Faculty of Science

Department of Physics

Classical relativistic Coulomb-type problems: stable and unstable orbits

Klassieke relativistische Coulomb-type problemen: stabiele en onstabiele banen

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1 Voorwoord

Deze thesis vormt de afsluiter van mijn masteropleiding in de Fysica aan de Universiteit Antwerpen. Tijdens de voorbije maanden heb ik de kans gekregen om mijzelf te verdiepen in het bestuderen van banen voor relativistisch en niet-relativistisch deeltjes in verschillende Coulomb-type potentialen. Graag wil ik van deze gelegenheid gebruik maken om een aantal mensen te bedanken die het schrijven van deze masterthesis mogelijk hebben gemaakt. In de eerste plaats gaat mijn dank uit naar mijn promotor professor François Peeters en co-promotor Ben Van Duppen, die mij de mogenlijkheid gegeven hebben om mezelf te verdiepen in dit interessant onderwerp. Ik dank hen hierbij voor de tijd die ze beschikbaar gesteld hebben om mijn vragen te beantwoorden, maar ook voor hun steun en geduld. Een speciaal woordje dank gaat naar mijn familie en vrienden waarvan ik wijze raad heb mogen ontvangen. Tot slot bedank ik nog mijn vriendin Sara en mijn neef Franco die hun best hebben gedaan om mijn thesis na te lezen en mij moreel hebben ondersteund tijdens het schrijfproces.

2 Abstract (Nederlands)

In deze masterthesis worden de verschillende mogelijke banen voor een relativistisch en niet-relativistisch deeltje in Kepler/Coulomb-type potentialen bestudeerd. Er worden bewegingsvergelijkingen opgesteld en opgelost voor zulke deeltjes in verschillende potentialen. De bewegingsvergelijkingen worden zowel analytische als numeriek opgelost. Er wordt een stabiliteitsanalyse gemaakt om de voorwaarde voor stabiele gebonden toestanden te vinden en hierbij worden de mogelijke banen voor beide deeltjes gecatalogeerd. Als eindresultaat voor ieder probleem wordt er een fasediagram gegeven dat een overzicht geeft voor de verschillende mogelijke banen voor beide deeltjes.

Bij het introduceren van de relativistische natuur van het deeltje in een Kepler/Coulomb potentiaal, worden er onstabiele banen gevonden waarbij het deeltje spiraalt en invalt in de oorspong van de Kepler/Coulomb potentiaal, terwijl zijn energie en angulaire momentum worden behouden. Dit fenomeen wordt "*atomic collapse*" genoemd en wordt meestal doormiddel van kwantum-relativistische mechanica behandeld. In de inleiding worden er sommige verbanden gelegd tussen kwantum-relativistisch en klassiek "*atomic collapse*" en er wordt uitgelegd hoe "*atomic collapse*" experimenteel kan geobserveerd worden. Verder wordt er gekeken hoe de banen van beide deeltjes worden beïnvloed door de Kepler/Coulomb potentiaal aan te passen $V \sim 1/r^n$. Door de macht $n = 2$ te nemen, wordt de potentiaal veel sterker en meer singulier dan de Kepler/Coulomb potentiaal. Hierbij wordt er geconcludeerd dat een niet-relativistisch deeltje bij een kritiek angulair momentum onstabiele banen vertoont zoals atomic collapse, terwijl voor angulair moment boven deze kritische waarde we stabiele banen zoals parabool/hyperbool baan vinden. Vanuit de stabiliteitsanalyse wordt er gevonden dat hier geen stabiele gebonden toestanden mogelijk zijn en het kritieke angulaire momentum fungeert als limiet waarbij onder deze waarde het deeltje "*atomic collapse*" ondervindt en erboven verstrooiingstoestanden. De situatie voor een relativistisch deeltje is anders. Vanuit de stabiliteitsanalyse worden er stabiele gebonden toestanden gevonden die zich manifesteren als stabiele circulaire banen. Hierbij wordt geobserveerd dat het deeltje afhankelijk van zijn begin positie zich bevindt in stabiele banen zoals circulaire banen, paraboolachtige banen waarbij in de omgeving van de oorsprong van de potentiaal een lusvormige gedrag vertoont en vervolgens ontsnapt. Als onstabiele banen worden er "*atomic collapse*" geobserveerd, waarbij het deeltje ofwel plotseling op de kern valt ofwel eerst een aantal lussen uitvoert rond de kern en uiteindelijk op de kern valt. Nog een belangrijk resultaat is dat voor $n = 2$ het deeltje zich gedraagt als een niet-relativistisch deeltje in een Coulomb potentiaal. In die zin dat er een minimaal angulair momentum nodig is om stabiele circulaire banen te bekomen. Nu is spiraliseren in het centrum van de potentiaal fysisch niet mogelijk omdat het deeltje oneindig energie zou moeten hebben en er wordt gevonden dat spiraliseren op de kern in dit geval gepaard gaat met een verhoging van de snelheid tot de licht snelheid $v \to c$ terwijl in het niet-relativistische geval $v \to \infty$.

Nog een onderdeel van mijn thesis is het bekijken hoe de banen van deze deeltjes worden beïnvloed als de singulariteit in de potentialen weg gewerkt is. Hierbij wordt er in het algemeen dezelfde conclusie getrokken en dat is dat het deeltje kinetische energie wint wanneer de singulariteit van de potentiaal geregulariseerd wordt. Dit wilt zeggen dat de potentiaalterm in de energievergelijking kleiner wordt en dit leidt tot een verhoging in kinetische energie dewelke ervoor zorgt dat het deeltje verstrooiingsbanen zal volgen. Als laatste onderdeel van mijn thesis worden er de banen van relativistiche en niet-relativistische deeltjes bekeken in geval van de aanwezigheid van twee potentialen. Dit systeem is equivalent met een molecule. Hierbij worden er steeds dezelfde patronen

teruggevonden. Als het deeltje geplaatst wordt tussen de twee potentialen, zal het deeltje voor eeuwig bewegen rondom beide potentialen en blijft dus gebonden met deze. In geval van verstrooingsbanen, worden deze vervormd door de aanwezigheid van de twee potentialen en vervolgens ontsnapt het deeltje.

3 Abstract (English)

In this master thesis we investigate the orbits of a relativistic and non-relativistic particle in different Kepler/Coulomb-type potentials. If we consider a non-relativistic particle in a Kepler or Coulomb potential, we know that the orbits can be circular, elliptic, parabolic and hyperbolic Kepler orbits. When introducing the relativistic nature of the particle we also get unstable orbits like, a collapse into the origin of the central force while energy and angular momentum are conserved quantities. This is possible because of the γ-factor which expresses how the mass increases with velocity.

By making a stability analysis we conclude that a non-relativistic particle doesn't perform stable bounded orbits in potential $V \sim 1/r^2$ but a relativistic particle does. By studying a relativistic particle in $V \sim 1/r^2$ we conclude that this behaves as a non-relativistic particle in $V \sim 1/r$ potential. This means that this cannot perform a spiraling motion into the origin of the potential while its energy is conserved.

By removing the singularity in $r = 0$ in each case $V \sim 1/r$ and $V \sim 1/r^2$ for both particles, we find that both particles gains some kinetic energy which results in scattering states.

In the presence of the two-charge potential we find for each case the same patterns. When we place the particle between the two potentials the particle performs an infinite circular motion around the two potentials. When we have scattering states the trajectory deforms and the particle escapes from the potentials.

For a non-relativistic particle in case of the two-charge potential we find orbit with a precessing elliptic motion with one of the two charges as alternately focal point. Also are found for a non-relativistic particle scattering states where the orbit is being deformed because of the two-charge potential.

These last two behaviors are also found for a relativistic particle in $V \sim 1/r$ potential. The only difference in observed trajectories is that in the relativistic case the unstable orbits (those who causes atomic collapse) become bounded orbits in the presence of the two-charge potential.

Contents

4 Introduction

In this master thesis we investigate the orbits of a relativistic and non-relativistic particle in different Kepler/Coulomb-type potentials. Because of the relativistic nature of a particle, we know from relativity that the mass depends on the velocity of the particle. This is mathematically expressed as the γ-factor which expresses how the mass increases with velocity. If we consider a non-relativistic particle in a Kepler or Coulomb potential, we know that the orbits can be circular, elliptic, parabolic and hyperbolic Kepler orbits. We know that for a spherically symmetric system the total energy is a conserved quantity and is expressed as kinetic plus potential energy. So, depending on the position of the particle on the orbit, there is a teamwork between kinetic and potential energy in such a way that the total energy is conserved. When introducing the relativistic nature of the particle we also get unstable orbits like, a collapse into the origin of the central force while energy and angular momentum are conserved quantities. This is possible because the γ-factor in case of a decrease in radius (increase in negative sense of potential energy) will provide for an increase in kinetic energy such that both energy components compensate each other keeping the total energy constant. In order to describe the system we made a few important assumptions: [5]

1. The particles are treated in a classical manner, i.e. we don't take into account quantum mechanical effects

2. We study the behavior of these particles under the influence of spherically symmetric central forces (central potential $F(r) = -\nabla V(r)$) which lead to a system where energy and angular momentum are conserved quantities

3. Because we make use of central potentials we have to mention that the particles are not charged, this means that in time the particles cannot radiate or absorb energy. In this way we get a conserved system with conserved total energy

4.1 Classical and Bohr hydrogen atom

We can compare the above described particles with a charged electron which experiences a Coulomb potential orbiting around the nucleus of a hydrogen atom. From a classical point of view a charged electron follows a bounded trajectory around the nucleus. Because the electron is continuously attracted by the attractive force from a classical point of view we would expect that in time the electron will gradually lose some energy because of radiation until its kinetic energy is not large enough to compensate the attractive force, at this point we would expect that the atom collapses into the origin of the nucleus. But this never happens because the quantum mechanical nature of electrons, atoms ... tells us that the energy is quantified and depending on the energy the electron is trapped in an orbit around the nucleus. This means that the possible orbits for an orbiting electron are not continuous but discrete. This is Bohr's quantum theory where the atomic stability is a consequence of quantum zero-motion (motion of an electron in lowest possible energy level) of an electron which prevents it to fall in the nucleus. Because the kinetic energy in this case scales with p^2 and we know that the linear momentum scales with one over the length scale of the electron wave function we have the total energy of the electron as $E_e = \frac{\hbar^2}{2m_e r^2} - \frac{Ze^2}{r^2}$. Thus for small radius the kinetic energy dominates in the energy equation but for large radius the Coulomb term plays the important role. For a bound orbit at some minimum radius both energy components will balance each other. This minimum radius corresponds to the lowest energy which is given (if $Z = 1$) by the Bohr radius $r_B = \frac{\hbar^2}{Ze^2 m_e}$, with $\hbar = 1.0545718 \cdot 10^{-34} m^2$kg/s Planck's constant, m_e the electron mass, e the electron charge and Z nuclear charge. As conclusion we find: [3][9]

1. Classically, a charged particle cannot perform a stable bound orbit around the origin of a central force because it will fall into it while losing energy due to radiation of electro-magnetic energy.

2. Quantum mechanically, because the energy of the charged particle is quantized, each energy level corresponds to a bounded orbit around the origin of the central force. Here the particle can emit or absorb only quantified energy which will provide for a jump of the particle into another orbit. Because the energy is quantized there is a lowest possible energy level which corresponds with the closest orbit to the nucleus with radius r_B.

4.2 Quantum-relativistic hydrogen atom

By introducing relativity the situation becomes a bit different from section 4.1. Because the kinetic energy in this case scales with cp with c the speed of light and because the linear momentum scales with the inverse of the length scale of the electron wave function we have that $p \sim 1/r$ which leads to $cp \sim c\hbar/r$. Consequently the total energy of the relativistic electron becomes

$$E_{e_{rel}} = \sqrt{(cp)^2 + (mc^2)^2} - \frac{Ze^2}{r} = \sqrt{\left(\frac{c\hbar}{r}\right)^2 + (mc^2)^2} - \frac{Ze^2}{r}. \tag{1}$$

If we minimize the energy of the relativistic electron with respect to the radius $r = r_0$ we find:

$$\left.\frac{\partial E_{e_{rel}}}{\partial r}\right|_{r=r_0} = 0, \tag{2}$$

$$\left.\frac{\partial E_{e_{rel}}}{\partial r}\right|_{r=r_0} = 0 = -\frac{(c\hbar)^2}{r_0^3\sqrt{\left(\frac{c\hbar}{r_0}\right)^2 + (mc^2)^2}} + \frac{Ze^2}{r_0^2}, \tag{3}$$

$$\Rightarrow \sqrt{1 + \left(\frac{mc}{\hbar}\right)^2 r_0^2} = \frac{c\hbar}{Ze^2}. \tag{4}$$

Because the term $(mc/\hbar)^2 r_0^2$ contains real quantities we have that the left hand side of Eq. (4) is larger than 1 consequently we get stable solutions only in the case $Z < \hbar c/e^2 \approx \alpha^{-1} = 137$. In other words the nuclear charge should be less than a certain critical value in order to find stable solutions. The nucleus with a charge larger than the critical charge is called, "*super-heavy atomic nucleus*" and the unstable solution is called "*atomic collapse*". These unstable solutions physically mean that the electron orbiting a super-heavy atomic nucleus will no longer perform a stable bounded orbit around it, like explained in previous section in Bohr's quantum theory, but the electron will spiral into it while losing its energy, schematically shown in Fig. 1. This unstable character was also found in the quantum mechanical problem with the same condition, i.e. $Z_c = 1/\alpha$, and was predicted in the year 1952 but was never observed. The only possible way to observe atomic collapse is by creating a charged super-heavy atomic nucleus such that by surpassing a certain charge threshold the resulting strong Coulomb field causes an atomic collapse where the electron wave function component shows the behavior to fall into it. In the next section is explained how atomic collapse was observed in graphene in 2013. [9][7][8]

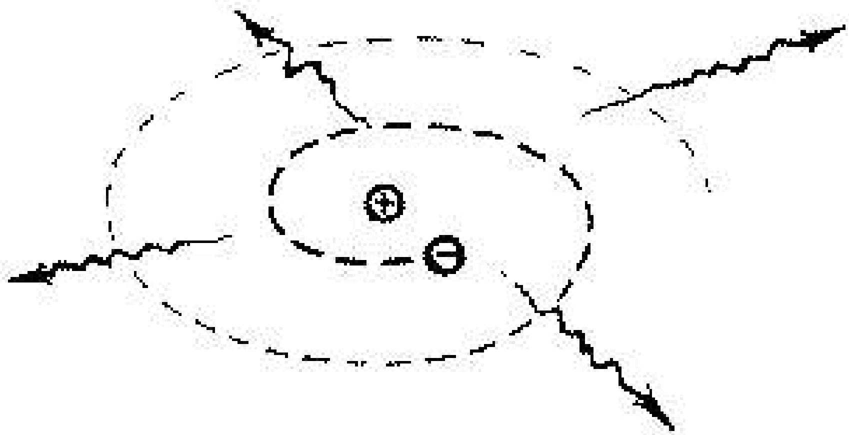


Figure 1: When an electron orbits around a super-heavy atomic nucleus the electron wave function collapses into an unstable state where the electron falls into the nucleus creating electron-positron pairs.[16]

4.3 Atomic collapse in graphene

Graphene is a very special nanomaterial which consists of a single atomic layer of carbon atoms arranged in a hexagonal lattice and bound together by strong sp^2 bonds. The layers are held together by van der Waals interaction leading to graphite. Graphene has unique properties such as high electron mobility which tells us how quickly an electron can move through it. If we look at the dispersion relation (relations between energy and linear momentum/ wave vector) of undoped graphene in Fig. 2 we notice that the valence and conduction bands join together at points in the Brillouin zone. These special points are also known as K and K' and also called Dirac points. [9]

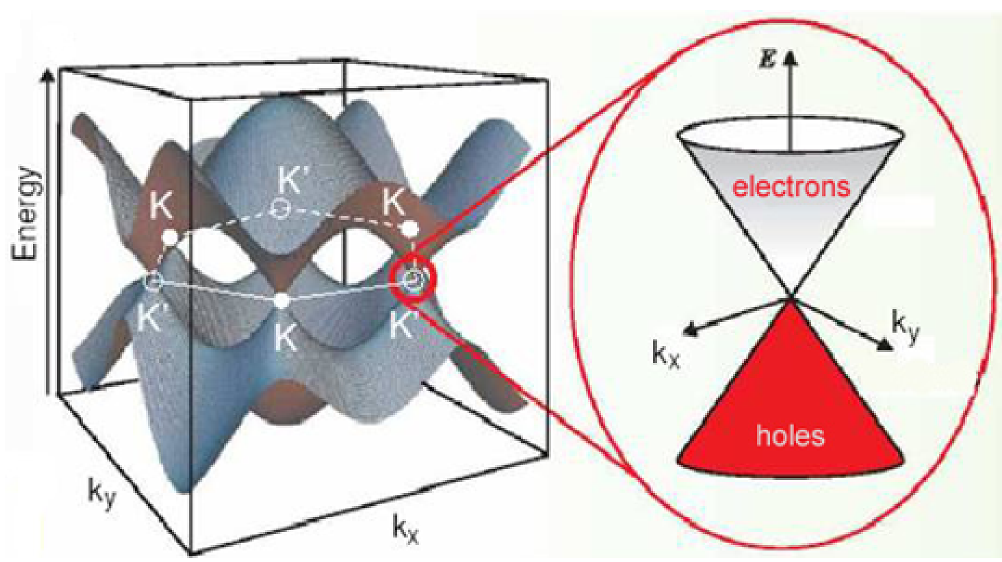


Figure 2: The band structure of undoped graphene.[10]

By looking at Fig. 2 we see that near the Dirac points the conduction band has a linear dispersion resulting in Dirac cones. A good model to describe the carriers (in this case electrons or holes) around this region is the Dirac equation, which joins the principles of quantum mechanics and special relativity. This means that electrons in graphene behaves as massless relativistic fermions which move with the Fermi velocity. The dispersion relation around the Brillouin zone is thus:

$$E = \pm v_F|p| = \hbar v_F|k| \tag{5}$$

with $v_F = 10^6 m/s$ the Fermi velocity. Because the electrons in graphene behaves as massless relativistic fermions near the Brillouin zone we can use graphene to simulate several quantum-relativistic phenomena such as Klein tunneling and atomic collapse. [9][7][8]

4.3.1 Experimental observation of atomic collapse

Atomic collapse has been observed recently in graphene in 2013 by the group of Michael Crommie at Berkeley. In graphene the critical charge $Z_c = 1/\alpha = \hbar c/e^2 = 137$ has to be replaced by $Z^* = \epsilon \frac{\hbar v_F}{e^2}$ with v_F the Fermi velocity and ϵ a combination of the dielectric constant of graphene and of its surrounding. This makes Z^* of order unity. The super-heavy atomic nucleus was simulated by putting positively charged molecules on top of graphene (see Fig. 3). A scanning tunneling microscope (STM) made it possible to push 5 charged calcium dimers close together such that they together act as a nucleus with charge +5e. In Fig. 3 the atomic collapse electron cloud is shown and is seen in the surrounding of the 5 charged calcium dimers by the enhanced density of states that was measured by the STM probe. [9][7][8]

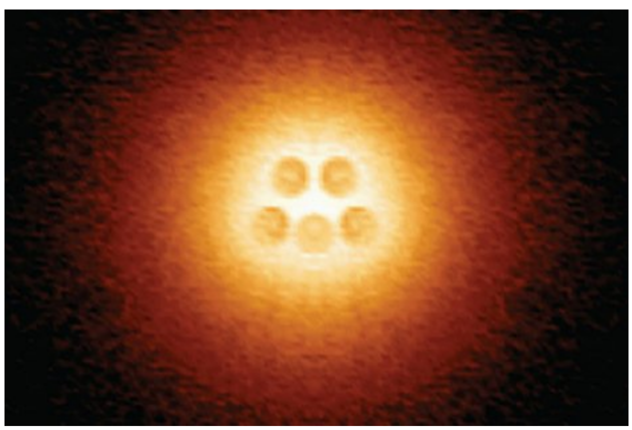


Figure 3: Atomic collapse in graphene.

