0, \quad j = 1, \ldots, k-1, \; i = 0, 1, 2, \ldots,$$

then the function $g(z)$ defined by

$$d_i = c_{ki}, \qquad g(z) = \sum_{i=0}^{\infty} d_i z^i \tag{8}$$

is ultimately 1-normal and satisfies $g(z^k) = f(z)$. The poles of f are the kth roots of the poles of g.

An example of this situation occurs when dealing with

$$f(z) = \frac{1}{\cosh(z^3)} + \frac{1}{1-z}$$

From the computation of the coefficients c_n, we see that

$$\lim_{n\to\infty} q_1^{(n)} = 1 / \lim_{n\to\infty} (c_n / c_{n+1}) = 1.$$

After eliminating the pole $z_1 = 1$, the sequence $C_n^{(1)}$ is clearly not 1-normal but has periodically appearing zeroes. However, the compressed series

$$g(z) = \sum_{i=0}^{\infty} d_i z^i, \qquad d_{2i} = C^{(1)}_{3\times 2i}, \qquad d_{2i+1} = C^{(1)}_{3\times 2i+1}$$

is ultimately 1-normal. The qd-table computed with the coefficients $C_n^{(1)}$ (of which we denote the entries with $e_m^{(n,1)}$ and $q_m^{(n,1)}$) delivers

$$\lim_{n\to\infty} e_1^{(n,1)} \neq 0, \qquad \lim_{n\to\infty} e_2^{(n,1)} = 0.$$

Hence the Hadamard polynomials have to be composed with the $q_1^{(n,1)}$ and the $q_2^{(n,1)}$. The sequence of Hadamard polynomials (also denoted with the superscript $^{(n,1)}$ instead of with the superscript $^{(n)}$) converges to

$$\lim_{n\to\infty} P_2^{(n,1)}(z) = 0.40528 + z^2.$$

In this way we find for $f(z)$ the poles

$$z_2^3 = \frac{i}{\sqrt{0.40528}}, \qquad z_3^3 = \frac{-i}{\sqrt{0.40528}}.$$

Altogether these are really good approximations for the poles of $f(z)$ given by

$$z_1 = 1, \qquad z_{j+1} = \sqrt[3]{\frac{\pi}{2}} e^{i\frac{(2j-1)\pi}{6}}, \quad j = 1, \ldots, 6.$$

To conclude, we point out that a more complete and thorough understanding of the classical qd-algorithm may lead to new developments in techniques that build on qd, such as its multipoint version [1], its multivariate version [2–4] and the latter's connection with Padé approximation [5,6,8].

7. Proofs

Proof of Theorem 2.2. Let us first prove the necessary conditions. We know that the functions $f_{s+i}(z)$ are ultimately 1-normal and that

$$\begin{cases} \lim_{n\to\infty} \dfrac{C_n^{(s)}}{C_{n+1}^{(s)}} = \cdots = \lim_{n\to\infty} \dfrac{C_n^{(s+t-1)}}{C_{n+1}^{(s+t-1)}} = z^*, \\ \lim_{n\to\infty} \dfrac{C_n^{(s+t)}}{C_{n+1}^{(s+t)}} \neq z^*. \end{cases}$$

We also know that z_{s+1} is a pole isolated in modulus. We don't know how to compute z_{s+1} or its multiplicity. The function $f_s(z)$ which is analytic in $B(0, |z_{s+1}|)$ can be written as

$$f_s(z) = \frac{P(z)}{(z_{s+1} - z)^k} + g(z),$$

where $g(z)$ is analytic in $B(0, |z|)$, and $P(z)$ is a polynomial of degree $k - 1$. Let us denote the pole of $f(z)$ that is strictly larger in modulus than z_{s+1} by $\tilde{z}$ (since we don't know the multiplicity of z_{s+1} yet, we can't denote it by z_{s+t+1}). The FPS representation of $f_s(z)$ is given by

$$f_s(z) = \sum_{n=0}^{\infty} C_n^{(s)} z^n,$$

with

$$C_n^{(s)} = \gamma(n) z_{s+1}^{-(n+1)} + b_n, \quad |b_n| < \mu\rho^n, \quad \mu > 0,\ \rho \in \left] \frac{1}{|\tilde{z}|}, \frac{1}{|z_{s+1}|} \right[.$$

Hence

$$\lim_{n\to\infty} \frac{C_n^{(s)}}{C_{n+1}^{(s)}} = z_{s+1},$$

and consequently $z_{s+1} = z^*$. If $k = 1$ then z_{s+1} cannot be a pole of $f_{s+1}(z)$ and hence

$$\lim_{n\to\infty} \frac{C_n^{(s+1)}}{C_{n+1}^{(s+1)}} \neq z^*.$$

This inequality should be interpreted as indicated in the discussion following Theorem 2.2. In case $t = 1$ the proof is finished. If $t > 1$ then $k = 1$ results in a contradiction for $\lim_{n\to\infty} C_n^{(s)}/C_{n+1}^{(s)}$. Hence $k > 1$ when $t > 1$. When performing the same computations as above for $f_{s+1}(z)$, we can conclude that $z_{s+2} = z_{s+1} = z^*$. This can be repeated till we obtain $z_{s+t} = \cdots = z_{s+1} = z^*$ and $k \geqslant t$. Since there is no contradiction with

$$\lim_{n\to\infty} \frac{C_n^{(s+t)}}{C_{n+1}^{(s+t)}} \neq z^*,$$

the process terminates and we can conclude that $k = t$.