Alternatively, the group of Prof. E. Andrei at Rutgers University was able to controllably charge a carbon vacancy in graphene and using STM to observe different atomic collapse states. [11] The same group was able to induce the atomic collapse state in pristine graphene by coating the STM tip with a particular metal. [12] The theoretical analysis of these experimental results was performed in the Phd thesis of D. Moldovan. [13]

4.4 Purpose and structure of the thesis

The purpose of this thesis is to present a classical description of atomic collapse and to investigate how it depends on the potential. It will be shown that atomic collapse is possible in different Kepler/Coulomb-type potentials by making a stability analysis in order to have stable bounded orbits and by figuring out the possible orbits of a relativistic and non-relativistic particle in these different potentials.

From quantum-relativistic physics we know that when the nucleus charge $Z > 1/\alpha$ the stable quantized bound orbits become unstable which leads to atomic collapse. This means that in order to describe atomic collapse with classical description we need to derive some conditions to have stable/unstable bounded orbits. A good example of a bounded orbit is a circular orbit which is only possible when the centripetal force is capable to compensate the attractive force of the Kepler/Coulomb potential. Therefore, we will investigate circular orbits in order to derive the conditions for stable and unstable orbits.

In the first part of the thesis we will look at a relativistic and non-relativistic particle in $V \sim 1/r^n$ potential with n=1,2. In each case we obtain the possible bound circular orbits and a classification of the possible trajectories is made. Afterwards we derive the equation of motions in the presence of these Kepler/Coulomb-type potentials and we figure out the possible orbits.

In the second part of the thesis I remove the singularity in $r = 0$ of the Kepler/Coulomb-type potentials and the impact on the orbits is investigated.

In the last part of the thesis I investigate what happens to the orbits of a relativistic and non-relativistic particle in the presence of two separate charges like in a molecular system.

5 Trajectories for relativistic and non-relativistic particle in a Kepler/Coulomb potential

The first part of this thesis is studying and rederive results from the paper [5], where the behavior of a classical relativistic particle in $V(r) = -\alpha_1/r$ potential was studied. Further, in this chapter we will extend these calculations and report some novel results which were not presented in the paper.

5.1 The mechanical problem

In this section we will compare the behavior of a non-relativistic and a relativistic particle in the Kepler/Coulomb potential $V(r) = -\alpha_1/\mathbf{r}$. It is important to know that these particles are treated in a classical manner and we don't take into account any quantum mechanical effects. In such a potential a particle experiences a force $\mathbf{F} = -\nabla V(r) = -\mathbf{e}_r \alpha_1/r^2$. We can find the equation of motion of a non-relativistic particle (momentum is $m\mathbf{v}$) by using the following Lagrangian: [1]

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}}^2 - \frac{\alpha_1}{\mathbf{r}}. \tag{6}$$

The factor $T = \frac{1}{2} m \dot{\mathbf{r}}^2$ describes the kinetic energy of the non-relativistic particle. The Euler-Lagrange (E-L) equation of motion tells us: [1]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}}\right) = \frac{\partial \mathcal{L}}{\partial \mathbf{r}} \Rightarrow \frac{d(m\mathbf{v})}{dt} = -\frac{\alpha_1}{r^2}\mathbf{r}. \tag{7}$$

Obviously we have the velocity $\dot{\mathbf{r}} = \mathbf{v}$. The previous result is nothing else than Newton's second law:

$$\mathbf{F} = m \cdot \mathbf{a} = \frac{d(m\mathbf{v})}{dt} = -\frac{\alpha_1}{r^2}\mathbf{r}. \tag{8}$$

If the Lagrangian is explicitly time independent, it is easy to proof that the total energy of the system is conserved. [1]

$$\frac{d\mathcal{L}(\mathbf{r}, \mathbf{v})}{dt} = \frac{\partial \mathcal{L}}{\partial \mathbf{r}}\mathbf{v} + \frac{\partial \mathcal{L}}{\partial \mathbf{v}}\frac{d}{dt}\mathbf{v}. \tag{9}$$

Using E-L equation of motion, we replace $\frac{\partial \mathcal{L}}{\partial \mathbf{r}}$ by $\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \mathbf{v}})$. We get:

$$\frac{d\mathcal{L}(\mathbf{r}, \mathbf{v})}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\right)\mathbf{v} + \frac{\partial \mathcal{L}}{\partial \mathbf{v}}\frac{d}{dt}\mathbf{v} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\mathbf{v}\right) \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\mathbf{v} - \mathcal{L}\right) = 0. \tag{10}$$

Making use of the Lagrangian in (6) we compute $\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = m\mathbf{v}$. In this way we find that the energy of the system is conserved:

$$\frac{d}{dt}\left(m\mathbf{v}^2 - \mathcal{L}\right) = \frac{d}{dt}(2T - (T - V)) = \frac{d}{dt}(T + V) = \frac{d}{dt}(E_{nr}) = 0. \tag{11}$$

We find here the definition of the non-relativistic energy: $E_{nr} = T + V = \frac{1}{2}m\mathbf{v}^2 - \frac{\alpha_1}{\mathbf{r}}$. [1] Note that this part was novel and was not reported in [5]. For a relativistic particle the situation is a bit different. We know from relativity that by introducing a small increase in velocity near the speed of light c, the mass of the relativistic particle is going to be affected and will increase. The increase of the mass is given by the gamma factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ and consequently we replace the mass of the particle m by $m/\sqrt{1-v^2/c^2}$. In this way the relativistic particle has a momentum $m\mathbf{v}/(\sqrt{1-v^2/c^2})$. Further we know from relativity that the frame of reference is very important and consequently the rest energy of the particle also. This means that the energy of the relativistic particle will be different from the energy of a non-relativistic particle but at the same time if we take the non-relativistic limit $c \to \infty$ the energy should be the same as the non-relativistic energy.(see Appendix A) The relativistic energy is given by:

$$E = \epsilon + mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\alpha_1}{r}. \tag{12}$$

Since the potential $V(r) = -\alpha_1/r$ leads to a spherical symmetric central force (the force is directed along the line which joins the object to the origin, $\mathbf{F} = -\mathbf{e}_r\alpha_1/r^2$), we have a conserved system which means that we have conserved quantities like energy and angular momentum [1][2]. This way we can show that the angular momentum ($\mathbf{L} = \mathbf{r} \times \mathbf{p}$) in this case is conserved:

$$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}\mathbf{r} \times \mathbf{p} + \mathbf{r} \times \frac{d}{dt}\mathbf{p} = 0. \tag{13}$$

Since $\mathbf{p} \sim \mathbf{v}$ and $\frac{d}{dt}\mathbf{r} = \mathbf{v}$ the first cross product vanishes as the two vectors are parallel ($sin(0) = sin(\pi) = 0$). The second term vanishes because of equation (8) which gives $\frac{d}{dt}\mathbf{p} \sim \mathbf{r}$. So in the non-relativistic case we have a conserved angular momentum $\mathbf{L}_{nr} = \mathbf{r} \times (m\mathbf{v})$, while in the relativistic case we have $\mathbf{L} = \mathbf{r} \times \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}$.

5.2 Angular momentum for circular orbits

For the special case of a circular orbit we know that the radius of the followed orbit is perpendicular to the velocity $\mathbf{v}$ of the particle. Consequently the angular momentum is perpendicular to the plane of the orbit. In this way, the angular momentum for a non-relativistic particle is given by $L_{nr} = rmv$ while for a relativistic particle $L = \frac{rmv}{\sqrt{1-\frac{v^2}{c^2}}}$.

5.2.1 Non-relativistic case

In order to have a stable circular orbit we know that the attractive force should be equal to the centripetal force, $m\frac{v^2}{r} = \frac{\alpha_1}{r^2}$. By combining this equation of motion with the found expression for the angular momentum, we find an expression for the radius and velocity of the particle in function of the angular momentum.

$$m\frac{v^2}{r} = \frac{\alpha_1}{r^2} \Rightarrow v^2 = \frac{\alpha_1}{mr}.$$

On the other hand:

$$L_{nr} = rmv \Rightarrow \frac{L_{nr}}{v} = rm.$$

By combining these expressions we find:

$$v = \frac{\alpha_1}{L_{nr}}; r = \frac{L_{nr}^2}{m\alpha_1}. \tag{14}$$

Notice that the angular momentum can take any values from zero to infinity. This means that in the non-relativistic case the particle can always follow a circular orbit unless the angular momentum becomes zero. [5]

5.2.2 Relativistic case

In the relativistic case the equation of motion becomes $\frac{m}{\sqrt{1-v^2/c^2}}\frac{v^2}{r} = \frac{\alpha_1}{r^2}$. Combined with $L = \frac{rmv}{\sqrt{1-\frac{v^2}{c^2}}}$ we find:

$$\frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^2}{r} = \frac{\alpha_1}{r^2} \Rightarrow v^2 = \frac{\alpha_1\sqrt{1-\frac{v^2}{c^2}}}{mr} \Rightarrow v = \frac{\alpha_1}{L}. \tag{15}$$

And combining the calculated velocity with the angular momentum, we find:

$$L = \frac{rm\alpha_1}{L\sqrt{1-(\frac{\alpha_1}{cL})^2}} \Rightarrow r = \frac{L^2}{\alpha_1 m}\sqrt{1-\left(\frac{\alpha_1}{cL}\right)^2}. \tag{16}$$

Now we can investigate the relation between angular momentum and radius which was not reported in [5]. From Fig. 4 we can clearly see that a relativistic particle follows stable circular orbits when the angular momentum is larger than the critical angular momentum $L_c = \alpha_1/c$. When the critical angular momentum is reached the radius becomes zero which physically means that the particle falls into the origin of the central force causing atomic collapse.

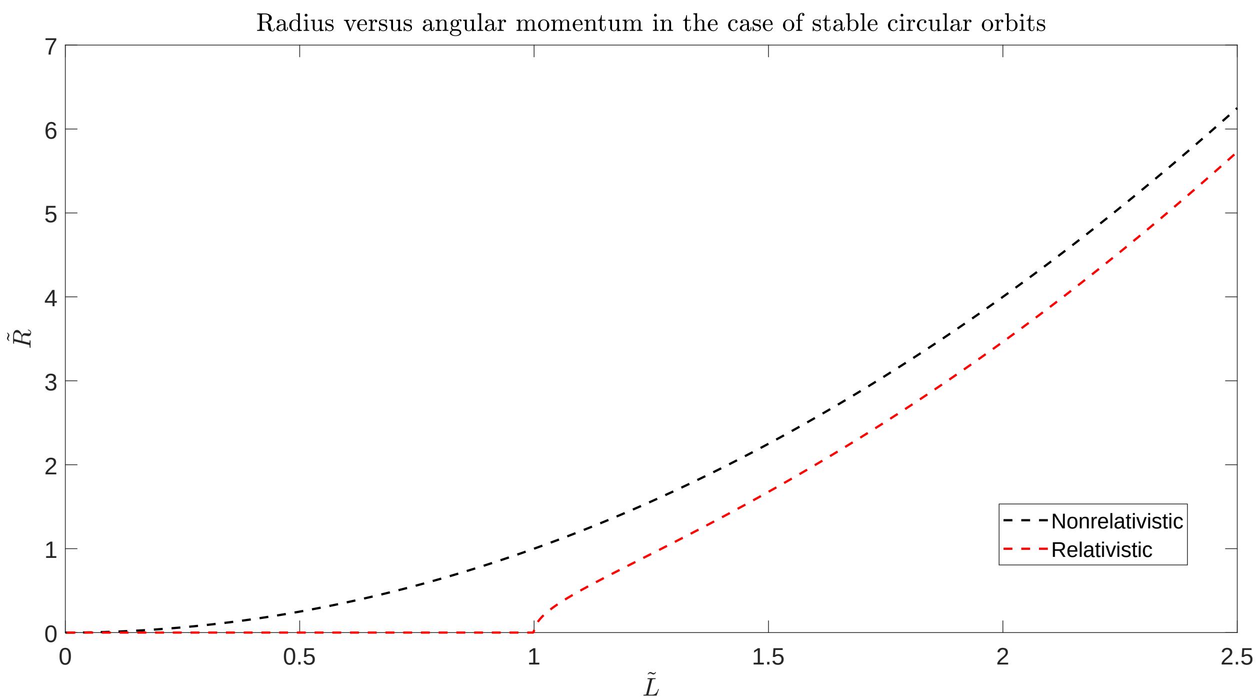


Figure 4: In non-relativistic case the radius becomes zero when the angular momentum is zero but in the relativistic case the radius becomes zero when the angular momentum becomes α_1/c with c the speed of light. This means that in order to have stable circular orbits in relativistic case the angular momentum should be larger than α_1/c. The plot is made by using dimensionless parameters $\tilde{R} = r/(\alpha_1/mc^2) = \tilde{L}^2\sqrt{1-\frac{1}{\tilde{L}^2}}$ and $\tilde{L} = Lc/\alpha_1$. (see Appendix B Table 8)

5.2.3 Role of power n for a general potential $V = kr^n$

In previous section we calculated the relations between angular momentum, radius and velocity in order to have stable circular orbits by using $V(r) = -\alpha_1/r$ potential. Now we consider the condition for stable circular orbits for a general potential of the form $V(r) = kr^n$, with k and n constant. This section was not reported in [5] and is novel. Again, in order to have stable circular orbits the centripetal force should be equal to attractive force:

$$\frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^2}{r} = nkr^{n-1} \Rightarrow v = \frac{nkr^{n+1}}{mvr}\sqrt{1-\frac{v^2}{c^2}} = \frac{nkr^{n+1}}{L}. \tag{17}$$

Where in the last equality in Eq. (17) we made use of the relativistic angular momentum $L = rmv/\sqrt{1-v^2/c^2}$. Combining Eq. (17) with expression for relativistic angular momentum we find:

$$L = \frac{rmv}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{L^2}{m} = \frac{nkr^{n+2}}{\sqrt{1-\frac{n^2k^2r^{2n+2}}{L^2c^2}}} \Rightarrow \frac{L^4}{m^2} = \frac{n^2k^2r^{2n+4}}{1-\frac{n^2k^2r^{2n+2}}{L^2c^2}}. \tag{18}$$

And:

$$\frac{L^4}{m^2} - \frac{L^4}{m^2}\frac{n^2k^2r^{2n+2}}{L^2c^2} = n^2k^2r^{2n+4} \Rightarrow \frac{L^4}{m^2} = n^2k^2r^{2n+2}\left(r^2 + \frac{L^2}{c^2m^2}\right). \tag{19}$$

In this way we get the following equation which describes the relation between radius and angular momentum:

$$\frac{L^4}{m^2} = n^2k^2r^{2n+2}\left(r^2 + \frac{L^2}{c^2m^2}\right). \tag{20}$$

Of course by inserting $n = -1$ and $k = -\alpha_1$ we have the same result as in Eq.(16) for a relativistic particle in the presence of a Coulomb potential. By taking the non-relativistic limit ($c \to \infty$) of equation (20) and by inserting $n = -1$ and $k = -\alpha_1$, we get again the found relation between angular momentum and radius in Eq. (14) for a non-relativistic particle. Now we can investigate which power n of the attractive potential (or attractive force because the relation $\mathbf{F} = -\nabla V(r)$ is allowed in order to have stable circular orbits. So, in order to have stable circular orbits about the origin of the coordinates frame, the particle experiences a general attractive central force $F = -\alpha_1/r^n$. The parameter r is the radius which measures the distance of the particle from the origin of the central force. We know that central forces which are spherical symmetric leads to a conserved system, consequently the total energy and angular momentum are conserved. Making use of this argument we find the definition of the potential field:

$$U(r) = \int_\infty^r F(r)dr = -\int_\infty^r \frac{\alpha_1}{r^n}dr = -\frac{1}{n-1}\frac{\alpha_1}{r^{n-1}}.$$

Since the angular momentum is conserved we make use of the angular momentum (see 5.2) to construct an effective potential, and consequently an effective force experienced by the particle on the circular orbit. [14][15] (see Table 1)

	Relativistic	Non-relativistic
(A)	$U_{eff}(r) = -\frac{1}{n-1}\frac{\alpha_1}{r^{n-1}} + \frac{L^2}{2m\gamma r^2}.$	$U_{eff}(r) = -\frac{1}{n-1}\frac{\alpha_1}{r^{n-1}} + \frac{L^2}{2mr^2}.$
(B)	$F_{eff}(r) = -\frac{\alpha_1}{r^n} + \frac{L^2}{m\gamma r^3}.$	$F_{eff}(r) = -\frac{\alpha_1}{r^n} + \frac{L^2}{mr^3}.$

Table 1: The effective potential and force. Notice in row (B) in both cases the second term on the right side of the equations is just the centrifugal force.

For some radius $r = \rho$ the particle will follow a circular orbit and consequently the effective force disappears ($F_{eff}(\rho) = 0$).

	Relativistic	Non-relativistic
(A)	$\frac{\alpha_1}{\rho^n} = \frac{L^2}{m\gamma\rho^3}.$	$\frac{\alpha_1}{\rho^n} = \frac{L^2}{m\rho^3}.$
(B)	$\frac{\alpha_1 m\gamma\rho^3}{L^2} = \rho^n.$	$\frac{\alpha_1 m\rho^3}{L^2} = \rho^n.$

Table 2: The top row (A) represents the equation of motion in the case of circular orbits where the effective force vanishes. The bottom row (B) is needed for further derivations.

The first row (A) of table 2 gives the same result found in 5.2.1 and 5.2.2. In order to have stable circular orbits the second derivative of the effective potential should be positive, which means that the zero point of the effective force (ρ) is a minimum. By calculating the second derivative of the effective potential in $r = \rho$ and combining this result with respect to row (B) of Table 2, we find the results presented in Table 3 :

	Relativistic	Non-relativistic
(A)	3-n>0	3-n>0

Table 3: By combining the second derivative of the effective potential with row (B) of Table 2 we find that the power n of the effective force should be less than 3.

The result in Table 3 tell's us that in order to have stable circular orbits the power n in the effective force should be smaller than 3.[14][15] Because we considered in the attractive potential that n is negative we find alternatively that n>-3. The fact that the power of the radius in the central force should be bigger than -3 means that the power of the potential should be at least $n = -1$ in order to have stable circular orbits because the relation $\mathbf{F} = -\nabla V(r)$. Interesting result is found when we put $n = -2$ in equation Eq. (20). In relativistic case we find some stable orbits, where we express the relation between angular momentum and radius in dimensionless units:

$$\tilde{R} = \frac{r}{\sqrt{\frac{\alpha_2}{mc^2}}} = \frac{\sqrt{2}\tilde{L}}{\sqrt{\tilde{L}^4 - 1}}. \tag{21}$$

By taking the non-relativistic limit ($c \to \infty$) the second term on the right side of Eq.(20) vanishes, this leads to:

$$\lim_{c\to\infty} \frac{L^4}{m^2} = \lim_{c\to\infty} n^2k^2r^{2n+2}\left(r^2 + \frac{L^2}{c^2m^2}\right) = n^2k^2r^{2n+4}. \tag{22}$$

When we put $n = -2$ and $k = -\alpha_2$ in Eq. (22) we clearly see that the r dependency vanishes, which physically means that we never find stable orbits in the case of $n = -2$ for a non-relativistic particle. This is what we expected because of the results found in Table 3. There we found that the power n in the central force should be bigger than -3. Eq. (20) is derived by using the potential $V(r) = kr^n$. When the power of the potential becomes $n = -2$ the particle experiences a central force proportional to r^{-3} because the relation $\mathbf{F} = -\nabla V(r)$ which generate non stable circular orbits.