Let us now prove the sufficient conditions. With z_{s+1} being a pole isolated in modulus and of multiplicity t, we can write $f_s(z)$ as

$$f_s(z) = \frac{P(z)}{(z_{s+1} - z)^t} + g(z),$$

where the function $g(z)$ is analytic in $B(0, |z_{s+t+1}|)$, and $P(z)$ is a polynomial of degree t. Possibly $z_{s+t+1} = \infty$. Then for $i = 0, \ldots, t - 1$:

$$C_n^{(s+i)} = \gamma_i(n) z_{s+1}^{-(n+1)} + b_n, \quad |b_n| < \mu\rho^n, \quad \mu > 0,\ \rho \in \left] \frac{1}{|z_{s+t+1}|}, \frac{1}{|z_{s+1}|} \right[,$$

where the b_n are the FPS coefficients of $g(z)$, and $\gamma_i(n)$ is a polynomial of degree $t-i-1$ in n. Since

$$\lim_{n\to\infty}\left|z_{s+1}^{(n+1)}C_n^{(s+i)}\right| = +\infty,$$

the $C_n^{(s+i)}$ are nonzero from a certain n on, because

$$\exists N, \forall n > N\colon \left|C_n^{(s+i)}\right| > \left|z_{s+1}^{-(n+1)}\right|.$$

Since

$$\lim_{n\to\infty} b_n|z_{s+1}|^{n+1} = 0, \qquad \lim_{n\to\infty}\frac{\gamma(n)}{\gamma(n+1)} = 1,$$

we have

$$\lim_{n\to\infty}\frac{C_n^{(s)}}{C_{n+1}^{(s)}} = \lim_{n\to\infty} z_{s+1}\frac{\gamma(n)+b_n|z_{s+1}|^{n+1}}{\gamma(n+1)+b_{n+1}|z_{s+1}|^{n+2}} = z_{s+1}.$$

Consequently, using the first part of the proof, z_{s+1} is the pole smallest in modulus and of multiplicity $t-1$ of the function $f_{s+1}(z)$. This can be repeated t times in total. Finally we obtain that z_{s+1} is not a pole of $f_{s+t}(z)$, and hence that $f_{s+t}(z)$ is an analytic function in $B(0,|z_{s+t+1}|)$ with $|z_{s+t+1}| > |z_{s+1}|$. Then the convergence radius of its FPS expansion, which is given by

$$\lim_{n\to\infty}\left|\frac{C_n^{(s+t)}}{C_{n+1}^{(s+t)}}\right|$$

if this limit exists, is strictly larger than $|z_{s+1}|$ and so

$$\lim_{n\to\infty}\frac{C_n^{(s+t)}}{C_{n+1}^{(s+t)}} \neq z_{s+1}.$$

Again this inequality should be interpreted as indicated in the discussion following Theorem 2.2. This concludes the proof. □

Proof of Lemma 3.1. The determinant $\tilde{H}_k^{(n)}$ equals $\det(A_{kt}^{(n)}B_{tk})$ where the $k\times t$ and $t\times k$ matrices $A_{kt}^{(n)}$ and B_{tk} are respectively given by

$$A_{kt}^{(n)} = \begin{pmatrix} \gamma_1 e^{-i(n+1)\theta_1} & \gamma_2 e^{-i(n+1)\theta_2} & \dots & \gamma_t e^{-i(n+1)\theta_t} \\ \frac{1}{r}\gamma_1 e^{-i(n+2)\theta_1} & \frac{1}{r}\gamma_2 e^{-i(n+2)\theta_2} & \dots & \frac{1}{r}\gamma_t e^{-i(n+2)\theta_t} \\ \dots & \dots & \dots & \dots \\ \frac{1}{r^{k-1}}\gamma_1 e^{-i(n+k)\theta_1} & \frac{1}{r^{k-1}}\gamma_2 e^{-i(n+k)\theta_2} & \dots & \frac{1}{r^{k-1}}\gamma_t e^{-i(n+k)\theta_t} \end{pmatrix}, \tag{9}$$

$$B_{tk} = \begin{pmatrix} 1 & \frac{1}{r}e^{-i\theta_1} & \dots & \frac{1}{r^{k-1}}e^{-i(k-1)\theta_1} \\ 1 & \frac{1}{r}e^{-i\theta_2} & \dots & \frac{1}{r^{k-1}}e^{-i(k-1)\theta_2} \\ \dots & \dots & \dots & \dots \\ 1 & \frac{1}{r}e^{-i\theta_t} & \dots & \frac{1}{r^{k-1}}e^{-i(k-1)\theta_t} \end{pmatrix}. \tag{10}$$