5.2.4 Hydrogen atom

We can compare the described relativistic particle in a Kepler or Coulomb potential with respect to (charge is e) electron which performs a circular orbit around an hydrogen atom (atomic number is Z=1). In this case the potential becomes $V(r) = -\alpha_1/r = -Ze^2/r = -e^2/r$, and the limiting angular momentum becomes $L = \alpha_1/c = e^2/c$. We can rewrite this expression by multiplying nominator and denominator by $\hbar$ (Planck's constant= $1.0545718 \cdot 10^{-34} m^2$kg/s), we get $L = (e^2/\hbar c)\hbar \approx (1/137)\hbar$. From [3] we know that the energy and angular momentum of the hydrogen atom problem is quantified. This means that the system has a minimum energy which leads to a minimum angular momentum. From [3] we know that the angular momentum can be expressed in terms of the orbital quantum number l and this leads to $L = l\hbar$ (semiclassical point of view as considered by Bohr). In the lowest energy level we have the lowest angular momentum $L_{low} = \hbar$. This is one of the reasons why the electron cannot fall on the hydrogen atom. The limiting angular momentum is much smaller than the lowest possible angular momentum $L_{limit} < L_{low} \Rightarrow (1/137)\hbar < \hbar$.[5]

5.3 Relation between energy and velocity in the case of circular orbits for relativistic and non-relativistic cases

In Appendix B we find a relation between energy and velocity of the particles in dimensionless units for the special case of circular orbits. This section was not reported in [5] and is novel. We can compare both behaviors in a graph:

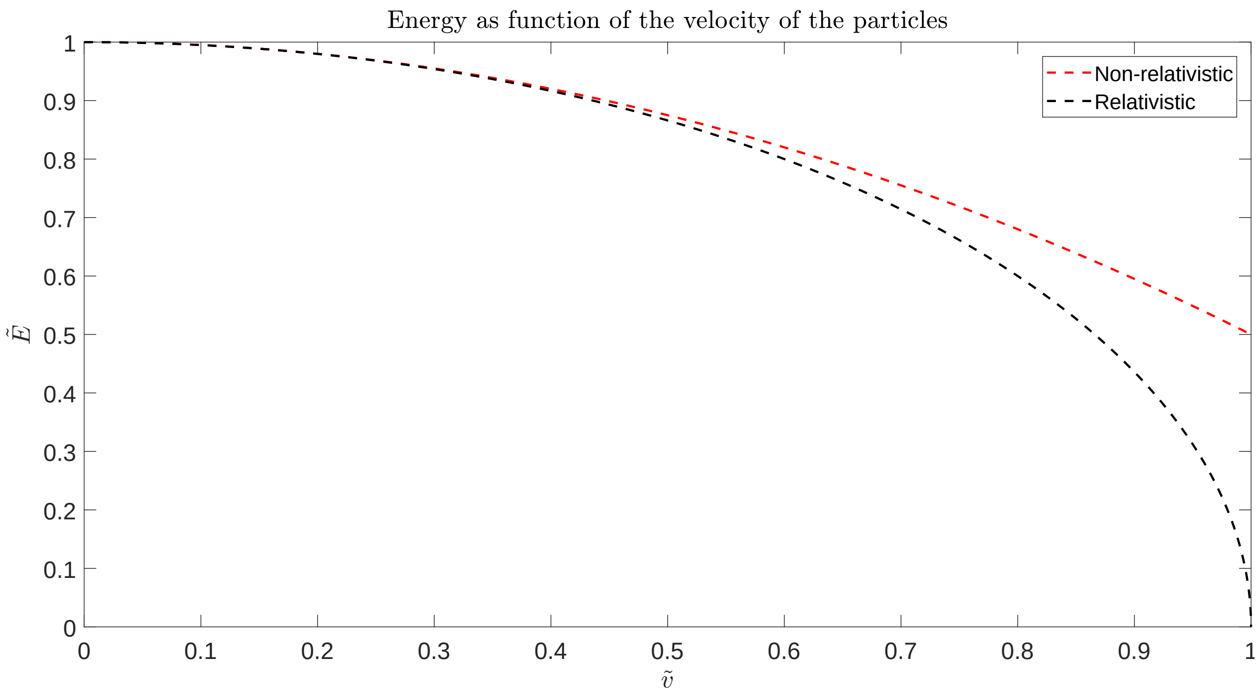


Figure 5: Comparing the dimensionless energy of a relativistic and non-relativistic particle as function of the dimensionless velocity, with $\tilde{E}_{rel} = E_{rel}/mc^2 = \sqrt{1-\tilde{v}^2}$, $\tilde{E}_{nrel} = E_{nrel}/mc^2 = 1 - \frac{1}{2}\tilde{v}^2$ and $\tilde{v} = v/c$. Notice that we put both particles on the same scale in order to compare them. Of course in non-relativistic case there is no limit on the speed of the particle.

In Appendix B Table 7 we find the relations between dimensionless radius and velocity which are expressed as $\tilde{R}_{rel} = \frac{1}{\tilde{v}^2}\sqrt{1-\tilde{v}^2}$ and $\tilde{R}_{nrel} = \frac{1}{\tilde{v}^2}$ for the special case of circular orbits. From Fig. 5 we can clearly see that the relativistic and non-relativistic energy takes two different behaviors. In relativistic case we see that for high velocities, near the speed of light, the energy becomes zero and at the same time the radius becomes zero. Physically this means that for high velocities near the speed of light the relativistic particle will spiral into the center of the central force. This behavior is not found for non-relativistic particles. On the other hand we see that both particles for low speed ($\tilde{v} \to 0$) the radius becomes infinitely large which leads to zero potential energy. In this case we get a free particle. The most important reason why the two particles takes different behavior in the limit $\tilde{v} \to 1$, is found in the relation between kinetic and potential energy for both particles. In non-relativistic case we find $T = -U/2$. This relation means that depending on the place of the particle on the orbit, part of the kinetic energy can be converted in potential energy and of course part of potential energy can be converted in kinetic energy. So, in this case for stable circular orbits the total amount of energy is constant and there is always a kinetic and potential energy term in the energy equation. On the other hand for a relativistic particle we have $T = -U$. This means that the kinetic/potential energy can be completely converted to potential/kinetic energy. Physically this fact means that the particle can fall into the center of the central force if the kinetic energy is completely converted into potential energy, and in the case that potential energy is completely converted into kinetic energy the particle becomes a free particle because it doesn't feel the potential of the central force at long distance from the origin of the attraction field. It is easy to proof that the relation $T = -U$ in relativistic case will convert up to $T = -\frac{U}{2}$ in non-relativistic case. In order to have circular orbits the centripetal force should be equal to attractive force:

$$\frac{m}{\sqrt{1 - v^2/c^2}} \frac{v^2}{r} = \frac{\alpha_1}{r^2}. \tag{23}$$

By making use of a Taylor expansion taking the non-relativistic limit $(1/c \to 0)$ of Eq. (23) we have:

$$\lim_{\frac{1}{c} \to 0} \frac{m}{\sqrt{1 - v^2/c^2}} \frac{v^2}{r} = \frac{\alpha_1}{r^2} \Rightarrow \lim_{\frac{1}{c} \to 0} mv^2 \left(1 + \frac{1}{2!} \frac{v^2}{c^2} + \dots \right) = \frac{\alpha_1}{r^2}. \tag{24}$$

This leads to the non-relativistic relation between kinetic and potential energy.

$$mv^2 = \frac{\alpha_1}{r^2} \Rightarrow T = -U/2. \tag{25}$$

5.4 Classification of trajectories

The motion of the relativistic particle is situated in a plane. It is easier to work in polar coordinates, so we can express the velocity and the angular momentum of the relativistic particle in polar coordinates. [5] The velocity is given by: (see Appendix C)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\widehat{r} + r\dot{\theta}\widehat{\theta}. \tag{26}$$

Consequently the angular momentum is given by: (see Appendix C)

$$\mathbf{L} = \mathbf{r} \times \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow L = \frac{mr^2\dot{\theta}}{\sqrt{1-\frac{(\dot{r}^2+r^2\dot{\theta}^2)}{c^2}}}. \tag{27}$$

By solving (27) to $\dot{\theta}$ we find: (see Appendix D)

$$\dot{\theta} = \sqrt{\frac{L^2c^2 - L^2\dot{r}^2}{L^2r^2 + m^2r^4c^2}}. \tag{28}$$

The energy of the relativistic particle in polar coordinates is given by:

$$E = \epsilon + mc^2 = \frac{mc^2}{\sqrt{1-\frac{\dot{r}^2+r^2\dot{\theta}^2}{c^2}}} - \frac{\alpha_1}{r}.$$

By filling the found expression for $\dot{\theta}$ in (28) in the expression of energy we find: (see Appendix E)

$$E = \frac{mc^2}{\sqrt{1-(\dot{r}/c)^2 - L^2(1-(\dot{r}/c)^2)/(L^2+m^2r^2c^2)}} - \frac{\alpha_1}{r}. \tag{29}$$

By solving equation (29) to $\dot{r}^2$ we find:

$$\Rightarrow \dot{r}^2 = c^2\left[1 - \left(1 + \frac{L^2}{m^2r^2c^2}\right)\left(\frac{mc^2}{E+(\alpha_1/r)}\right)^2\right]. \tag{30}$$

Further, in order to classify which orbit is allowed, we need some constraints on the energy of the particle. As the angular momentum and the energy are conserved quantities and we have restrictions on the angular momentum, we can express the energy in terms of angular momentum such that we find restrictions on the energy of the particle. The energy in expression (12) can be reduced in the form of a limiting angular momentum with speed of the relativistic particle equal to the speed of light. [5]

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\alpha_1}{r} \Rightarrow L_{limit} = \frac{rmc}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{r}{c}(E + \frac{\alpha_1}{r}). \tag{31}$$

Since the velocity of the particle should be less than the speed of light c, the angular momentum found in expression (27) is less than the limit angular momentum found in Eq. (31): [5]

$$L < L_{limit} \Rightarrow L = \frac{mr^2\dot{\theta}}{\sqrt{1-\frac{(\dot{r}^2+r^2\dot{\theta}^2)}{c^2}}} < \frac{rmc}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{r}{c}(E + \frac{\alpha_1}{r}). \tag{32}$$

From Eq. (32) we deduce the fact that $L < \frac{r}{c}(E + \frac{\alpha_1}{r})$ or equivalently $L - \frac{\alpha_1}{c} < \frac{r}{c}E$. Because from 5.2.2 we know that we have stable circular orbits when $L \geq \alpha_1$ and all quantities L, α_1, r, c, E are real, we find that in order to have stable circular orbits the total relativistic energy of the particle should be positive $E = \epsilon + mc^2 > 0$. [5] This fact doesn't exclude bounded orbits. We will clarify in section 5.10 how to get bounded states. Finally it is clear that the right hand side of Eq. (30) should be at least zero or positive. This means that the radial velocity is limited between $0 \leq \dot{r}^2 < c^2$. By using the definition of $\dot{r}$ in Eq. (30) in the inequality $0 \leq \dot{r}^2 < c^2$ we find the following inequality $-L^2c^2 < 0 \leq (E^2 - m^2c^4)r^2 + 2E\alpha_1 r + (\alpha_1^2 - L^2c^2) = Y(r)$. The fact that all quantities are real satisfies automatically the left hand side of the inequality. Now we can take a look at the r dependent function Y(r). This function tells us which relativistic orbit is allowed. Physically, the fact that the radial velocity is restricted, means that on the followed orbit we have some special points where $\dot{r} = 0$. That the radial velocity vanishes in these points means that the particle cannot move further from the origin of the central force, otherwise the restriction is not satisfied. This means that the particle in these points is forced to change its trajectory. When we have two turning points r_1 and r_2 we have a bounded trajectory like ellipses. Sometimes we have special cases. In a conserved energy system, when kinetic and potential energy are constant and do not change in time we have a circular orbit. In this case the two turning points are at the same radial distance which means that $r_1 = r_2$. The allowed orbits are situated in the region where $Y(r) \geq 0$ imposed of course by the condition $0 \leq \dot{r}^2 < c^2$. At first we can find the turning-points for the motion of the particle by solving $Y(r) = 0$. This is done by calculating first the discriminant which is given in this case by $D = (2\alpha_1 E)^2 - 4 \cdot (E^2 - m^2c^4) \cdot (\alpha_1^2 - L^2c^2)$ and finally we find the turning-points:

$$r_{turning-point} = \frac{\alpha_1 E \pm \sqrt{E^2\alpha_1^2 + (m^2c^4 - E^2)(\alpha_1^2 - L^2c^2)}}{m^2c^4 - E^2}. \tag{33}$$

Now, because we want to describe relativistic circular orbits we should have the two turning-points at the same radius. The only way to put both turning points on the same radius is to find the energy which provides for $\sqrt{E^2\alpha_1^2 + (m^2c^4 - E^2)(\alpha_1^2 - L^2c^2)} = 0$.

$$\Rightarrow E^2\alpha_1^2 + m^2c^4\alpha_1^2 - m^2c^4L^2c^2 - E^2\alpha_1^2 + E^2L^2c^2 = 0 \Rightarrow E = mc^2\sqrt{1 - \left(\frac{\alpha_1}{Lc}\right)^2}. \tag{34}$$

This expression is the energy in the case of circular orbits for a relativistic particle. Clear is that if we take non-relativistic limit $c \to \infty$ this energy should converge to the non-relativistic energy in the case of circular orbits. The relativistic energy is expressed as $E = \epsilon + mc^2$ and consequently the energy difference from the rest mass energy of the particle is $\epsilon = E - mc^2$. If we take from this expression the non-relativistic limit we get:

$$\lim_{\frac{1}{c}\to 0} \epsilon_{nrel} = \lim_{\frac{1}{c}\to 0}(E - mc^2) = \lim_{\frac{1}{c}\to 0} mc^2\sqrt{1 - \left(\frac{\alpha_1}{Lc}\right)^2} - mc^2 \approx \lim_{\frac{1}{c}\to 0} mc^2\left(1 - \frac{1}{2!}\frac{\alpha_1^2}{L^2c^2} + \dots\right) - mc^2. \tag{35}$$

$$\Rightarrow \lim_{\frac{1}{c}\to 0} \epsilon_{nrel} \approx -\frac{1}{2}\left(\frac{m\alpha_1^2}{L_{nrel}^2}\right). \tag{36}$$

Of course we made use of a Taylor expansion and the higher orders in (1/c) vanishes because of the limit. Now, by inserting the found non-relativistic angular momentum as function of velocity in

5.2.1, we get indeed the non-relativistic energy for circular orbits as function of the velocity (see also Appendix B):

$$\epsilon_{nrel} = -\frac{1}{2}mv^2. \tag{37}$$

Now that we found the energy which puts the turning points on the same radius we can analyze the function Y(r) as function of the radius. Depending on the values of the angular momentum and the energy we get different situations. From the definition of Y(r) we see that by using the energy from Eq. (34) and for $L > L_c = \alpha_1/c$ we have a sign change because the energy becomes less than the rest mass energy and consequently we get a negative parabolic plot. Also is the parabola shifted down in the $Y(r) < 0$ region because the shift term $(\alpha_1^2 - L^2c^2)$ is negative. When the angular momentum becomes equal to the critical angular momentum $L = L_c = \alpha_1/c$ we see that the energy becomes zero and the shift term $(\alpha_1^2 - L^2c^2)$ in the definition of Y(r) becomes zero. Here we get no solutions. When the angular momentum becomes less than the critical angular momentum, the shift term becomes positive and consequently the parabola is shifted in the $Y(r) > 0$ region. In this case the energy becomes imaginary and this leads of course to unstable orbits like explained in section 5.2.2. In order to make it easier we introduce dimensionless parameters (look at Appendix B) such as $\tilde{E} = E/mc^2 = \sqrt{1 - \frac{1}{\tilde{L}^2}}$, $\tilde{L} = Lc/\alpha_1$ and $\tilde{r} = r/(\alpha_1/mc^2)$. Using these parameters we rewrite the definition of Y(r) and turning points:

$$\tilde{r}_{turning-point} = \frac{\tilde{E} \pm \sqrt{\tilde{E}^2 + (1-\tilde{E}^2)(1-\tilde{L}^2)}}{1-\tilde{E}^2}. \tag{38}$$

And we get $Y(\tilde{r})$:

$$Y(\tilde{r}) = (\tilde{E}^2 - 1)\tilde{r}^2 + 2\tilde{E}\tilde{r} + (1 - \tilde{L}^2). \tag{39}$$

Below are given some classified trajectories.

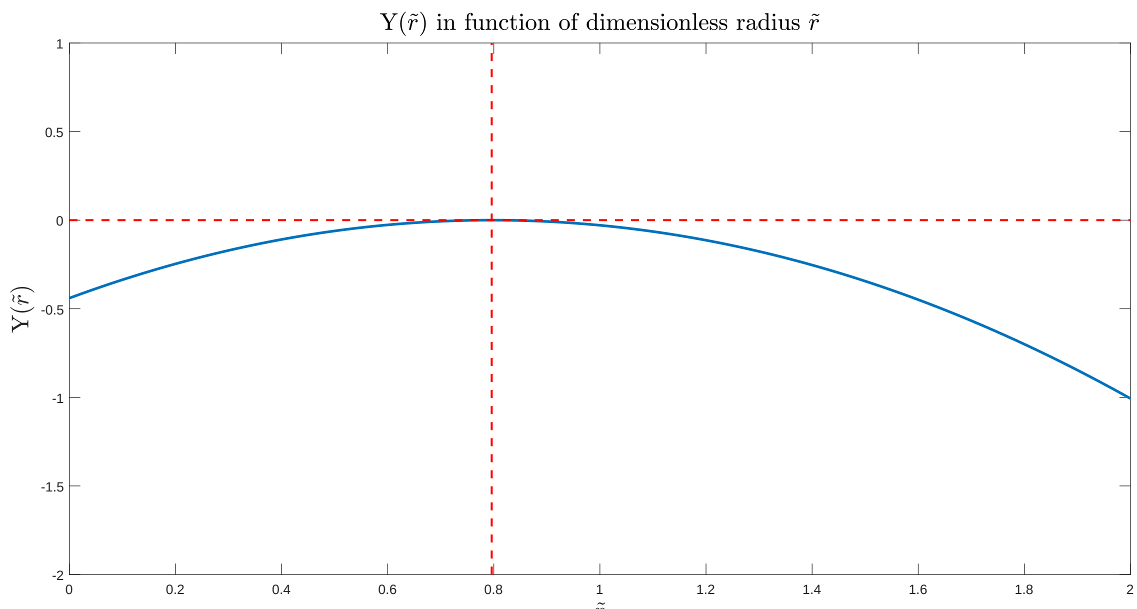


Figure 6: This graph shows the relation between $Y(\tilde{r})$ and dimensionless radius. We used the following parameters: $\tilde{L} = 1.2$, $\tilde{E} =\approx 0.55277$

From the above discussions and restrictions we know that the possible allowed orbits are in the region where the function $Y(\tilde{r})$ is positive. Most of the time in Fig. 6 the function $Y(\tilde{r})$ is negative because this plot is made by using the dimensionless energy $\tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}}$ in the case of circular orbits. This leads to a sign change in the quadratic term in the definition of $Y(\tilde{r})$. In this case we only have 1 possible orbit, which is a circular orbit with radius $\tilde{r}_{turning-point} \approx 0.79599$. The turning points are given by the point where the red dashed lines cross each other. Further we can investigate what happens by adding some energy to the system $\tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}} + \tilde{E}_0$. In this case the dimensionless energy will no longer provide to put both turning points on the same radius and consequently we cannot get circular orbits.

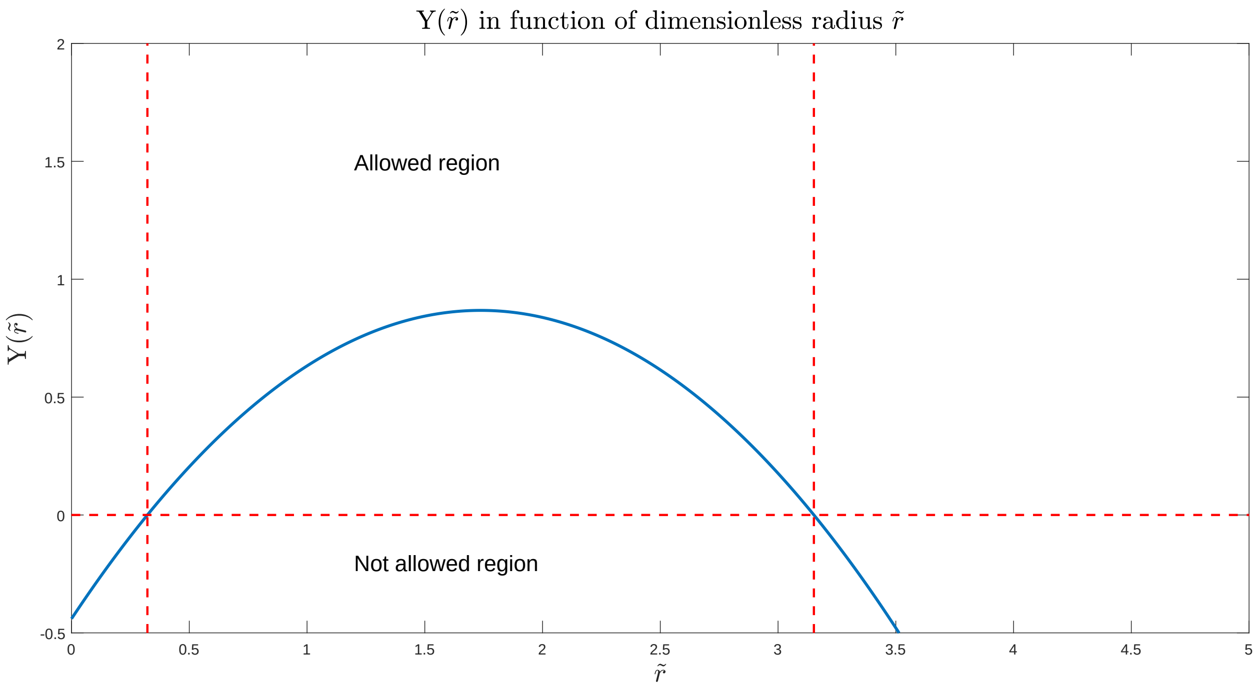


Figure 7: In this case we made use of $\tilde{L} = 1.2$ and $\tilde{E}_0 = 0.2$, which leads to the energy $\tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}} + 0.2 \approx 0.75277$. Here we get the following turning points: $\tilde{r}_{turning-point1} \approx 3.15219$ and $\tilde{r}_{turning-point2} \approx 0.32212$. Clear to see is that we get two turning points which are not at the same radius. This physically means that the only allowed orbit in this case is an ellipse.

Further by adding some more energy, physically means that the particle gains more kinetic energy. This leads to an unbounded orbit where we only get one turning point. We visualize this orbit as a parabolic or hyperbolic Kepler orbit see Fig. 8. (see section 5.10)

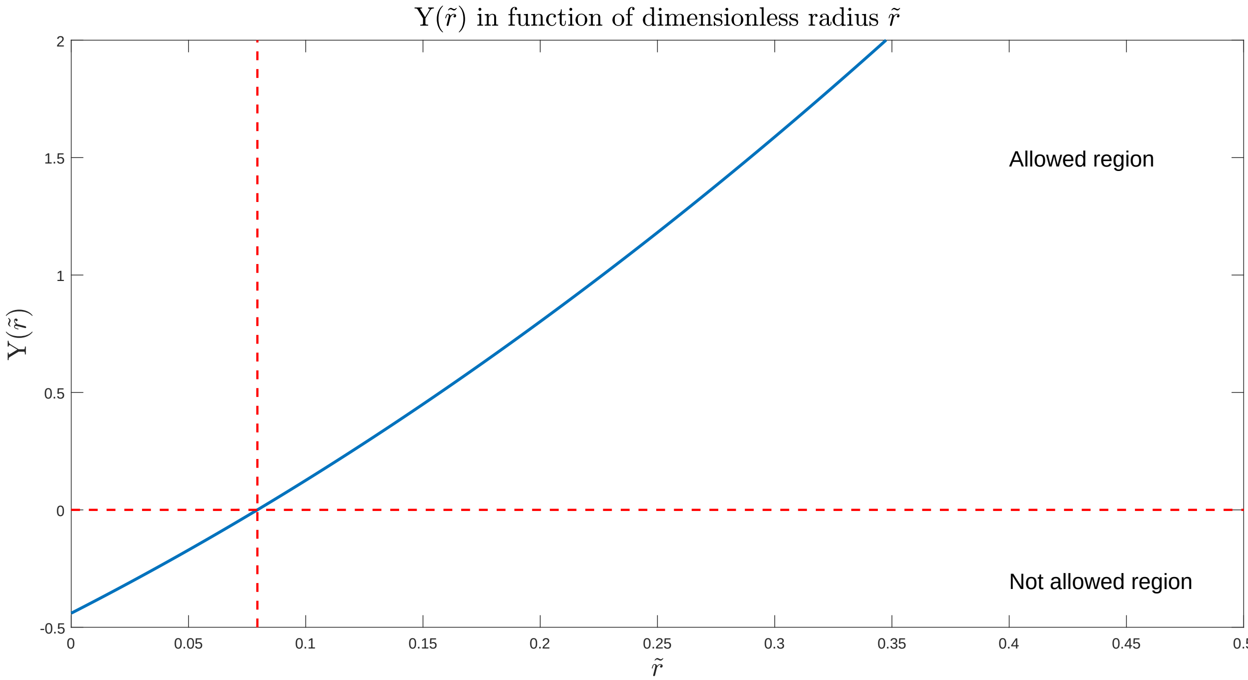


Figure 8: In this case we made use of $\tilde{L} = 1.2$ and $\tilde{E}_0 = 2$, which leads to the energy $\tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}} + 2 \approx 2.55277$. Here we get the following turning points: $\tilde{r}_{turning-point1} \approx 0.07937$ and $\tilde{r}_{turning-point2} \approx -1.00485$. Clear to see is that we get two turning points which are not at the same radius. Of course the negative turning point leads to a non physical solution. This means physically that the only allowed orbit in this case is an hyperbolic/parabolic orbit.

5.5 Turning points analysis for a non-relativistic particle

In this section the turning points of a non-relativistic particle in $V(r) = -\alpha_1/r$ potential are investigated. Of course by taking the non-relativistic limit ($1/c \to 0$) of the function $Y(r) = (E^2-m^2c^4)r^2+2E\alpha_1 r+(\alpha_1^2-L^2c^2)$ we find a non-relativistic version of it $y(r) = E_{nr}r^2+\alpha_1 r-L_{nr}/2m$ [5]. Important to notice is that the analysis in the case of both particles is the same and we present this analysis in order to be complete. Here the turning points are expressed as:

$$r_{turning-point} = \frac{-\alpha_1 \pm \sqrt{1+\frac{2E_{nr}L_{nr}^2}{m\alpha_1^2}}}{2E_{nr}^2}. \tag{40}$$

By making use of dimensionless parameters $\tilde{E}_{nr} = E_{nr}/mc^2 = -(1/2\tilde{L}_{nr}^2)$, $\tilde{r} = r/\alpha_1/mc^2$ and $\tilde{L}_{nr} = L_{nr}c/\alpha_1$ we rewrite the found expressions. The function y(r) becomes:

$$\tilde{y}(\tilde{r}) = \tilde{E}_{nr}\tilde{r}^2 + \tilde{r} - \tilde{L}_{nr}/2. \tag{41}$$

With $\tilde{y} = y(mc^2/\alpha_1^2)$. The turning points becomes:

$$\tilde{r}_{turning-point} = \frac{-1 \pm \sqrt{1+2\tilde{E}_{nr}\tilde{L}_{nr}^2}}{2\tilde{E}_{nr}^2}. \tag{42}$$

Now we can visualize the function $\tilde{y}$ as function of the dimensionless radius $\tilde{r}$.

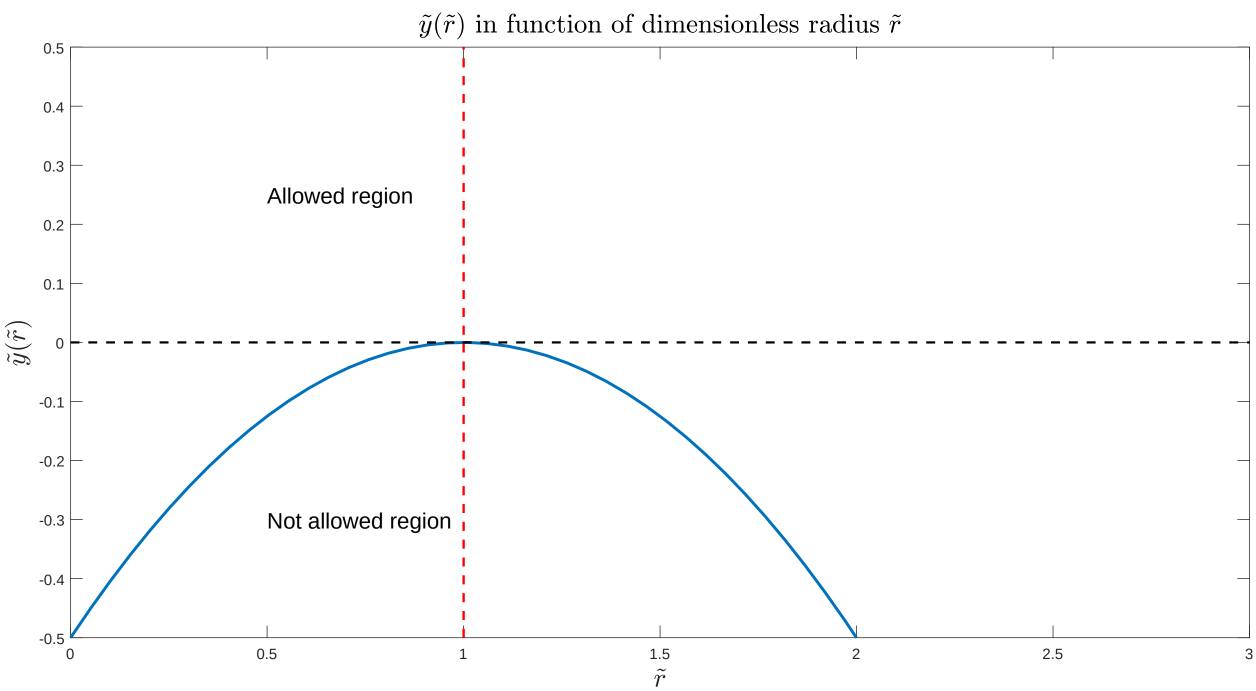


Figure 9: In this case we made use of $\tilde{L} = 1$ and $\tilde{E} = -(1/2\tilde{L}^2) = -0.5$. Because we used the energy which leads to both turning points at the same radius, we get a circular orbit. The turning points are on the same radius $\tilde{r}_{turning-point} = 1$ and in the plot is given by the point where the black and red dashed lines crosses.

Further we can investigate what happens by adding some energy to the system $\tilde{E} = -\frac{1}{2\tilde{L}^2} + \tilde{E}_0$. In this case the dimensionless energy will no longer result to have both turning points at the same radius and consequently we do not obtain circular orbits. As a result we will get elliptic orbits with two different turning points.

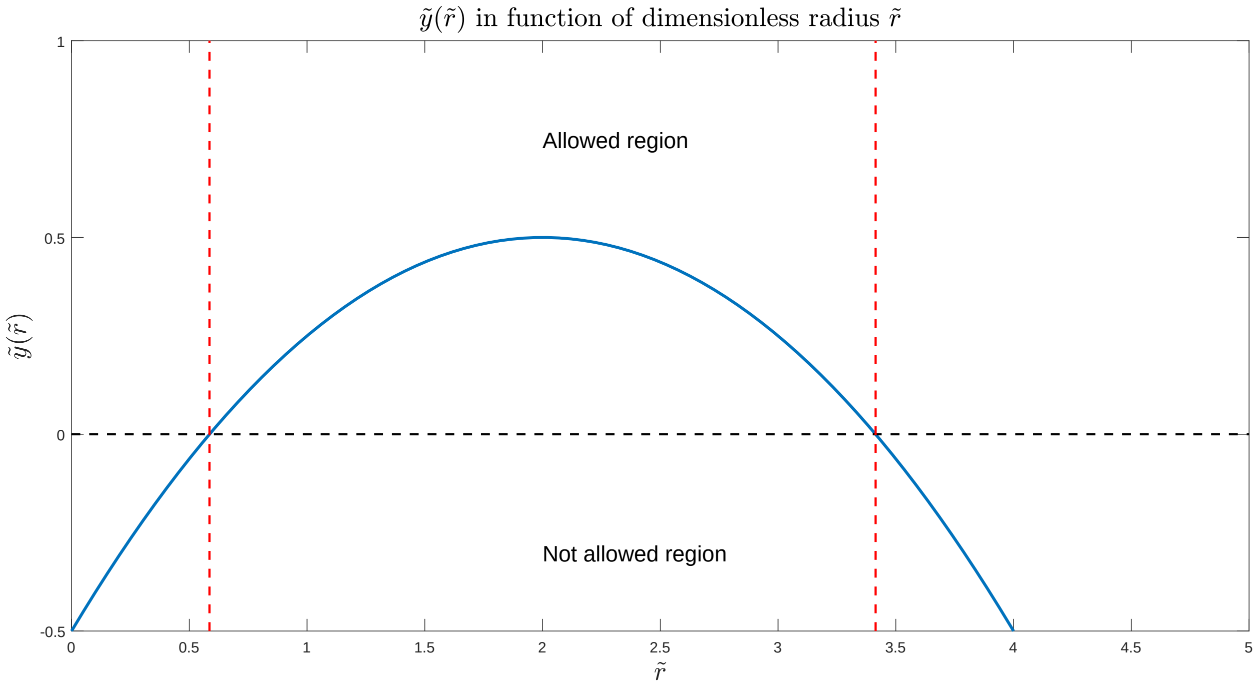


Figure 10: In this case we made use of $\tilde{L} = 1$ and $\tilde{E}_0 = 0.25$, which leads to the energy $\tilde{E} = -\frac{1}{2\tilde{L}^2} + 0.25 = -0.25$. Here we get the following turning points: $\tilde{r}_{turning-point1} \approx 0.58579$ and $\tilde{r}_{turning-point2} \approx 3.41421$.

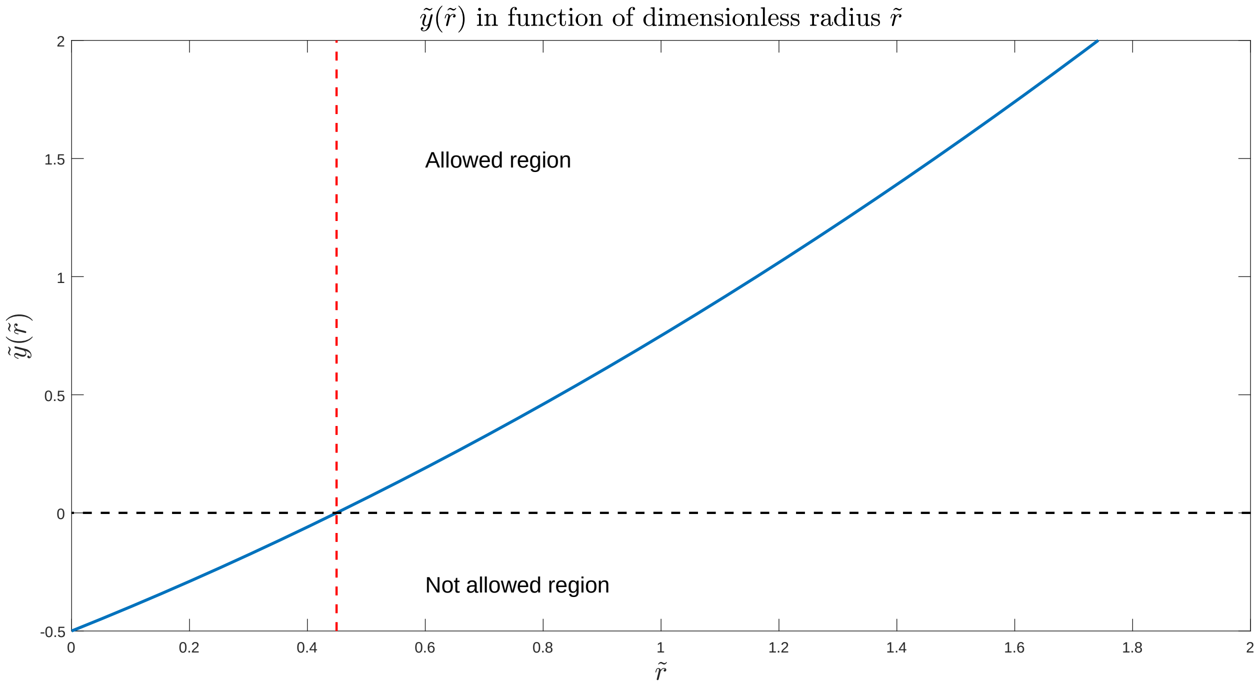


Figure 11: In this case we made use of $\tilde{L} = 1$ and $\tilde{E}_0 = 0.75$, which leads to the energy $\tilde{E} = -\frac{1}{2\tilde{L}^2} + 0.75 = 0.25$. Here we get the following turning points: $\tilde{r}_{turning-point1} \approx 0.44949$ and $\tilde{r}_{turning-point2} \approx -4.44949$. Clear to see is that we get two turning points which are not at the same radius. Of course the negative turning point leads to a non physical solution. This physically means that the only allowed orbit in this case is when we have one turning point like in the case of an hyperbolic/parabolic orbit.