This is easily verified since for all $\mu,\nu = 1,\dots,k$ the entries $\tilde{c}_{n+\mu+\nu-2}$ of $\tilde{H}_k^{(n)}$ satisfy:

$$\begin{aligned} \tilde{c}_{n+\mu+\nu-2} &= \frac{1}{r^{\mu+\nu-2}}\sum_{j=1}^{t}\gamma_j e^{-i(n+\mu+\nu-1)\theta_j} \\ &= \sum_{j=1}^{t}\left(\frac{\gamma_j e^{-i(n+\mu)\theta_j}}{r^{\mu-1}}\right)\left(\frac{e^{-i(\nu-1)\theta_j}}{r^{\nu-1}}\right) \\ &= \sum_{j=1}^{t} a_{\mu j}^{(n)} b_{j\nu}. \end{aligned}$$

It is clear that $\operatorname{rank}(A_{kt}^{(n)}) = k$ and $\operatorname{rank}(B_{tk}) = k$. Let $\tilde{\mathcal{H}}_k^{(n)}$ denote the matrix of which $\tilde{H}_k^{(n)}$ is the determinant. Then

$$\tilde{\mathcal{H}}_k^{(n)} = A_{kt}^{(n)} B_{tk},$$
$$\operatorname{rank}\big(\tilde{\mathcal{H}}_k^{(n)}\big) \leqslant \min\big(\operatorname{rank}\big(A_{kt}^{(n)}\big), \operatorname{rank}(B_{tk})\big) = k.$$

The determinant $\tilde{H}_k^{(n)}$ also equals

$$\tilde{H}_k^{(n)} = \left(\frac{1}{r}\cdots\frac{1}{r^{k-1}}\right)^2 \Delta_k^{(n)} = \frac{1}{r^{k(k-1)}}\Delta_k^{(n)}, \tag{11}$$

with

$$\Delta_k^{(n)} = \begin{vmatrix} \sum_{j=1}^t \gamma_j(n+1)e^{-i(n+1)\theta_j} & \dots & \sum_{j=1}^t \gamma_j(n+k)e^{-i(n+k)\theta_j} \\ \sum_{j=1}^t \gamma_j(n+2)e^{-i(n+2)\theta_j} & \dots & \sum_{j=1}^t \gamma_j(n+k+1)e^{-i(n+k+1)\theta_j} \\ \vdots & & \vdots \\ \sum_{j=1}^t \gamma_j(n+k)e^{-i(n+k)\theta_j} & \dots & \sum_{j=1}^t \gamma_j(n+2k-1)e^{-i(n+2k-1)\theta_j} \end{vmatrix}.$$

The expression $\gamma_i(n)$ is a polynomial in n of degree $m_i - 1$ with m_i the multiplicity of z_i. The determinant $\Delta_k^{(n)}$ equals (we again denote $\gamma_i(n)$ by γ_i)

$$\Delta_k^{(n)} = \sum_{j_1,\dots,j_k\in\{1,\dots,t\}} \begin{vmatrix} \gamma_{j_1}e^{-i(n+1)\theta_{j_1}} & \dots & \gamma_{j_k}e^{-i(n+k)\theta_{j_k}} \\ \gamma_{j_1}e^{-i(n+2)\theta_{j_1}} & \dots & \gamma_{j_k}e^{-i(n+k+1)\theta_{j_k}} \\ \vdots & & \vdots \\ \gamma_{j_1}e^{-i(n+k)\theta_{j_1}} & \dots & \gamma_{j_k}e^{-i(n+2k-1)\theta_{j_k}} \end{vmatrix},$$

where the $j_1,\dots,j_k$ are mutually distinct. It is sufficient to continue the proof for γ_i constant. Each vector $(j_1,\dots,j_k)$ represents an injection $\sigma:\{1,\dots,k\}\to\{1,\dots,t\}$. Let $\mathbb{A}_k$ be the set of all such injections. Then

$$\begin{aligned} \Delta_k^{(n)} &= \sum_{\sigma\in\mathbb{A}_k} (\gamma_{j_1}\cdots\gamma_{j_k})e^{-i(n+1)(\theta_{j_1}+\dots+\theta_{j_k})} \begin{vmatrix} 1 & \dots & e^{-i(k-1)\theta_{j_k}} \\ e^{-i\theta_{j_1}} & \dots & e^{-i(k)\theta_{j_k}} \\ \vdots & & \vdots \\ e^{-i(k-1)\theta_{j_1}} & \dots & e^{-i(2k-2)\theta_{j_k}} \end{vmatrix} \\ &= \sum_{\sigma\in\mathbb{A}_k} (\gamma_{j_1}\cdots\gamma_{j_k})e^{-i(n+1)(\theta_{j_1}+\dots+\theta_{j_k})}e^{-i(\theta_{j_2}+2\theta_{j_3}+\dots+(k-1)\theta_{j_k})} \begin{vmatrix} 1 & \dots & 1 \\ e^{-i\theta_{j_1}} & \dots & e^{-i\theta_{j_k}} \\ \vdots & & \vdots \\ e^{-i(k-1)\theta_{j_1}} & \dots & e^{-i(k-1)\theta_{j_k}} \end{vmatrix}. \end{aligned}$$

We denote $\gamma_\sigma = \gamma_{j_1}\cdots\gamma_{j_k}$ and $\theta_\sigma = \theta_{j_1}+\dots+\theta_{j_k}$, $\Theta_\sigma = \theta_{j_2}+2\theta_{j_3}+\dots+(k-1)\theta_{j_k}$ and

$$\Delta_\sigma = \begin{vmatrix} 1 & \dots & 1 \\ e^{-i\theta_{j_1}} & \dots & e^{-i\theta_{j_k}} \\ \vdots & & \vdots \\ e^{-i(k-1)\theta_{j_1}} & \dots & e^{-i(k-1)\theta_{j_k}} \end{vmatrix}.$$

Then

$$\Delta_k^{(n)} = \sum_{\sigma\in\mathbb{A}_k} \gamma_\sigma e^{-i(n+1)\theta_\sigma} e^{-i\Theta_\sigma}\Delta_\sigma. \tag{12}$$

Assume $\Delta_k^{(n)} = 0$ from some value for n on. Then a regular system of at least $\operatorname{card}(\mathbb{A}_k)$ linear homogeneous equations with unknowns $X_\sigma = \gamma_\sigma e^{-i\Theta_\sigma}\Delta_\sigma$ exists, implying $X_\sigma = 0$ and hence $\gamma_i = 0$. Consequently

$$\tilde{c}_n = \frac{1}{r^{n+1}}\big(\gamma_1 e^{-i(n+1)\theta_1} + \dots + \gamma_t e^{-i(n+1)\theta_t}\big) = 0,$$

from some value for n on, which is not the case. So $\Delta_k^{(n)} = 0$ at most for a periodic subsequence of n-values, the period being strictly less than $\operatorname{card}(\mathbb{A}_k)$. Therefore $\Delta_k^{(n)} \neq 0$ for an infinite number of n-values and the sequence $\{\Delta_k^{(n)}\}_n$ oscillates because of (12). We conclude that the sequence $\{\tilde{H}_k^{(n)}\}_n$ does not converge.

For $k = t$, the matrices $A_{tt}^{(n)}$ and B_{tt} are square and

$$\tilde{H}_k^{(n)} = \det\big(A_{tt}^{(n)} B_{tt}\big) = \det\big(A_{tt}^{(n)}\big)\det(B_{tt}) \neq 0. \qquad \square$$

Proof of Theorem 3.2. We have seen that the coefficient c_n in the FPS development of $f(z)$ is given by

$$c_n = \frac{1}{r^{n+1}}\big(\gamma_1 e^{-i(n+1)\theta_1} + \cdots + \gamma_t e^{-i(n+1)\theta_t} + O\big(h^{n+1}\big)\big),$$

where $h = \rho r$ with $\rho \in \,]\frac{1}{|z_{t+1}|}, \frac{1}{r}[$. Here $z_{t+1} = \infty$, if z_t is the last pole. In any case $|h| < 1$. Let $\mathcal{H}_k^{(n)}$ denote the matrix of which $H_k^{(n)}$ is the determinant. Then

$$\mathcal{H}_k^{(n)} = \frac{1}{r^{n+1}}\big(A_{kt}^{(n)} B_{tk} + O\big(h^{n+1}\big) R_k^{(n)}(h,r)\big),$$

where $A_{kt}^{(n)}$ and B_{tk} are given by (9) and (10) and where

$$R_k(h,r) = \left(\frac{O(h^{\mu+\nu-2})}{r^{\mu+\nu-2}}\right)_{1 \leqslant \mu,\nu \leqslant k}.$$

Consequently

$$H_k^{(n)} = \frac{1}{r^{k(n+1)}}\big(\det\big(A_{kt}^{(n)} B_{tk}\big) + O\big(h^{n+1}\big)\big) = \frac{1}{r^{k(n+1)}}\big(\tilde{H}_k^{(n)} + O\big(h^{n+1}\big)\big).$$

For $e_k^{(n)}$ given by

$$e_k^{(n)} = \frac{H_{k+1}^{(n)} H_{k-1}^{(n+1)}}{H_k^{(n+1)} H_k^{(n)}},$$

we obtain

$$\begin{aligned} e_k^{(n)} &= \frac{r^{k(n+2)+k(n+1)}(\tilde{H}_{k+1}^{(n)} + O(h^{n+1}))(\tilde{H}_{k-1}^{(n+1)} + O(h^{n+2}))}{r^{(k+1)(n+1)+(k-1)(n+2)}(\tilde{H}_k^{(n+1)} + O(h^{n+2}))(\tilde{H}_k^{(n)} + O(h^{n+1}))} \\ &= r\tilde{e}_k^{(n)} + O\big(h^{n+1}\big), \end{aligned}$$

with

$$\tilde{e}_k^{(n)} = \frac{\tilde{H}_{k+1}^{(n)} \tilde{H}_{k-1}^{(n+1)}}{\tilde{H}_k^{(n+1)} \tilde{H}_k^{(n)}}.$$

According to Lemma 3.1 the sequence $\{\tilde{H}_k^{(n)}\}_n$ oscillates. From (11) we find that

$$\tilde{e}_k^{(n)} = \frac{1}{r^2}\frac{\Delta_{k+1}^{(n)} \Delta_{k-1}^{(n+1)}}{\Delta_k^{(n+1)} \Delta_k^{(n)}},$$

with $\Delta_k^{(n)}$ given by (12), also oscillates and hence, for $1 \leqslant k \leqslant t$, the sequence $\{e_k^{(n)}\}_n$ does not converge. $\square$