5.6 Orbit equations

We can find the orbit equations $r(\theta)$ by expressing the momentum in polar coordinates:[5]

$$\mathbf{p} = \widehat{r}p_r + \widehat{\theta}p_\theta = \frac{m(\dot{r}\widehat{r} + \widehat{\theta}r\dot{\theta})}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m\dot{r}\widehat{r}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m\widehat{\theta}r\dot{\theta}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m\dot{r}\widehat{r}}{\sqrt{1-\frac{v^2}{c^2}}} + \widehat{\theta}\frac{L}{r}. \tag{43}$$

The last equality is found by making use of Eq.(27). Because we will need later the following expression, we calculate p_r/p_θ.[5]

$$\frac{p_r}{p_\theta} = \frac{\frac{m\dot{r}}{\sqrt{1-\frac{v^2}{c^2}}}}{\frac{mr\dot{\theta}}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{1}{r}\frac{\dot{r}}{\dot{\theta}} = \frac{1}{r}\frac{dr}{d\theta}. \tag{44}$$

By using $p_\theta = L/r$ found in expression (43) we finally get: [5]

$$p_r = \frac{L}{r^2}\frac{dr}{d\theta}. \tag{45}$$

By using expression (43) we can rewrite the relativistic energy expressed in (12) up to $E = \sqrt{p^2c^2 + m^2c^4} - \alpha_1/r$. (see Appendix F) Now we can rewrite the expression $E = \sqrt{p^2c^2 + m^2c^4} - \alpha_1/r$ up to: [5]

$$\left(E + \frac{\alpha_1}{r}\right)^2 = p^2c^2 + m^2c^4 = \underbrace{\frac{m^2\dot{r}^2}{1-\frac{v^2}{c^2}}}_{=(p_r)^2}c^2 + \frac{L^2c^2}{r^2} + m^2c^4. \tag{46}$$

Combining (46) with (45) gives the following differential equation: [5]

$$\left(E + \frac{\alpha_1}{r}\right)^2 = \left(\frac{L}{r^2}\frac{dr}{d\theta}\right)^2 c^2 + \frac{L^2c^2}{r^2} + m^2c^4. \tag{47}$$

By introducing the r-inverse variable $u = 1/r \Rightarrow du/d\theta = -(1/r^2)dr/d\theta$ we can rewrite the differential equation in (47) up to: [5]

$$\left(E + \alpha_1 u\right)^2 = \left(\frac{du}{d\theta}\right)^2 L^2c^2 + u^2L^2c^2 + m^2c^4. \tag{48}$$

If we differentiate (48) with respect to θ we find:[5]

$$\alpha_1 E\frac{du}{d\theta} + \alpha_1^2\frac{du}{d\theta}u = L^2c^2u\frac{du}{d\theta} + L^2c^2\frac{d^3u}{d^3\theta}. \tag{49}$$

By dividing (49) by $du/d\theta$ and rewriting the equation, we find a second order linear differential equation which has as solution the orbits of the relativistic particle. [5]

$$\frac{d^2u}{d^2\theta} + \left[1 - \left(\frac{\alpha_1}{Lc}\right)^2\right]u - \frac{E\alpha_1}{(Lc)^2} = 0. \tag{50}$$

It is clear that we get different solutions depending on the value of the angular momentum L. If $L = \alpha_1/c$ we need to solve the following differential equation: [5]

$$\frac{d^2u}{d^2\theta} - \frac{E}{\alpha_1} = 0. \tag{51}$$

The solution of (51) is easily found in this way:[5]

$$u(\theta) = \int_{\theta_0}^{\theta}\int_{\theta_0}^{\theta}\frac{E}{\alpha_1}d^2\theta = \frac{E}{2\alpha_1}(\theta-\theta_0)^2 + c'. \tag{52}$$

The constant c' is calculated by introducing the solution $u(\theta)$ in (48) (for $L = \alpha_1/c$) this leads to $c' = (m^2c^4 - E^2)/(2E\alpha_1)$. [5] This means that the complete solution of Eq.(51) is given by:

$$u(\theta) = \frac{E}{2\alpha_1}(\theta-\theta_0)^2 + \frac{m^2c^4 - E^2}{2E\alpha_1}. \tag{53}$$

In the case of $L > \alpha_1/c$ we can solve (50) by finding the characteristic equation for the homogeneous solution, and for the particular solution we can use the Wronskian. We get the following general solution:

$$u(\theta) = c_1 \cdot e^{i\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)} + c_2 \cdot e^{-i\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)} + \frac{E\alpha_1}{L^2c^2-\alpha_1^2}. \tag{54}$$

By taking the real part of the homogeneous solution, (54) is simplified up to:

$$u(\theta) = (c_1+c_2)\cdot cos\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right) + \frac{E\alpha_1}{L^2c^2-\alpha_1^2}. \tag{55}$$

By introducing (55) in (48) we compute $c_1 + c_2$. It appears to be $c_1 + c_2 = \sqrt{\frac{E^2L^2c^2-m^2c^4(L^2c^2-\alpha_1^2)}{(L^2c^2-\alpha_1^2)^2}}$ (see Appendix G), in this way we find the orbit equation in the case of $L > \alpha_1/c$:

$$u(\theta) = \sqrt{\frac{E^2L^2c^2-m^2c^4(L^2c^2-\alpha_1^2)}{(L^2c^2-\alpha_1^2)^2}}\cdot cos\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right) + \frac{E\alpha_1}{L^2c^2-\alpha_1^2}. \tag{56}$$

Finally we find the solution for the orbit equation in the case of $L < \alpha_1/c$. In this case (50) becomes

$$\frac{d^2u}{d^2\theta} - \left[\left(\frac{\alpha_1}{Lc}\right)^2 - 1\right]u - \frac{E\alpha_1}{(Lc)^2} = 0. \tag{57}$$

Which leads to the solution:

$$u(\theta) = \sqrt{\frac{E^2L^2c^2+m^2c^4(\alpha_1^2-L^2c^2)}{(\alpha_1^2-L^2c^2)^2}}\cdot cosh\left(\sqrt{(\frac{\alpha_1}{Lc})^2-1}(\theta-\theta_0)\right) - \frac{E\alpha_1}{\alpha_1^2-L^2c^2}. \tag{58}$$

Because it is easier to work with dimensionless expressions we make use of Appendix B to rewrite the found $u(\theta)$ solutions in dimensionless form. Note that this part is novel and is not reported in [5]

$$u(\theta) = \frac{mc^2}{\alpha_1}\sqrt{\frac{\tilde{E}^2\tilde{L}^2-(\tilde{L}^2-1)}{(\tilde{L}^2-1)^2}}\cdot cos\left(\sqrt{1-\frac{1}{\tilde{L}^2}}(\theta-\theta_0)\right) + \frac{\tilde{E}}{\tilde{L}^2-1}\frac{mc^2}{\alpha_1}. \tag{59}$$

With $\tilde{E} = E/mc^2$, $\tilde{L} = Lc/\alpha_1$. We see here that the scale of the solution of the orbit equations $u(\theta)$ is determined by mc^2/α_1 which by consulting Appendix B we see that the units of the radius is given in this case by meters. In order to find the begin conditions of the particles we rewrite the relation between angular momentum and radius from Eq.(16) in dimensionless form such that the radius is expressed as $r = \tilde{L}^2\sqrt{1 - \frac{1}{\tilde{L}^2}\frac{\alpha_1}{mc^2}}$.

5.7 Non-relativistic limit: $c \to \infty$

Of course by taking the non-relativistic limit we have to obtain the well known Kepler orbits. Eq.(59) provides us the stable relativistic orbits and if we take the limit $1/c \to 0$ we find the following solution:[5] (see Appendix I)

$$u(\theta) = \frac{m\alpha_1}{L_{nr}^2}\sqrt{1 + \frac{2L_{nr}^2 E_{nr}}{m\alpha_1^2}}cos(\theta - \theta_0) + \frac{m\alpha_1}{L_{nr}^2}. \tag{60}$$

We have to notice that by using the found energy (in the case of circular orbits) in Eq.(36) we find that the square root becomes zero which leads to the vanishing of θ dependency, which indeed means that here we get circular orbits. When the energy becomes less than $-m\alpha_1^2/(2L_{nr}^2)$ part of the solution becomes imaginary which leads to some deviations in trajectories and to non physical solutions. When the energy becomes bigger than $-m\alpha_1^2/(2L_{nr}^2)$ the radius changes in time which leads to ellipses in the case of bound states, and of course to parabolic or hyperbolic orbits in the case of scattering states. Here we define also some quantities such that it makes it easier to visualize our results. We define $\tilde{L}_{nr} = \frac{L_{nr}c}{\alpha_1}$ and $\tilde{E}_{nr} = \frac{E_{nr}}{mc^2}$. In this way the solution in Eq.(60) is rewritten as:

$$u(\theta) = \frac{1}{\tilde{L}_{nr}^2}\frac{mc^2}{\alpha_1}\sqrt{1 + 2\tilde{L}_{nr}^2\tilde{E}_{nr}}cos(\theta - \theta_0) + \frac{1}{\tilde{L}_{nr}^2}\frac{mc^2}{\alpha_1}. \tag{61}$$

Further, we express some found relations in function of these new quantities. The relation between radius and angular momentum found in Eq.(14) becomes:

$$r = \tilde{L}_{nr}^2\frac{\alpha_1}{mc^2}. \tag{62}$$

In this way we find the begin position of the particle when $\theta = 0°$. Here we get $u_0 = 1/r_0$ and from this relation we find the begin polar angle (θ_0) between the x-axis (horizontal axis) and the position vector which joins the particle in the begin position to the origin of the central force. Finally, the energy which puts the turning points at the same radius or in other words the energy which provides to have circular orbits, found in Eq.(36) becomes:

$$\tilde{E}_{nr} = -\frac{1}{2}\left(\frac{1}{\tilde{L}_{nr}^2}\right). \tag{63}$$

Note that we put both solutions, relativistic and non-relativistic on the same scale such that we can compare the results. Note that this last part where we made use of dimensionless parameters is novel and was not reported in [5]

5.8 Solution of orbit equation in non-relativistic case

In this subsection we visualize some solutions of the orbit equations. When we make use of the energy in Eq.(63) we get a circular orbit because the square root in the solution Eq.(61) vanishes.

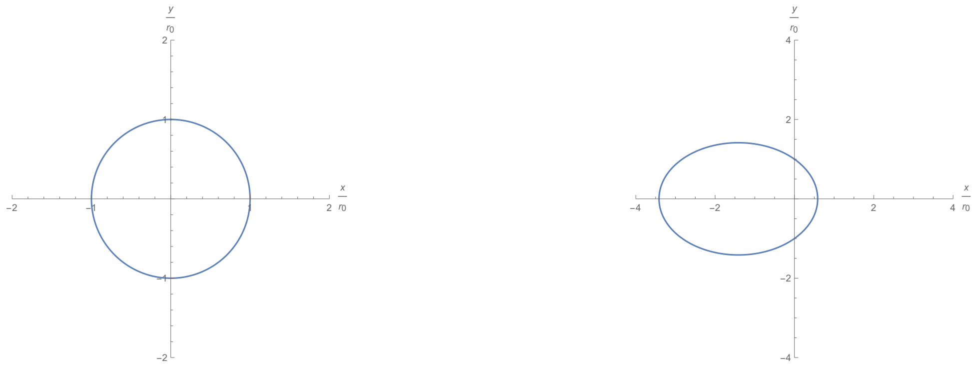


Figure 12: On the left we have a circular orbit made by making use of parameters: $\tilde{L}_{nr}^2 = 1$,$\tilde{E} = -1/(2\tilde{L}_{nr}^2) = -0.5$. On the right we made use of the parameters: $\tilde{L}_{nr}^2 = 1$,$\tilde{E} = -0.25$. In the left case we chose to start in the begin position such that $\theta_0 = 0°$ and the polar angle θ is varied in both cases in [0,2π]. In the right case we calculate the begin polar angle by solving the equation $u_0 = 1/r_0$ for $\theta = 0°$. In this case $r_0 = \tilde{L}^2 = 1\alpha_1/mc^2$. From here we find that $\theta_0 = \pm 1.5708$. The Cartesian coordinates x and y are calculated in meters (m) and also the begin position r_0 is given in meters.

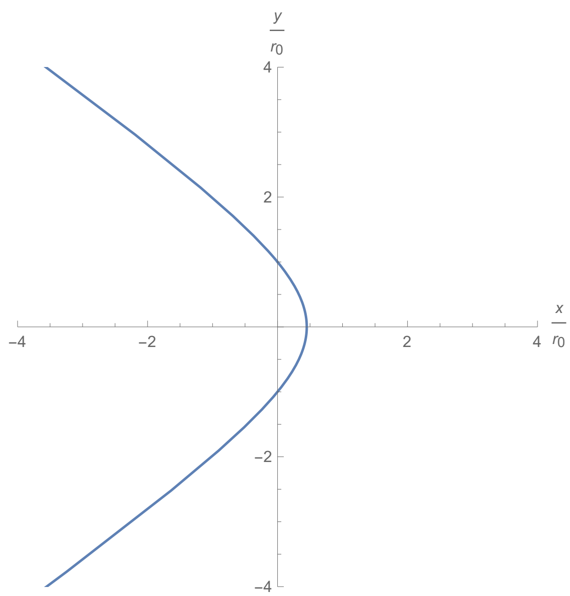


Figure 13: In this case we have a parabolic orbit made by using the following parameters: $\tilde{L}_{nr}^2 = 1$,$\tilde{E} = -1/(2\tilde{L}_{nr}^2) + 0.75 = 0.25$. Like explained before we find the begin position by solving $u_0 = 1/r_0$ in $\theta = 0°$, this leads to $\theta_0 = \pm 1.5708°$ and the polar angle is varied in [0,2π]. The Cartesian coordinates and r_0 are given in meters.

5.9 Phase diagram for a non-relativistic particle in $V(r) = -\alpha_1/r$ potential

In this section we present a dimensionless phase diagram. This is a good review for the derived solutions for a non-relativistic particle in $V(r) = -\alpha_1/r$ potential. Depending on the values of angular momentum and energy we get different trajectories. In the figure the numbers refer to some characteristic orbits that are very similar to the orbits shown in the previous section.

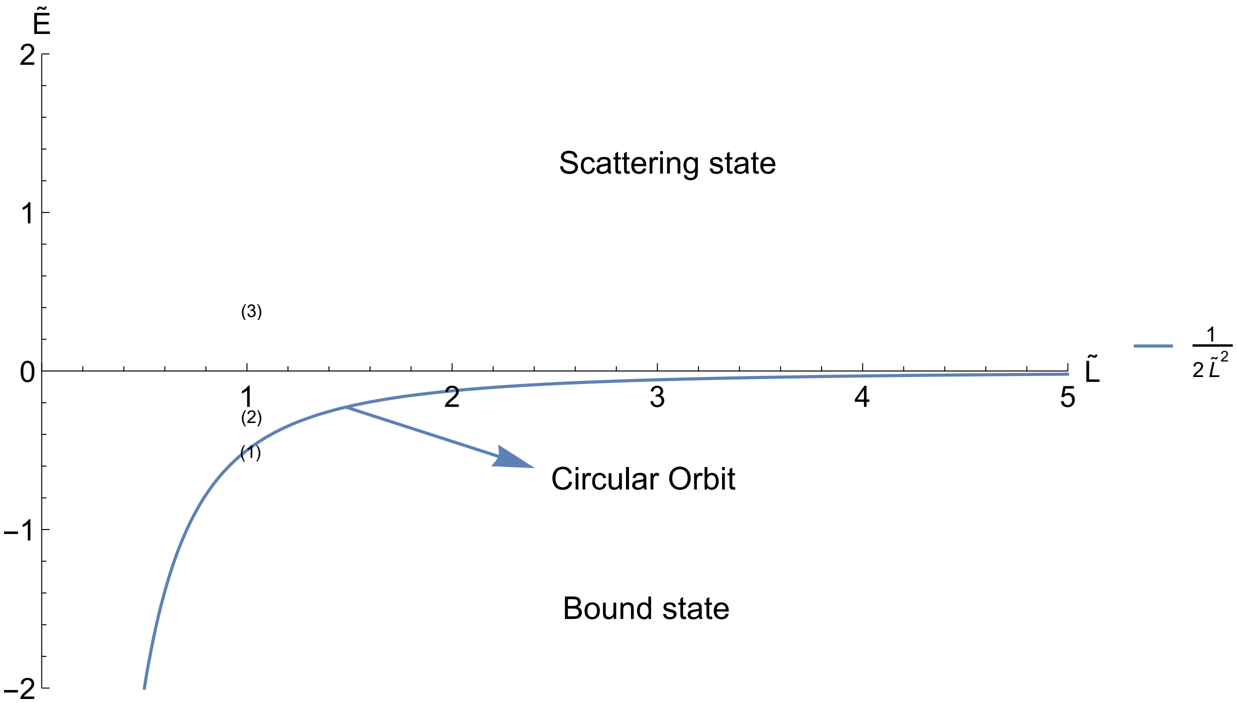


Figure 14: Phase diagram for a non-relativistic particle in $V(r) = -\alpha_1/r$ potential. With $\tilde{L}_{nr} = \frac{L_{nr}c}{\alpha_1}$ and $\tilde{E}_{nr} = \frac{E_{nr}}{mc^2}$. The blue curve is the found relation between dimensionless energy as function of dimensionless angular momentum in the special case of circular orbits found in Eq. (63).

Clear to see from Fig.14 is that there is no critical angular momentum for a non-relativistic particle in $V(r) = -\alpha_1/r$ potential. This means that the velocity in non-relativistic case can take any value from zero to infinity. When we use the energy of circular orbits we get of course a circular orbit because the square root in solution Eq. (61) disappears and consequently we get a constant radius which does not change in time causing a circular orbit. A characteristic orbit for (1) in Fig.14 is given by the left solution in Fig.12. When the energy becomes bigger than $-\frac{1}{2}\frac{1}{\tilde{L}_{nr}^2}$ the particle gains some kinetic energy and the orbit of the particle is going to be affected. Here we get ellipses with of course two different turning points in comparison with the case in (1). A characteristic orbit for (2) is given by the right orbit in Fig.12. Finally, if we add more energy to the system the particle gets more kinetic energy which will provide for an escaping from the attractive force. These are the scattering states, which means that the particle gets trapped by the central force and afterwards changes its trajectory and escapes from the potential. A characteristic orbit for (3) is given by Fig.13. Note that this section was novel and was not reported in [5].

5.10 Solutions of orbit equations in relativistic case

From the above results we can deduce that the relativistic particle will have different behaviors for different values of the angular momentum. By solving the orbit equations in polar coordinates (r,θ) and afterwards transforming these solutions in Cartesian coordinates (x,y), we find five different trajectories (bound and unbound orbits) for a relativistic particle. Further by making a dimension analysis we find that the radius is given in meters and consequently the solutions given in Cartesian coordinates x and y are given in meters. (see Appendix H)

5.10.1 L$\gg \alpha_1$/c and E>mc^2

The fact that the angular momentum is much larger than the critical angular momentum means that the speed of the particle is near the speed of light. Because the energy is rewritten such that $E = \epsilon + mc^2$, with ϵ the energy difference from the particle rest energy (mc^2), we deduce that $E > mc^2$ means that the kinetic energy term from ϵ is in this case bigger than the potential energy term. This fact leads to a scattering state of the particle. In the figure below we see that the particle comes from infinity and gets trapped by the central force situated in the origin. The particle will turn around the origin and afterwards is scattered back to infinity. This trajectory is similar to the parabolic Kepler orbit, see Fig. 15.