Proof of Theorem 3.3. The FPS coefficients c_n in the series development of $f(z)$ are given by

$$c_n = \sum_{j=1}^{t} \gamma_j(n) z_j^{-(n+1)} + b_n,$$

with $\gamma_j(n)$ a polynomial of degree $m_j - 1$ in n, where m_j is the multiplicity of z_j, and with $|b_n| < \mu\rho^n$, where $\mu > 0$ and $\rho \in \,]\frac{1}{|z_{t+1}|}, \frac{1}{|z_t|}[$. Again $z_{t+1} = \infty$ if z_t is the last pole. With z_u being the pole of maximal multiplicity, c_n/c_{n+1} can be written as

$$\begin{aligned} \frac{c_n}{c_{n+1}} &= r\frac{\gamma_1(n)e^{-i(n+1)\theta_1} + \cdots + \gamma_t(n)e^{-i(n+1)\theta_t} + b_n r^{n+1}}{\gamma_1(n+1)e^{-i(n+2)\theta_1} + \cdots + \gamma_t(n+1)e^{-i(n+2)\theta_t} + b_{n+1} r^{n+2}} \\ &= r\frac{\gamma_u(n)}{\gamma_u(n+1)} e^{i\theta_u} \frac{\frac{\gamma_1(n)}{\gamma_u(n)} e^{-i(n+1)(\theta_1-\theta_u)} + \cdots + 1 + \cdots + \frac{\gamma_t(n)}{\gamma_u(n)} e^{-i(n+1)(\theta_t-\theta_u)} + \frac{b_n}{\gamma_u(n)} r^{n+1}}{\frac{\gamma_1(n+1)}{\gamma_u(n+1)} e^{-i(n+2)(\theta_1-\theta_u)} + \cdots + 1 + \cdots + \frac{\gamma_t(n+1)}{\gamma_u(n+1)} e^{-i(n+2)(\theta_t-\theta_u)} + \frac{b_{n+1}}{\gamma_u(n+1)} r^{n+2}}. \end{aligned}$$

Now for all $j \neq u$:

$$\lim_{n\to\infty} \frac{\gamma_j(n)}{\gamma_u(n)} e^{-i(n+1)(\theta_j-\theta_u)} = 0,$$
$$\lim_{n\to\infty} \frac{\gamma_u(n)}{\gamma_u(n+1)} = 1,$$
$$\lim_{n\to\infty} \frac{b_n r^n}{\gamma_u(n)} = 0. \qquad \square$$

Proof of Lemma 3.5. We first prove part (a).

With s poles of maximal multiplicity among the t equimodular poles, we have

$$c_n = \frac{1}{r^{n+1}} \sum_{j=1}^{t} \gamma_j(n) e^{-i(n+1)\theta_j} + b_n,$$

with $\partial\gamma_1(n) = \cdots = \partial\gamma_s(n) = m_s - 1$ and $\partial\gamma_j(n) = m_j - 1 < m_s - 1$ for $j = s+1, \ldots, t$. Let α_k denote the coefficient of n^{m_s-1} in $\gamma_k(n)$, and this for $k = 1, \ldots, s$. Then c_n can be rewritten as

$$c_n = \frac{n^{m_s-1}}{r^{n+1}} \left(\underbrace{\sum_{k=1}^{s} \left(\alpha_k + \frac{\gamma_k(n) - \alpha_k n^{m_s-1}}{n^{m_s-1}} \right) e^{-i(n+1)\theta_k}}_{\mathrm{I}} + \underbrace{\sum_{k=s+1}^{t} \frac{\gamma_k(n)}{n^{m_s-1}} e^{-i(n+1)\theta_k}}_{\mathrm{II}} + \frac{r^{n+1} b_n}{n^{m_s-1}} \right),$$

with $\partial(\gamma_k(n) - \alpha_k n^{m_s-1}) \leqslant m_s - 2$ for $k = 1, \ldots, s$, and with $\partial\gamma_k(n) \leqslant m_s - 2$ for $k = s+1, \ldots, t$. If we define

$$\gamma := \max\Bigl(\max_{1\leqslant k\leqslant s} \partial\bigl(\gamma_k(n) - \alpha_k n^{m_s-1}\bigr), \max_{s+1\leqslant k\leqslant t} \partial\gamma_k(n) \Bigr) = m_s - \kappa, \quad \kappa \geqslant 2,$$
$$\tau := m_s - 1 - \gamma,$$

then $\gamma \leqslant m_s - 2$ and $\tau \geqslant 1$. We conclude that

$$c_n = \frac{n^{m_s-1}}{r^{n+1}} \left(\alpha_1 e^{-i(n+1)\theta_1} + \cdots + \alpha_s e^{-i(n+1)\theta_s} + O\left(\frac{1}{n^\tau}\right) \right).$$

For part (b) the proof is as follows.

We have t equimodular poles of modulus r say, and among these s of them have the same maximal multiplicity, for simplicity m_s ($m_s > m_{s+1}$).

From expression I, we extract $\sum_{j=1}^{s} \alpha_j e^{-i(n+1)\theta_j}$ and from expression II $\frac{\alpha_{s+1}}{n^{m_s-m_{s+1}}} e^{-i(n+1)\theta_{s+1}}$, the rest of I and II being of order $O(1/n^\tau)$.