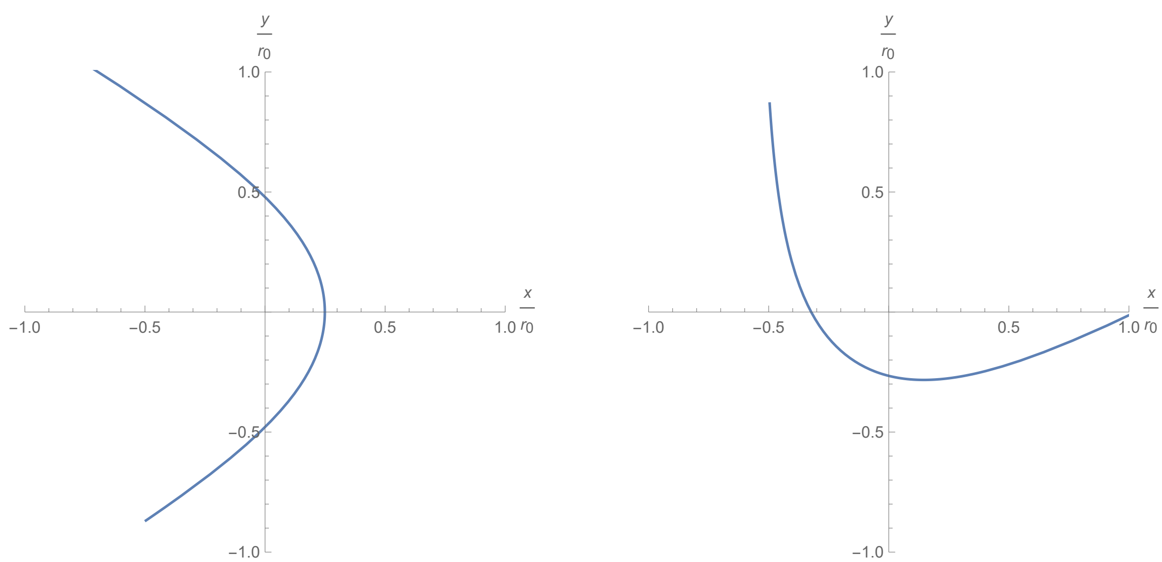


Figure 15: Solution of orbit equation for parameters: $\tilde{L} = 2$,$\tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}} + 1 \approx 1.36603$. The polar angle is varied in $\in [0\pi, 1.5\pi]$. The begin radius r_0 is given in meters and the Cartesian coordinates are calculated in meters. The left orbit begins in polar angle $\theta_0 \approx 2.08967°$ and the right orbit in $\theta_0 \approx -2.08967°$.

The begin conditions of the particle are easily found in the following manner. At the beginning by inserting angular momentum $\tilde{L} = 2$ we find the begin position of the particle which is found by using the relation between angular momentum and radius $r = \tilde{L}^2\sqrt{1 - \frac{1}{\tilde{L}^2}}\frac{\alpha_1}{mc^2}$. Here we get $r_0 = 3.4641\alpha_1/mc^2$. Further, because the angular momentum in this case is larger than the critical angular momentum we make use of the solution given in Eq. (59) in order to find the begin polar angle θ_0. By inserting the begin polar angle $\theta = 0$, we find:

$$u_0 = \frac{1}{r_0} = \frac{mc^2}{\alpha_1}\sqrt{\frac{\tilde{E}^2\tilde{L}^2 - (\tilde{L}^2-1)}{(\tilde{L}^2-1)^2}} \cdot cos\left(\sqrt{1-\frac{1}{\tilde{L}^2}}(-\theta_0)\right) + \frac{\tilde{E}}{\tilde{L}^2-1}\frac{mc^2}{\alpha_1}. \tag{64}$$

By inserting all parameters this equation leads to $\theta_0 = \pm 2.08967°$. Because we get two begin polar angles for the given position r_0, this means that we get two solutions and in other words we get two different orbits. By plotting both solutions we see from Fig. 15 that the only difference from each other is that the particle travels from another begin point on the orbit.

5.10.2 L$\geq \alpha_1$/c and E>mc^2

In the case that $L \to \alpha_1/c$, from infinity, the particle makes loops around the origin of the central force and is scattered back to infinity. The more the angular momentum approaches to the critical value the more the particle makes loops around the origin. From Fig. 4 we deduce that for $L \to \alpha_1/c$ the radius becomes small (the particle is situated around the origin) and from Fig. 5 we deduce $v \to c$. In this case the energy is still larger than the rest energy, this means that the kinetic energy (positive) still dominates. But for some time the particle is trapped by the central force which means that the potential energy for a moment dominates. Afterwards the kinetic energy gets the control and leads to the escaping of the particle from the central force to infinity. Of course this kind of orbit is not allowed for a non-relativistic particle, where only parabolic/hyperbolic orbits are found.

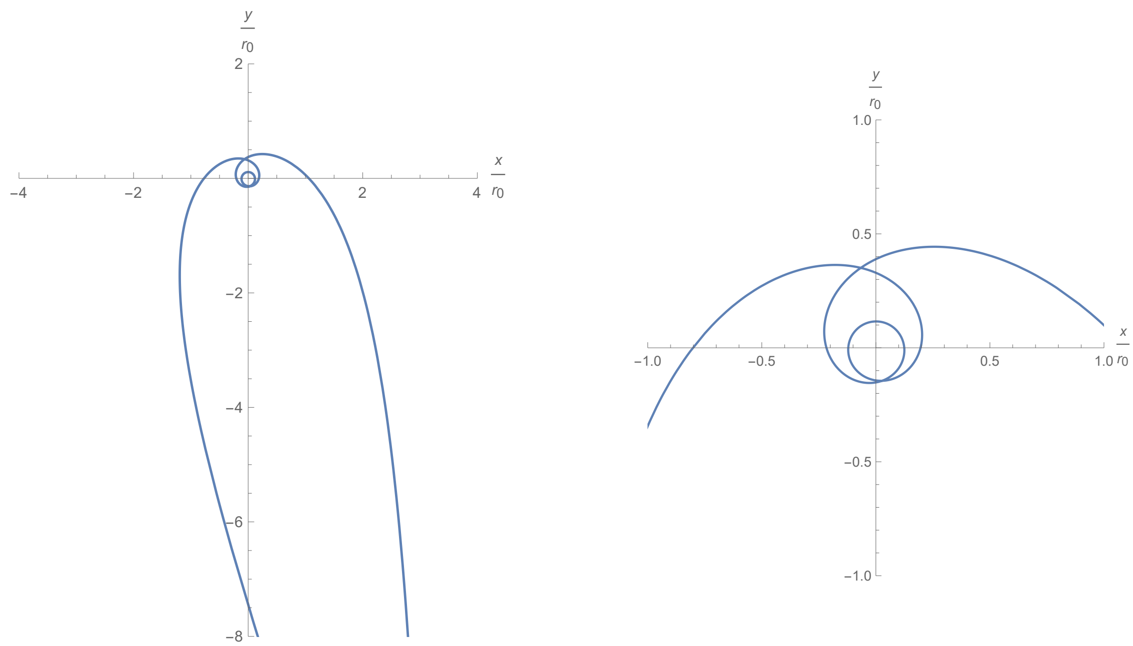


Figure 16: Solution of orbit equation for parameters: $\tilde{L} = 1.05, \tilde{E} = \sqrt{1-\frac{1}{\tilde{L}^2}} + 1 \approx 1.30491$. The polar angle is varied in $\in [0.5\pi, 7\pi]$. The begin radius r_0 is given in meters and the Cartesian coordinates are calculated in meters. The begin polar angle is $\theta_0 \approx -7.97772°$.

5.10.3 L$\geq \alpha_1$/c and E<mc^2

In this case the total energy is smaller than the rest energy. This means that the potential energy dominates in the term ϵ which leads to a bounded orbit, in contrast to the unbounded orbits discussed above. In this case the relativistic particle will loop around the origin in a precessing motion which leads to precessing ellipses around the origin of the central force. This motion is similar to the motion of the planet Mercury where it precesses at the periapsis which leads to an orbit similar as a rosette. Here we need to notice an important fact. When we have stable bounded orbits (those that do not plug into the origin) the potential energy in ϵ takes the control and consequently the total relativistic energy becomes less than the rest mass energy $E = \epsilon + mc^2 < mc^2$. In order to have stable circular orbits the total relativistic energy should be positive (see chapter 5.4), we have the fact that $\epsilon = T + V$ should be bigger than $-mc^2$.

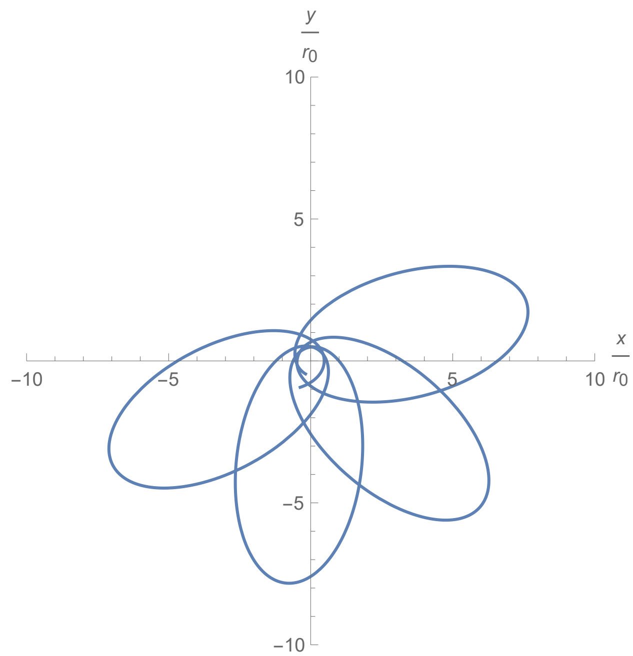


Figure 17: Solution of orbit equation for parameters: $\tilde{L} = 2$,$\tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}} + 0.09999 \approx 0.96603$ and $\theta_0 \approx 1.94899°$. The begin radius r_0 is given in meters and the Cartesian coordinates are calculated in meters. The polar angle is varied in $\in [0\pi, 10\pi]$.

The begin conditions of the particle are easily found in the following manner. At the beginning by inserting angular momentum $\tilde{L} = 2$ we find the begin position of the particle which is found by using the relation between angular momentum and radius $r = \tilde{L}^2\sqrt{1 - \frac{1}{\tilde{L}^2}\frac{\alpha_1}{mc^2}}$. Here we get $r_0 = 3.4641\alpha_1/mc^2$. Further, because the angular momentum in this case is larger than the critical angular momentum we make use of the solution given in Eq.(59) in order to find the begin polar angle θ_0. By inserting the polar angle $\theta = 0$, we find:

$$u_0 = \frac{1}{r_0} = \frac{mc^2}{\alpha_1}\sqrt{\frac{\tilde{E}^2\tilde{L}^2 - (\tilde{L}^2 - 1)}{(\tilde{L}^2 - 1)^2}} \cdot cos\left(\sqrt{1 - \frac{1}{\tilde{L}^2}}(-\theta_0)\right) + \frac{\tilde{E}}{\tilde{L}^2 - 1}\frac{mc^2}{\alpha_1}. \tag{65}$$

By inserting all parameters this equation leads to:

$$\theta_0 \approx \pm 1.94899°. \tag{66}$$

From Eq. (66) we deduce that we get two solutions which means that we get two different orbits. One of them is of course given in Fig. (17). Below we have the solution for $\theta_0 = -1.94899°$.

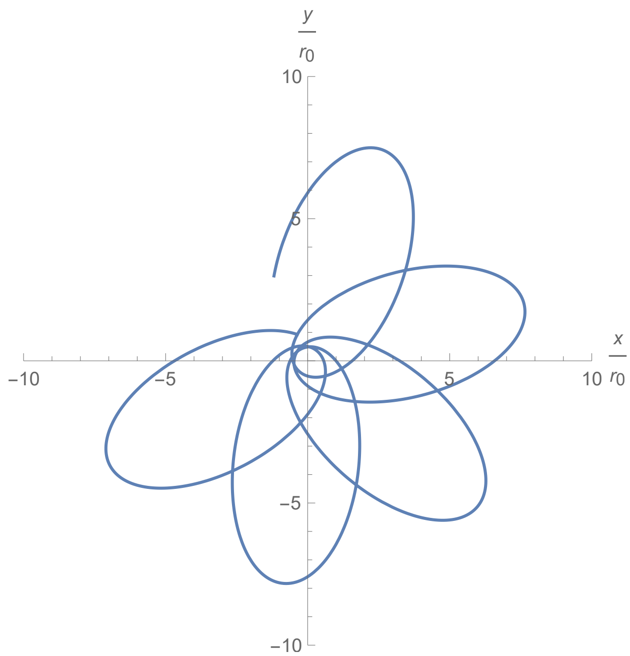


Figure 18: Solution of orbit equation for parameters: $\tilde{L} = 2, \tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}} + 0.09999 \approx 0.96603$ and $\theta_0 \approx -1.94899°$. The polar angle is varied in $\in [0\pi, 10\pi]$.

5.10.4 L< α_1/c and E>mc^2

From Fig.4 we know that the critical angular momentum is equal to α_1/c. But we get interesting results for L< α_1/c. In the figure below the total energy is bigger than the rest energy which means that the kinetic energy takes the control of the motion of the particle and consequently the motion of the particle becomes unbound. The interesting fact is that the particle comes from infinity and approaching to the origin of the central force, this will spiral around the origin and finally when the kinetic energy is totally converted in potential energy the particle will fall into the center of the force. Remember that the total sum of the energy is constant, and like explained before this behavior is possible because of the relation between kinetic and potential energy $T = -U$. Further, by analyzing the energy expression it is easy to understand how the total energy is kept constant in this case. The energy is expressed as such $E_{rel} = mc^2/\sqrt{1 - \frac{v^2}{c^2}} - \frac{\alpha_1}{\mathbf{r}}$. When the particle approaches the origin the radius becomes very small which leads to an increase in negative sense of the potential energy. At the same time when the radius approaches to zero the particle moves faster with v→c (see Fig. 5 and see appendix B Table 7). This fact leads to an increase in positive sens of the kinetic energy which is compensated by the potential energy, in this way keeping the total energy constant.

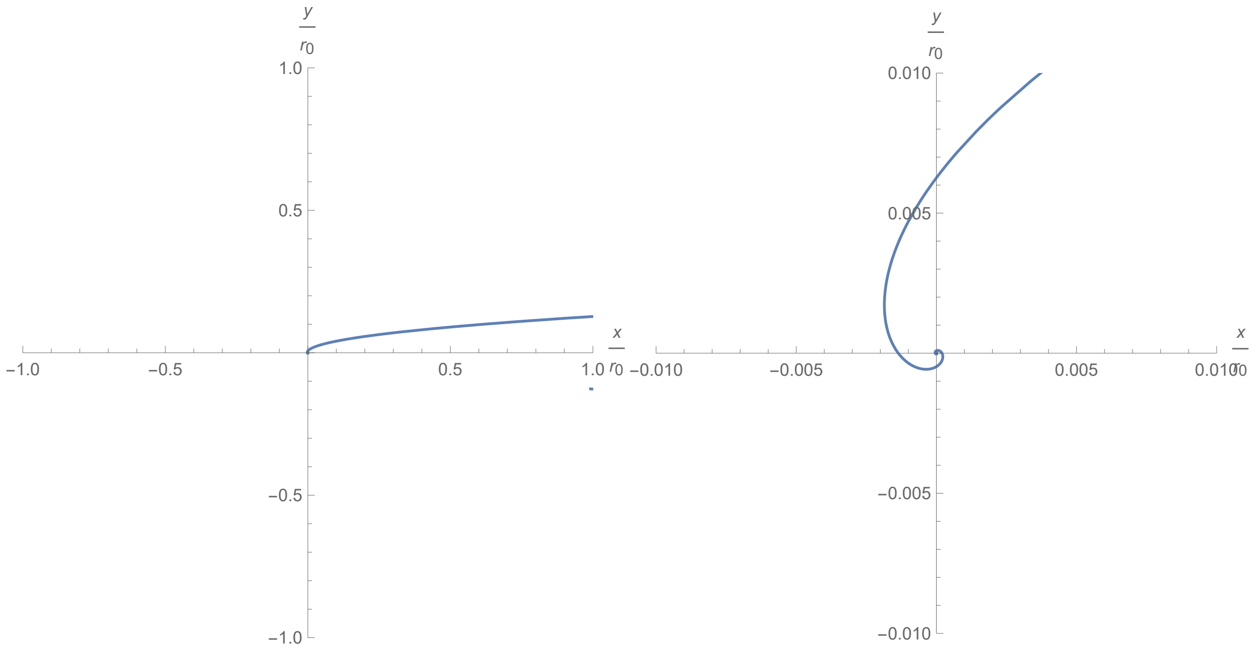


Figure 19: Solution of orbit equation for parameters: $\tilde{L} = 0.9$, $\tilde{E} = 1$ and $\theta_0 \approx 0.127259°$. The polar angle is varied in $\in [0\pi, 10\pi]$. The right figure is an enlargement of the left orbit. By looking closer to the surroundings of the origin of the central force we notice that the particle spirals around it in a few loops. Of course by taking a larger range for the polar angle we will see that the particle will fall into the center. The begin radius r_0 is given in meters and the Cartesian coordinates are calculated in meters.

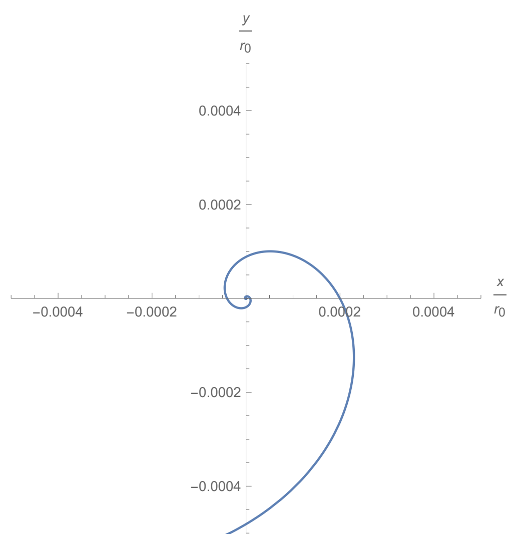


Figure 20: This figure is an enlargement of Fig.19. By looking closer to the surroundings of the origin of the central force we notice that the particle spirals around it in a few loops. Of course by taking a larger range for the polar angle we will see that the particle will fall into the center.

In this case we need to be careful. When the angular momentum is less than the critical angular momentum we cannot use the found relation between angular momentum and radius in order to find the begin position of the particle, because $r = \tilde{L}^2\sqrt{1-\frac{1}{\tilde{L}^2}\frac{\alpha_1}{mc^2}}$ leads to a non physical position. So, what I do in this case, I choose a certain begin position r_0, and from this I find the begin polar angle of the particle. In Fig. 19 I choose to set the begin position of the particle in $r_0 = 100\alpha_1/mc^2$. From this, we make use of the same strategy as in the case of section 5.10.3 and we calculate the begin polar angle:

$$u_0 = \frac{1}{r_0} = \frac{mc^2}{\alpha_1}\sqrt{\frac{\tilde{E}^2\tilde{L}^2 - (\tilde{L}^2-1)}{(\tilde{L}^2-1)^2}} \cdot cos\left(\sqrt{1-\frac{1}{\tilde{L}^2}}(-\theta_0)\right) + \frac{\tilde{E}}{\tilde{L}^2-1}\frac{mc^2}{\alpha_1}. \tag{67}$$

Which leads to $\theta_0 = \pm 0.127259°$. Because the relativistic nature of the particle leads to this interesting results we show graphically that the total relativistic energy is a conserved quantity. Firstly, we find an expression for the velocity of the relativistic particle. Here we assumed that the total relativistic energy is a conserved quantity, so we find from definition of relativistic energy in dimensionless units the velocity: (see Appendix B)

$$\tilde{E} = \frac{1}{\sqrt{1-\tilde{v}^2}} - \frac{1}{\tilde{R}}. \tag{68}$$

$$\Rightarrow \tilde{v} = \sqrt{1 - \frac{1}{(\tilde{E}+\frac{1}{\tilde{r}})^2}}. \tag{69}$$

With $\tilde{r} = r/r_0$. Now by plotting the energy as function of the polar angle θ, we find that the total relativistic energy is conserved.