Then we can write for c_n, with $b = m_s - m_{s+1}$,

$$c_n = \frac{n^{m_s-1}}{r^{n+1}} \left(\sum_{j=1}^{s} \alpha_j e^{-i(n+1)\theta_j} + \frac{\alpha_{s+1}}{n^b} e^{-i(n+1)\theta_{s+1}} + O\left(\frac{1}{n^\tau}\right) \right),$$

and for $c_{n+\mu+\nu-2}$ in $H_{s+1}^{(n)}$,

$$c_{n+\mu+\nu-2} = \frac{n^{m_s-1}}{r^{n+1}} \left(\frac{n+\mu+\nu-2}{n} \right)^{m_s-1} \chi_{\mu\nu}^{(n)},$$
$$\chi_{\mu\nu}^{(n)} = \sum_{j=1}^{s} \alpha_j e^{-i(n+1)\theta_j} + (n+\mu+\nu-2)^{-b} \frac{\alpha_{s+1}}{r^{\mu+\nu-2}} e^{-i(n+\mu+\nu-1)\theta_{s+1}} + O\left(\frac{1}{(n+\mu+\nu-2)^\tau} \right) \frac{1}{r^{\mu+\nu-2}}.$$

We have

$$\left(\frac{n+\mu+\nu-2}{n} \right)^{m_s-1} = 1 + (m_s-1)(\mu+\nu-2) O\left(\frac{1}{n}\right),$$
$$(n+\mu+\nu-2)^{-b} = n^{-b} \left[1 - b(\mu+\nu-2) O\left(\frac{1}{n}\right) \right].$$

Let us denote

$$a_{\mu j}^{(n)} := \frac{1}{r^{\mu-1}} \alpha_j e^{-i(n+\mu)\theta_j}, \quad j = 1, \ldots, s,$$
$$a_{\mu,s+1}^{(n)} := \frac{1}{r^{\mu-1}} n^{1-\kappa} \alpha_{s+1} e^{-i(n+\mu)\theta_{s+1}},$$
$$b_{j\nu} := \frac{1}{r^{\nu-1}} e^{-i(\nu-1)\theta_j}, \quad j = 1, \ldots, s,$$
$$b_{s+1,\nu} := \frac{1}{r^{\nu}} e^{-i(\nu-1)\theta_{s+1}},$$
$$A^{(n)} := \left(a_{\mu\nu}^{(n)}\right)_{1 \leqslant \mu,\nu \leqslant s+1},$$
$$B := (b_{\mu\nu})_{1 \leqslant \mu,\nu \leqslant s+1}.$$

Then $c_{n+\mu+\nu-2}$ can be rewritten as

$$c_{n+\mu+\nu-2} = \frac{n^{m_s-1}}{r^{n+1}} \left(\sum_{j=1}^{s+1} a_{\mu j}^{(n)} b_{j\nu} + O\left(\frac{1}{n}\right) R_{\mu\nu}^{(n)} \right),$$

with

$$R_{\mu\nu}^{(n)} = (\mu+\nu-2)\left[-bn^{-b} \frac{\alpha_{s+1}}{r^{\mu+\nu-2}} e^{-i(n+\mu+\nu-1)\theta_{s+1}} + O\left(\frac{n}{(n+\mu+\nu-2)^{\tau}}\right) \frac{1}{r^{\mu+\nu-2}} + (m_s-1)\chi_{\mu\nu}^{(n)} \right].$$

Let us have a closer look at the rest term $R_{\mu\nu}^{(n)}$:

$$\lim_{n\to\infty} -bn^{-b} \frac{\alpha_{s+1}}{r^{\mu+\nu-2}} e^{-i(n+\mu+\nu-1)\theta_{s+1}} = 0,$$
$$O\left(\frac{n}{(n+\mu+\nu-2)^{\tau}}\right) \frac{1}{r^{\mu+\nu-2}} = O\left(\frac{1}{n^{\tau-1}}\right) \frac{1}{r^{\mu+\nu-2}}, \quad \tau - 1 \geqslant 0,$$
$$\lim_{n\to\infty} (n+\mu+\nu-2)^{-b} \frac{\alpha_{s+1}}{r^{\mu+\nu-2}} e^{-i(n+\mu+\nu-1)\theta_{s+1}} = 0.$$

Hence for a constant $D(\mu,\nu,r)$ and an integer N,

$$\max_{n>N} \left| R_{\mu\nu}^{(n)} \right| < D(\mu,\nu,r).$$

With $R_{s+1}^{(n)}$ defined by $R_{s+1}^{(n)} = (R_{\mu\nu}^{(n)})_{1 \leqslant \mu,\nu \leqslant s+1}$, we can write

$$\mathcal{H}_{s+1}^{(n)} = \frac{n^{m_s-1}}{r^{n+1}} \left(A^{(n)} B + O\left(\frac{1}{n}\right) R_{s+1}^{(n)} \right),$$

and consequently

$$H_{s+1}^{(n)} = \frac{n^{(s+1)(m-1)}}{r^{(s+1)(n+1)}} \left(\det\left(A^{(n)}\right) \det(B) + O\left(\frac{1}{n}\right) \right).$$