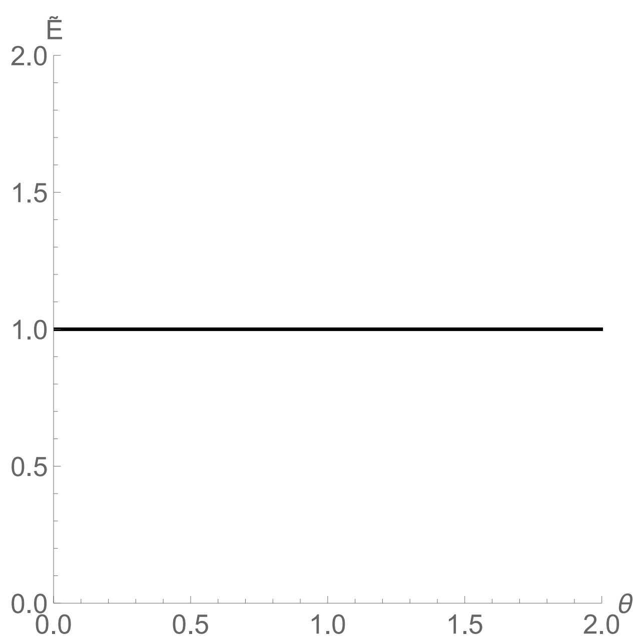


Figure 21: Conserved energy in the case of situation in Fig.19.

Further, we can investigate in this case what happens if we place the particle very close to the origin of the force. In this cases we can use a backward strategy where we first chose a begin polar angle and afterwards we find the begin position of the particle in Cartesian coordinates. We choose to put the begin angle $\theta_0 = 10°$ which leads to $r_0 = 0.00639319\alpha_1/mc^2$.

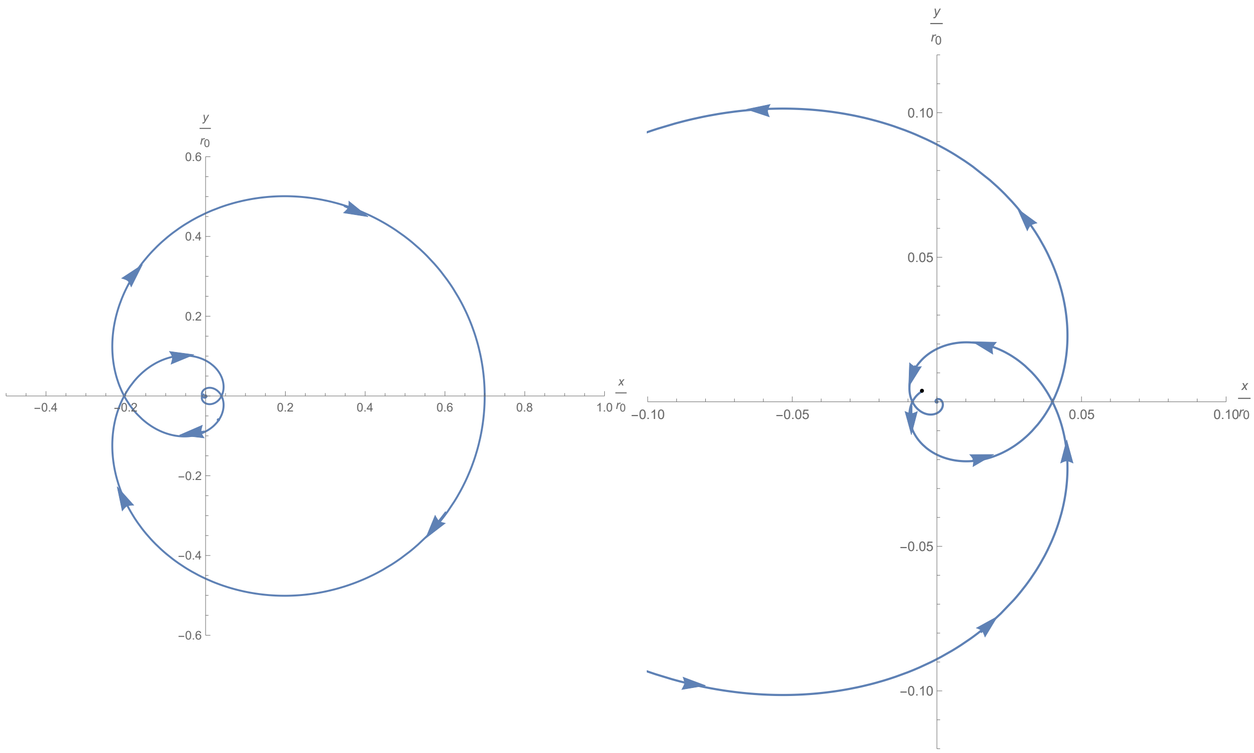


Figure 22: Solution of orbit equation for parameters: $\tilde{L} = 0.9$, $\tilde{E} = 0.2$ and $\theta_0 \approx 10°$. The polar angle is varied in $\in [0\pi, 10\pi]$. The right figure is an enlargement of the left orbit. By looking closer to the surroundings of the origin of the central force we notice that the particle spirals around it in a few loops. Of course by taking a larger range of the polar angle we will see that the particle will fall into the center. The begin radius r_0 is given in meters and the Cartesian coordinates are calculated in meters.

Here I draw some arrows just to visualize better the behavior of the particle. The start point of the particle is to see in the right figure of Fig. 22 and is given by the black dot. From the start position of the particle we see from Fig. 22 that the particle spirals out of the origin and afterwards makes a big loop around it and finally spirals into the center. Of course this behavior together with the behavior found in Fig. 19 are not possible in non-relativistic cases, because this is a pure effect of the relativistic nature of the particle.

5.10.5 Solution of orbit equations and turning points

As last orbit for a relativistic particle in $V(r) = -\alpha_1/r$ potential we have circular orbits. Here we make use of the energy for circular orbits (energy which puts both turning points at the same radius) to make Fig. 23.

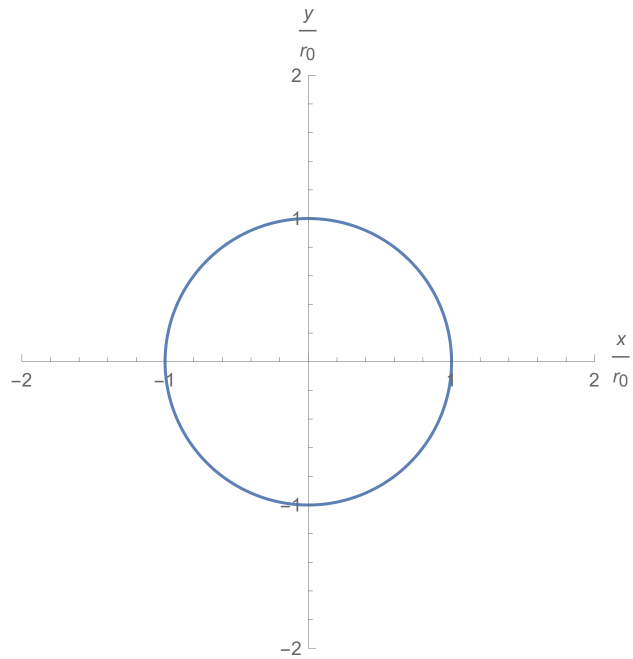


Figure 23: Solution of orbit equation for parameters: $\tilde{L} = 1.1$, $\tilde{E} = \sqrt{1 - \frac{1}{\tilde{L}^2}} = 0.416598$ and $\theta_0 \approx 0°$. The polar angle is varied in $\in [0, 2\pi]$.

5.11 Phase diagram for a relativistic particle in $V(r) = -\alpha_1/r$ potential

In this section a dimensionless phase diagram is presented. This is a good review for the derived solutions for a relativistic particle in a $V(r) = -\alpha_1/r$ potential. Depending on the values of the angular momentum and energy we get different trajectories which most of the time are very different from these of a non-relativistic particle. In the figure the numbers between brackets refer to some characteristic orbits that are very similar to the orbits shown in the previous chapter.

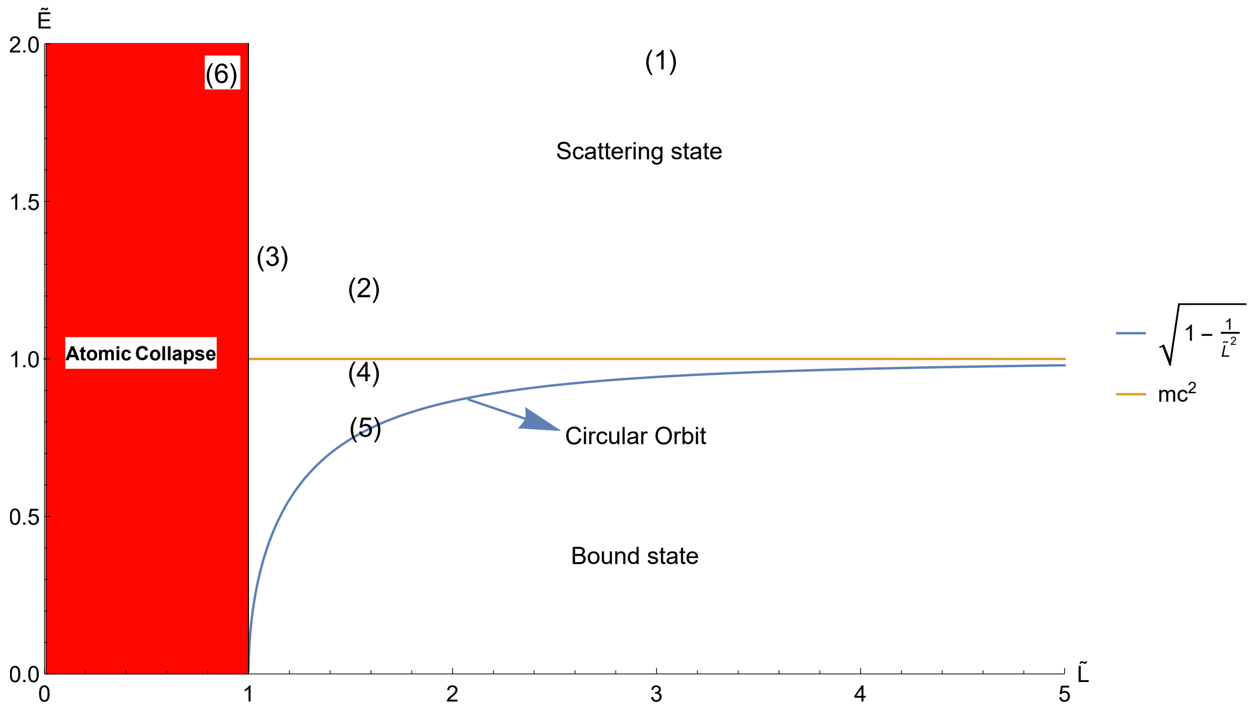


Figure 24: Phase diagram for a relativistic particle in $V(r) = -\alpha_1/r$ potential. With $\tilde{L}_{rel} = \frac{L_{rel}c}{\alpha_1}$ and $\tilde{E}_{rel} = \frac{E_{rel}}{mc^2}$. The blue curve is the found relation between dimensionless energy as function of dimensionless angular momentum in the special case of circular orbits found in Eq. (34).

In Fig. 24 we see that for angular momentum less than the critical angular momentum $\tilde{L} < 1$ we get atomic collapse which means that the particle falls into the origin of the central force. This is a pure consequence of the relativistic nature of the particle. A characteristic orbit for an atomic collapse given by number (6) in Fig. 24 is shown in Fig. 19 where depending on the begin position of the particle we get different solutions such as Fig. 22. Every possible orbit with energy lower than the rest energy is a bounded orbit. On the blue curve we get circular orbits, here an example of a characteristic orbit of (5) in Fig. 24 is found in Fig. 23. If we are in the bound state region and we add some energy to the particle we are in the region of (4) in Fig. 24. In this case we get precessing ellipses around the origin. A characteristic orbit for (4) is given in Fig. 17. When the energy is larger than the rest mass energy we have scattering states. We have seen from the previous results that by approaching the limiting angular momentum the particle performs more and more loops around the origin before escaping the attractive field. A characteristic orbit for (1) is the hyperbolic/parabolic Kepler orbit such as seen in Fig. 15. Finally for (2) and (3) we have a characteristic orbit like Fig. 16.

6 Relativistic particle in $V(r) = -\alpha_2/r^2$ potential

In this chapter we consider a relativistic particle in $V(r) = -\alpha_2/r^2$ potential which is an extension of [5] and contains novel results. We will solve again the equation of motion in the case of circular orbits (in this case $mv^2/(\sqrt{1-(v/c)^2}r) = 2\alpha_2/r^3$) and from this calculation we get the relation between angular momentum and radius. Because it is easier we make this relation dimensionless. Afterwards we will make again a classification of trajectories like we did in 5.4 for a relativistic particle in $V(r) = -\alpha_1/r$ potential such that we find the conditions to have stable bounded/ circular orbits. Finally we will write down the equation of motion for a relativistic particle in $V(r) = -\alpha_2/r^2$ potential and we will study the orbits of the particle for different cases.

6.1 Radius vs angular momentum

In equation (20) in chapter 5.2.3 we obtained a general relation between angular momentum and radius for a general potential $V(r) = kr^n$. In this case by inserting $k = -\alpha_2$ and $n = -2$ we get the relation between angular momentum and radius for a relativistic particle in $V(r) = -\alpha_2/r^2$ potential:

$$\tilde{R} = \frac{r}{\sqrt{\frac{\alpha_2}{mc^2}}} = \frac{\sqrt{2}\tilde{L}}{\sqrt{\tilde{L}^4 - 1}} \tag{70}$$

With $\tilde{L} = L/\sqrt{2m\alpha_2}$. The angular momentum depends now on the mass of the particle. This result is not surprising and is explained in chapter 9. Now we can investigate the relation between angular momentum and radius in Fig. 25:

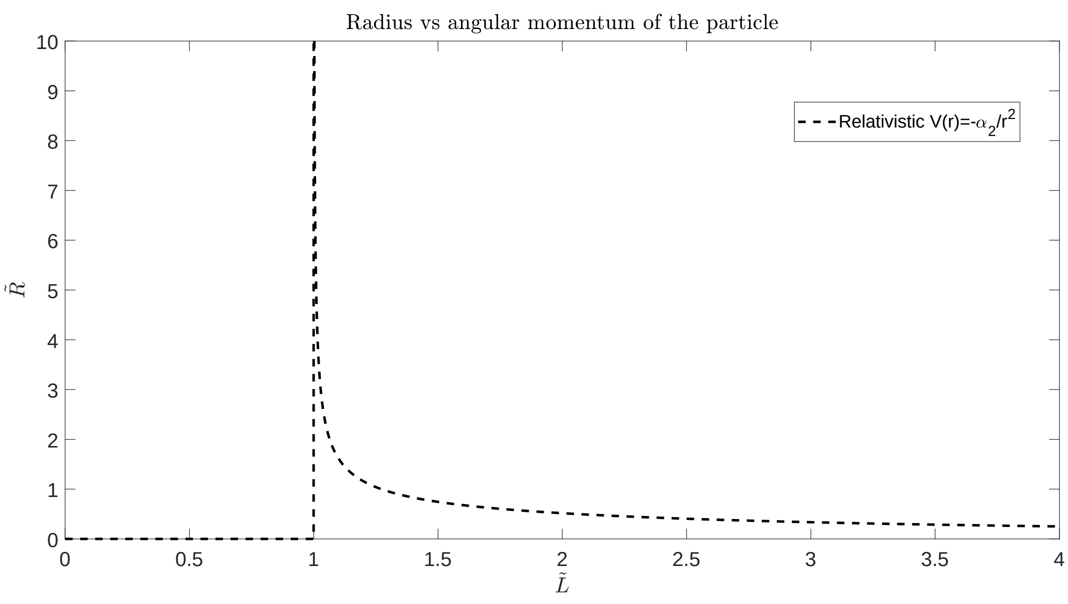


Figure 25: Dimensionless relation between angular momentum and radius.

It is clear from Fig. 25 that the radius diverges when the angular momentum becomes $L = L_c = \sqrt{2m\alpha_2}$. From Fig. 25 we can distinct two different critical angular momenta. The critical angular momentum $L_c^{(1)} = \sqrt{2m\alpha_2}$ will provide for a divergence of the radius while the angular momentum $L \to \infty$ leads to an atomic collapse. For angular momentum less than the critical angular momentum we don't find stable circular orbits while for $L > L_c$ we find stable circular orbits.