Now

$$\det\left(A^{(n)}\right) = \alpha_1 \cdots \alpha_{s+1} \frac{1}{r^{s(s+1)/2}} \frac{1}{n^b} e^{-i(n+1)(\theta_1+\cdots+\theta_s+\theta_{s+1})} V,$$

with

$$V = \begin{vmatrix} e^{-i\theta_1} & \ldots & e^{-i\theta_s} & e^{-i\theta_{s+1}} \\ e^{-i2\theta_1} & \ldots & e^{-i2\theta_s} & e^{-i2\theta_{s+1}} \\ \vdots & & \vdots & \vdots \\ e^{-i(s+1)\theta_1} & \ldots & e^{-i(s+1)\theta_s} & e^{-i(s+1)\theta_{s+1}} \end{vmatrix}.$$

So

$$\det(A^{(n)}) = \alpha_1 \cdots \alpha_{s+1} \frac{1}{r^{s(s+1)/2}} \frac{1}{n^b} e^{-i(n+1)\theta} V,$$
$$\theta = \theta_1 + \cdots + \theta_s + \theta_{s+1}.$$

Since $\det(B)$ is independent of n, we obtain in this case

$$H_{s+1}^{(n)} = \frac{n^{(s+1)(m_s-1)}}{r^{(s+1)(n+1)}}\left(\frac{A}{n^b}e^{-i(n+1)\theta} + O\left(\frac{1}{n}\right)\right), \qquad A = \frac{\alpha_1 \cdots \alpha_{s+1} V}{r^{s(s+1)/2}}. \qquad \square$$

Proof of Theorem 3.6. We first focus on the e-values. Using Lemma 3.5(a), the $c_{n+\mu+\nu-2}$ are given by

$$\begin{aligned} c_{n+\mu+\nu-2} &= \frac{(n+\mu+\nu-2)^{m_s-1}}{r^{n+\mu+\nu-1}}\left(\sum_{j=1}^{s}\alpha_j e^{-i(n+\mu+\nu-1)\theta_j} + O\left(\frac{1}{(n+\mu+\nu-1)^\tau}\right)\right) \\ &= \frac{n^{m_s-1}}{r^{n+1}}\left(\frac{n+\mu+\nu-2}{n}\right)^{m_s-1}\left(\sum_{j=1}^{s}\alpha_j e^{-i(n+\mu+\nu-1)\theta_j} + O\left(\frac{1}{(n+\mu+\nu-1)^\tau}\right)\right) \\ &= \frac{n^{m_s-1}}{r^{n+1}}\left(\sum_{j=1}^{s}\alpha_j e^{-i(n+\mu+\nu-1)\theta_j} + O\left(\frac{1}{n}\right)R_{\mu\nu}^{(n)}\right), \end{aligned}$$

with

$$R_{\mu\nu}^{(n)} = \frac{1}{r^{\mu+\nu-2}}O\left(\frac{n}{(n+\mu+\nu-1)^\tau}\right) + (m_s-1)(\mu+\nu-2)\left(\sum_{j=1}^{s}\alpha_j e^{-i(n+\mu+\nu-1)\theta_j} + O\left(\frac{1}{(n+\mu+\nu-1)^\tau}\right)\right)$$

and

$$\max_{n>N}\left|R_{\mu\nu}^{(n)}\right| < D(\mu,\nu,r),$$

for a constant $D(\mu,\nu,r)$ and an integer N. If we define the matrix $R_k^{(n)} = (R_{\mu\nu}^{(n)})_{1\leqslant\mu,\nu\leqslant k}$, then the $k\times k$ Hankel matrix $\mathcal{H}_k^{(n)}$ can be written as

$$\mathcal{H}_k^{(n)} = \frac{n^{m_s-1}}{r^{n+1}}\left(A_{ks}^{(n)}B_{sk} + O\left(\frac{1}{n}\right)R_k^{(n)}\right),$$

where the matrices $A_{ks}^{(n)}$ and B_{sk} are defined in the proof of Lemma 3.1. From this we can conclude that

$$H_k^{(n)} = \frac{n^{k(m_s-1)}}{r^{k(n+1)}}\left(\tilde{H}_k^{(n)} + O\left(\frac{1}{n}\right)\right), \quad \tau \geqslant 1,\ k=1,\ldots,s,$$

where the determinant $\tilde{H}_k^{(n)}$ is also given in Lemma 3.1. Hence

$$e_k^{(n)} = \left(\frac{n}{n+1}\right)^{m_s-1} r\frac{\tilde{H}_{k+1}^{(n)}\tilde{H}_{k-1}^{(n+1)} + O(\frac{1}{n})}{\tilde{H}_k^{(n)}\tilde{H}_k^{(n+1)} + O(\frac{1}{n})}, \quad k=1,\ldots,s-1,$$

with $\{\tilde{H}_k^{(n)}\}_n$ oscillating, as detailed in Lemma 3.1. Hence we can conclude as in the proof of Theorem 3.2 that for $1\leqslant k\leqslant t$, the sequence $\{e_k^{(n)}\}_n$ does not converge.

Note that for $m_s = 1$ the expression above resembles the one obtained for $e_k^{(n)}$ in Theorem 3.2. To show that $\lim_{n\to\infty} e_s^{(n)} = 0$, we use the result obtained in Lemma 3.5 for the determinant $H_{s+1}^{(n)}$:

$$e_s^{(n)} = \frac{H_{s+1}^{(n)}H_{s-1}^{(n+1)}}{H_s^{(n)}H_s^{(n+1)}} = \rho_n\left(\frac{A}{n^b}e^{-i(n+1)\theta} + O\left(\frac{1}{n}\right)\right)\frac{(\tilde{H}_{s-1}^{(n+1)} + O(\frac{1}{(n+1)}))}{(\tilde{H}_s^{(n)} + O(\frac{1}{n}))(\tilde{H}_s^{(n+1)} + O(\frac{1}{(n+1)}))},$$

with A and b given in Lemma 3.5, and with

$$\rho_n = \frac{n^{(s+1)(m_s-1)}(n+1)^{(s-1)(m_s-1)}}{r^{(s+1)(n+1)+(s-1)(n+2)}}\frac{r^{s(n+1)+s(n+2)}}{n^{s(m_s-1)}(n+1)^{s(m_s-1)}} = r\left(\frac{n}{n+1}\right)^{m_s-1}.$$

Let us now turn to the sequence of Hadamard polynomials. We introduce the notation

$$\tilde{c}_{n+\mu+\nu-2} = \sum_{j=1}^{s}\frac{\alpha_j}{r^{\mu+\nu-2}}e^{-i(n+\mu+\nu-1)\theta_j},$$

and construct a Hankel determinant $\tilde{H}_s^{(n)}$ of size s with these $\tilde{c}_n$. The Hadamard polynomial associated with these $\tilde{c}_n$ is

$$\tilde{P}_s^{(n)}(z) = \frac{\tilde{H}_s^{(n)}(z)}{\tilde{H}_s^{(n)}},$$

where $\tilde{H}_s^{(n)}(z)$ is defined accordingly. The polynomial $\tilde{P}_s^{(n)}(z)$ is monic of degree s. Similar to [17, p. 626] we can check that

$$\tilde{P}_s^{(n)}(z) = \prod_{j=1}^{s}\left(z - e^{-i\theta_j}/r\right).$$

Let $\mathcal{H}_s^{(n)}(z)$ and $\tilde{\mathcal{H}}_s^{(n)}(z)$ respectively denote the matrices of which $H_s^{(n)}(z)$ and $\tilde{H}_s^{(n)}(z)$ are the determinants. Then

$$\mathcal{H}_s^{(n)}(z) = \frac{n^{m_s-1}}{r^{n+1}}\left(\tilde{\mathcal{H}}_s^{(n)}(z) + O\left(\frac{1}{n}\right)\tilde{R}_{s+1}^{(n)}\right),$$

with $\tilde{R}_{s+1}^{(n)}$ given by

$$\tilde{R}_{\mu\nu}^{(n)} = R_{\mu\nu}^{(n)}, \quad 1 \leqslant \mu \leqslant s+1,\ 1 \leqslant \nu \leqslant s,$$
$$\tilde{R}_{\mu,s+1}^{(n)} = 0,$$

and consequently

$$H_s^{(n)}(z) = \frac{n^{s(m_s-1)}}{r^{s(n+1)}}\left(\tilde{H}_s^{(n)}(z) + O\left(\frac{1}{n}\right)\pi^{(n)}(z)\right),$$

with $\pi^{(n)}(z)$ given by

$$\pi^{(n)}(z) = \sum_{i=0}^{s}\pi_i^{(n)}z^i,$$
$$\max_{n>N}\left|\pi_i^{(n)}\right| < p_i,$$

where N is fixed. So if s is a critical multiplicity index, then

$$H_s^{(n)} = \frac{n^{s(m_s-1)}}{r^{s(n+1)}}\left(\tilde{H}_s^{(n)} + O\left(\frac{1}{n}\right)\right),$$

and for the Hadamard polynomial

$$P_s^{(n)}(z) = \frac{\tilde{H}_s^{(n)}(z) + O(\frac{1}{n})\pi^{(n)}(z)}{\tilde{H}_s^{(n)} + O(\frac{1}{n})}.$$

Hence

$$\left|P_s^{(n)}(z) - \prod_{j=1}^{s}\left(z - e^{-i\theta_j}/r\right)\right| = O\left(\frac{1}{n}\right)\frac{\pi^{(n)}(z) - \prod_{j=1}^{s}(z - e^{-i\theta_j}/r)}{\tilde{H}_s^{(n)} + O(\frac{1}{n})},$$

with, for N fixed,

$$\max_{n>N}\left|\frac{\pi^{(n)}(z) - \prod_{j=1}^{s}(z - e^{-i\theta_j}/r)}{\tilde{H}_s^{(n)} + O(\frac{1}{n})}\right| = \left|R(z)\right|,$$

where $R(z)$ is a polynomial of at most degree s with coefficients independent of n. On every compact set $K \subset \mathbb{C}$ we then have

$$\max_{z\in K}\left|P_s^{(n)}(z) - \prod_{j=1}^{s}\left(z - e^{-i\theta_j}/r\right)\right| = O\left(\frac{1}{n}\right)\max_{z\in K}\left|C(z)\right|,$$

which converges to zero for n tending to infinity. □

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