6.2 Classification of trajectories

In order to classify which orbit is allowed, we can make the same calculation as in 5.4, based on the fact that the radial velocity should be less than the speed of light $0 \leq \dot{r}^2 < c^2$ with:

$$\dot{r}^2 = c^2 \left[1 - \left(1 + \frac{L^2}{m^2 r^2 c^2} \right) \left(\frac{mc^2}{E + (\alpha_2 / r^2)} \right)^2 \right].$$

From the above restriction we find:

$$-L^2 c^2 r^2 < 0 \leq (E^2 - m^2 c^4) r^4 + r^2 (2\alpha_2 E - L^2 c^2) + \alpha^2 = Y(r).$$

By calculating the zero point of the function Y(r), we find the turning-points where the radial velocity of the particle becomes zero. The fact that the radial velocity of the particle becomes zero implies that the particle is forced to change trajectory in the turning-points:

$$r_{turning,\pm} = \sqrt{\frac{-(2\alpha_2 E - L^2 c^2) \pm \sqrt{(2\alpha_2 E - L^2 c^2)^2 - 4\alpha_2^2 (E^2 - m^2 c^4)}}{2(E^2 - m^2 c^4)}}. \tag{71}$$

Of course we get closed (circular) orbits if the two turning points are at the same radius. The only way to put the two turning-point at the same radius is to find a relation between energy and angular momentum such that: $\sqrt{(2\alpha_2 E - L^2 c^2)^2 - 4\alpha_2^2 (E^2 - m^2 c^4)} = 0$. By working out this expression we get: $E = L^2 c^2 / (4\alpha_2) + m^2 c^2 \alpha_2 / L^2$. From 6.1 we find that the critical angular momentum is equal to $L_c^{(1)} = \sqrt{2m\alpha_2}$. Combining the critical angular momentum with the found expression of the energy as function of L we get: $E = mc^2$. This result combining with expression (71) clarify again why the radius diverges at the critical angular momentum. But when the angular momentum becomes very large and approaches the second critical angular momentum $L_c^{(2)} \to \infty$ we predict that the particle follows stable circular orbits very close to the origin. However, physically the particle cannot fall into the origin with the assumption/conditions to have circular orbits, because otherwise the energy of the particle should be infinite and this is clearly not physically possible. One could make a graphical turning points analysis. Because it is easier to work with dimensionless parameters we make use of $\tilde{E} = E/mc^2$ and $\tilde{L} = L/\sqrt{2m\alpha_2}$ to rewrite the found energy which provides for circular orbits, turning points and the function $Y(r)$. The turning points in dimensionless units are expressed as:

$$\tilde{r}_{turning,\pm} = \frac{r_{turning,\pm}}{\sqrt{\alpha_2 / mc^2}} = \sqrt{\frac{-2(\tilde{E} - \tilde{L}^2) \pm \sqrt{4(\tilde{E} - \tilde{L}^2)^2 - 4(\tilde{E}^2 - 1)}}{2(\tilde{E}^2 - 1)}}. \tag{72}$$

The dimensionless energy in the case of circular orbits is expressed as:

$$\tilde{E} = \frac{E}{mc^2} = \frac{1}{mc^2} \left(\frac{L^2 c^2}{4\alpha_2} + \frac{m^2 c^2 \alpha_2}{L^2} \right) = \frac{L^2}{4m\alpha_2} + \frac{2m\alpha_2}{2L^2} = \frac{\tilde{L}^2}{2} + \frac{1}{2\tilde{L}^2} = \frac{\tilde{L}^4 + 1}{2\tilde{L}^2}. \tag{73}$$

The function $Y(r)$ becomes:

$$\tilde{Y}(\tilde{r}) = (\tilde{E}^2 - 1)\tilde{r}^4 + 2\tilde{r}^2 (\tilde{E}^2 - \tilde{L}^2) + 1. \tag{74}$$

Further we can make the same calculation as made in 5.4 and investigate which relativistic energy is allowed to this system. In 5.4 we get $L - \frac{\alpha_1}{c} < \frac{r}{c}E$ this means that in order to have stable circular orbits the total relativistic energy should be positive because from Fig. 4 we deduce that for $L > \alpha_1/c$ we get stable circular orbits. But if the total energy, in this case, becomes negative we get $\frac{\alpha_1}{c} - L > \frac{r}{c}|E|$. Because all quantities are real and positive we deduce that for stable circular orbits ($L > \alpha_1/c$) the inequality is not satisfied. This means that in order to have stable orbits, the total energy for a relativistic particle in $V(r) = -\alpha_1/r$ potential cannot be negative. The situation for a relativistic particle in $V(r) = -\alpha_2/r^2$ is a bit different. Here the total relativistic energy is expressed as $E = mc^2/\sqrt{1-(v/c)^2} - \alpha_2/r^2$ and consequently $(r/c)\cdot(E + \alpha_2/r^2) = L_{lim} = mcr/\sqrt{1-(v/c)^2}$. Because the limit angular momentum (angular momentum for particles which travels with speed of light) should be bigger than the angular momentum of the particle, we get:

$$L < L_{lim} = L < \frac{r}{c}(E + \alpha_2/r^2). \tag{75}$$

$$\Rightarrow L - \frac{\alpha_2}{rc} < \frac{r}{c}E. \tag{76}$$

Because from Fig. 25 we deduce that for $L > L_c^{(1)}$ we have stable circular orbits and at the same time for high angular momentum the radius becomes small, the terms $L - \frac{\alpha_2}{rc}$ in left hand side of the inequality (76) compensate with each other or becomes negative because for high angular momentum the radius becomes extremely small, see Fig. 25. When the total relativistic energy becomes negative we get an inequality which is satisfied because the same reason as explained in the previous case from inequality (76) in the case of $E > 0$. We get:

$$\Rightarrow \frac{\alpha_2}{rc} - L > \frac{r}{c}|E|. \tag{77}$$

6.3 Energy of a relativistic particle in $V(r) = -\alpha_2/r^2$ potential

In this section we visualize the behavior of the energy of the relativistic particle in $V(r) = -\alpha_2/r^2$ potential. The energy is expressed as:

$$E = \epsilon + mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{\alpha_2}{r^2}. \tag{78}$$

We make it dimensionless:

$$\tilde{E} = \frac{E}{mc^2} = \frac{1}{\sqrt{1 - \tilde{v}^2}} - \frac{1}{\tilde{R}^2}. \tag{79}$$

With: $\tilde{v} = v/c$ and $\tilde{R} = r/\sqrt{(\alpha_2/mc^2)}$. Notice that in this case for a relativistic particle in $V(r) = -\alpha_2/r^2$ potential we are using other units in comparison to the previous cases. Here in order to make radius dimensionless we have $\tilde{R}^2 = r^2/(\alpha_2/mc^2)$ which by making a dimension analysis we have:

$$\frac{r^2}{\frac{\alpha_2}{mc^2}} \sim \left[\frac{m^2}{\frac{\alpha_2}{kg \cdot \frac{m^2}{s^2}}} \right]. \tag{80}$$

From (80) we notice that in order to make $\tilde{R}$ dimensionless we should use kgm^4/s^2 units for α_2. From the equation of motion for circular orbits in this case expressed as $mv^2/(\sqrt{1-(v/c)^2}r) = 2\alpha_2/r^3$ we find a relation between velocity and radius:

$$\frac{mrv}{\sqrt{1 - v^2/c^2}} = L = \frac{2\alpha_2}{rv} \Rightarrow v = \frac{2\alpha_2}{rL}. \tag{81}$$

With L the relativistic angular momentum expressed as $L = mrv\gamma$. We rewrite the expression (81) by inserting the relativistic angular momentum and by making these expressions dimensionless with, $\tilde{v} = v/c$ and $\tilde{R} = r/\sqrt{(\alpha_2/mc^2)}$ such that a relation between dimensionless velocity and radius is found:

$$\tilde{v}^2 = \frac{2}{\tilde{R}^2\gamma} = \frac{2\sqrt{1 - \tilde{v}^2}}{\tilde{R}^2} \Rightarrow \frac{\tilde{v}^2}{2\sqrt{1 - \tilde{v}^2}} = \frac{1}{\tilde{R}^2}. \tag{82}$$

From this relation we have to investigate two limits. When $\tilde{v} \to 0$ the left hand side of Eq. (82) becomes zero which means that the radius becomes infinity. In the case $\tilde{v} \to 1$ the left hand side becomes infinity which means that the radius becomes zero. Now, combining previous results we find dimensionless energy in function of the velocity.

$$\tilde{E} = \frac{1}{\sqrt{1 - \tilde{v}^2}} - \frac{\tilde{v}^2}{2\sqrt{1 - \tilde{v}^2}} = \frac{(2 - \tilde{v}^2)\sqrt{1 - \tilde{v}^2}}{2(1 - \tilde{v}^2)}. \tag{83}$$

Eq. (83) express the dimensionless energy as function of dimensionless velocity for a relativistic particle in $V(r) = -\alpha_2/r^2$ potential in the special case of circular orbits which is graphically shown in Fig. 26.

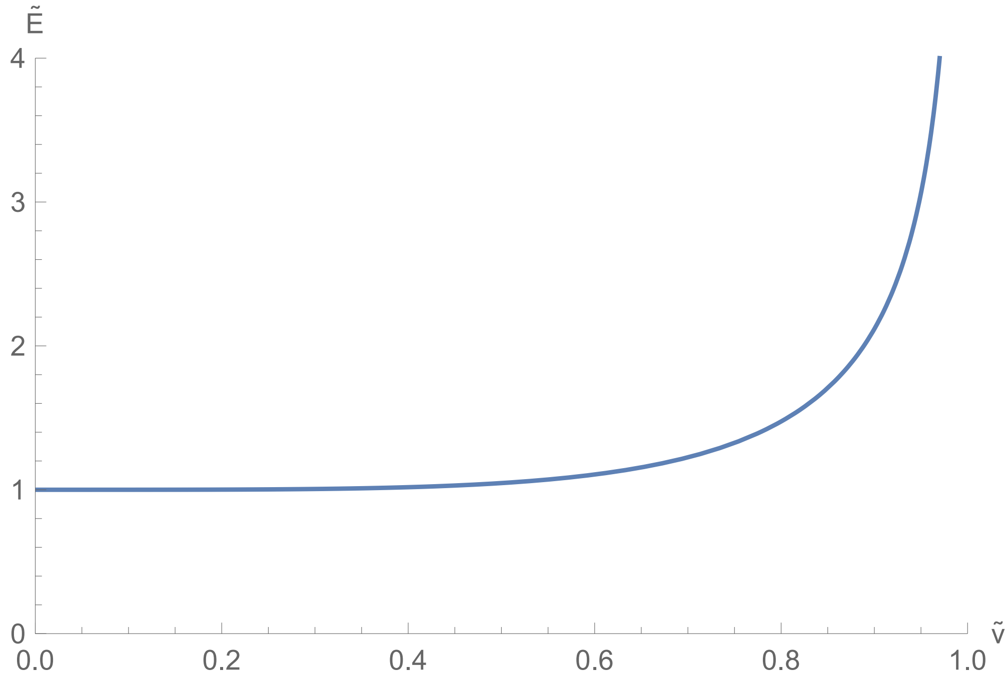


Figure 26: Dimensionless energy as function of dimensionless velocity. The dimensionless velocity is of course constrained in [0,1] because $\tilde{v} = v/c$ the speed of light is the upper limit for the velocity.

From Fig. 26 we see that for small $\tilde{v}$ the energy is constant and as in previous chapter the energy becomes the rest energy of the particle mc^2. For $\tilde{v} \to 1$ we see that the energy diverges. Physically this means that when the particle's velocity is small, kinetic and potential energy compensate each other (note here we are in non-relativistic regime) but at a certain moment when the gamma factor decides to make the difference, this relation breaks and at a certain moment the particle begins to move faster and faster while the radius becomes smaller and smaller. Note that near the region where $\tilde{v} \to 1$ the energy approaches infinity. This fact implies that here we don't get a spiraling motion around the origin while energy is a conserved quantity because in order to have such trajectory the energy should be infinity which is clearly not physically possible. This behavior is very different from the case of a relativistic particle in $V(r) = -\alpha_1/r$. There the γ factor ensures for a compensations between kinetic and potential energy but here we don't get such compensation because the potential energy is more singular and dominates in the energy equation. Note that the relation between kinetic and potential energy is in this case $T = -2U$. Interesting result to mention is that the relativistic particle in $V(r) = -\alpha_2/r^2$ behaves like a non-relativistic particle in $V(r) = -\alpha_1/r$ potential, in the sense that in order to find a spiraling motion where the particle falls into the origin the energy should be infinity. Of course in the case of a non-relativistic particle this only happens when the velocity becomes infinity but it happens here when the velocity is near the speed of light c. Because I scaled each problem, where relativistic and non-relativistic particle are present in different potentials, I plot the energy as function of the velocity for all these problems in a graph, look at Fig. 27. Further, like we did in section 5.3 where we studied the behavior of the energy in the special case of circular orbits of a relativistic and non-relativistic particle in $V(r) = -\alpha_1/r$ potential, we can investigate the relation between kinetic and potential energy in the presence of the potential $V(r) = -\alpha_2/r^2$. In the presence of the potential $V(r) = -\alpha_2/r^2$ the particle experiences a central attractive force equal to $F = -(2\alpha_2/r^3)\mathbf{e}_r$. In order to have stable circular orbits the centripetal force should be equal to the attractive force:

$$\frac{m}{\sqrt{1 - v^2/c^2}} \frac{v^2}{r} = \frac{2\alpha_2}{r^3}. \tag{84}$$

From Eq. (84) we get the relation between kinetic and potential energy of a relativistic particle in the presence of $V(r) = -\alpha_2/r^2$ potential $T = -2U$. By making use of a Taylor expansion taking the non-relativistic limit $(1/c \to 0)$ of Eq. (84) we have:

$$\lim_{\frac{1}{c}\to 0} \frac{m}{\sqrt{1 - v^2/c^2}} \frac{v^2}{r} = \frac{2\alpha_2}{r^3} \Rightarrow \lim_{\frac{1}{c}\to 0} mv^2 \left(1 + \frac{1}{2!}\frac{v^2}{c^2} + \dots\right) = \frac{2\alpha_2}{r^2}. \tag{85}$$

This leads to the non-relativistic relation between kinetic and potential energy in presence of $V(r) = -\alpha_2/r^2$ potential:

$$mv^2 = \frac{2\alpha_2}{r^2} \Rightarrow T = -U. \tag{86}$$

In chapter 7 we will look at a non-relativistic particle in the presence of $V(r) = -\alpha_2/r^2$ potential. As last part of this section we can look at the dimensionless energy as function of dimensionless velocity of the particles in the different case like we did in Fig. (5).

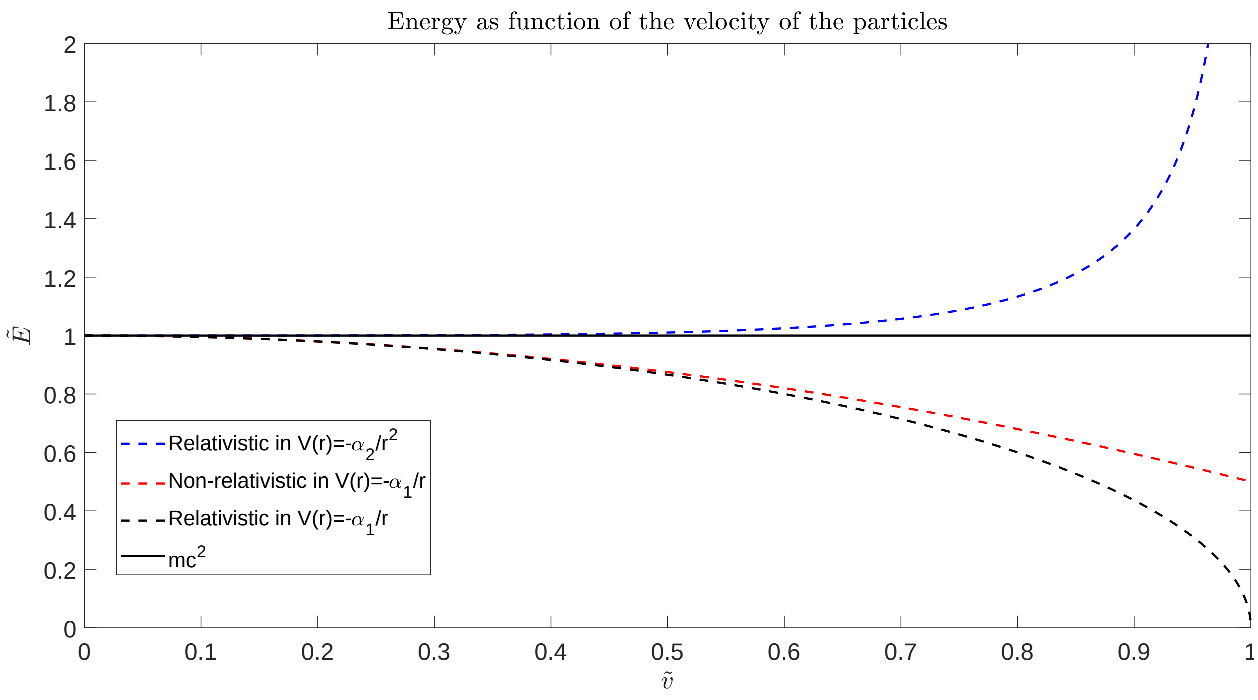


Figure 27: Dimensionless energy as function of dimensionless velocity. The dimensionless velocity is of course constrained in [0,1] because $\tilde{v} = v/c$ the speed of light is the upper limit for the velocity.

6.4 Orbit equation

Now we derive an equation of motion for a relativistic particle in $V(r) = -\alpha_2/r^2$ potential by making use of the same calculation and procedure like we did in 5.6. We make use of equation (48) but the potential is replaced by $V(r) = V(1/u) = -\alpha_2 u^2$:

$$\left(E + \alpha_2 u^2\right)^2 = \left(\frac{du}{d\theta}\right)^2 L^2c^2 + u^2L^2c^2 + m^2c^4. \tag{87}$$

With $u = 1/r \rightarrow du/d\theta = -1/r^2 \cdot dr/d\theta$. If we differentiate the previous expression and afterwards dividing the expression by du/dθ we find:

$$\frac{2E\alpha_2 u}{L^2c^2} + \frac{2\alpha_2^2 u^3}{L^2c^2} = \frac{d^2u}{d\theta^2} + u.$$

Finally we get the equation of motion of a relativistic particle in $V(r) = -\alpha_2/r^2$ potential:

$$\frac{d^2u}{d\theta^2} + u\left(1 - \frac{2E\alpha_2}{L^2c^2}\right) - \frac{2\alpha_2^2u^3}{L^2c^2} = 0. \tag{88}$$

Because it is easier to work with dimensionless units we introduce dimensionless parameters and rewrite Eq. (87) and Eq. (88) in dimensionless units. So we make use of $\tilde{E} = E/mc^2$, $\tilde{L} = L/\sqrt{2m\alpha_2}$, $\tilde{r} = r/\sqrt{\frac{\alpha_2}{mc^2}}$ to rewrite Eq. (87) as:

$$\left(\tilde{E} + \frac{1}{\tilde{r}^2}\right)^2 = 2\left(\sqrt{\frac{\alpha_2}{mc^2}}\frac{1}{r^2}\frac{dr}{d\theta}\right)^2\tilde{L}^2 + \frac{2}{\tilde{r}^2}\tilde{L}^2 + 1. \tag{89}$$

Here we make use of the substitution $\tilde{u} = 1/\tilde{r}$ which leads to:

$$\frac{d\tilde{u}}{d\theta} = \frac{d}{d\theta}\frac{1}{\tilde{r}} = \frac{d}{d\theta}\left(\frac{1}{\frac{r}{\sqrt{\frac{\alpha_2}{mc^2}}}}\right) = -\sqrt{\frac{\alpha_2}{mc^2}}\frac{1}{r^2}\frac{dr}{d\theta}. \tag{90}$$

By substituting Eq.(90) in Eq.(89) we get a dimensionless equation:

$$\left(\tilde{E} + \tilde{u}^2\right)^2 = 2\tilde{L}^2\left(\frac{d\tilde{u}}{d\theta}\right)^2 + 2\tilde{L}^2\tilde{u}^2 + 1. \tag{91}$$

In order to find the orbit equation we make use of the same procedure as before, that is, we differentiate this expression and we divide it by $d\tilde{u}/d\theta$ which leads to the following orbit equation:

$$\frac{d^2\tilde{u}}{d\theta^2} + \tilde{u}\left(1 - \frac{\tilde{E}}{\tilde{L}^2}\right) - \frac{\tilde{u}^3}{\tilde{L}^2} = 0. \tag{92}$$

We will solve this equation of motion numerically. We insert the begin conditions by setting the particle in begin position r_0 at a begin polar angle θ_0 which means that $\tilde{u}_0(\theta_0) = 1/\tilde{r}_0(\theta_0)$. As second condition we use the derived equation based on the fact that energy and angular momentum are conserved quantities $(\tilde{E} + \tilde{u}^2)^2 = 2\tilde{L}^2(d\tilde{u}/d\theta)^2 + 2\tilde{L}^2\tilde{u}^2 + 1$.

We get an interesting result when the energy becomes negative $E \ll -mc^2$. In this case we get repulsive orbits. The particle, approaching the origin of the potential begins to repulse and changes in trajectory. The reason why the particle repulses when the energy becomes $E \ll -mc^2$ can be found in the original equation of motion. If the energy becomes negative we get (see Appendix F for the definition of relativistic energy):

$$-|E| = \sqrt{p^2c^2 + m^2c^4} - \alpha_2/r^2. \tag{93}$$

$$\Rightarrow -(|E| - \alpha_2/r^2) = \sqrt{p^2c^2 + m^2c^4}. \tag{94}$$

$$\Rightarrow (|E| - \alpha_2/r^2)^2 = p^2c^2 + m^2c^4. \tag{95}$$

Finally by filling the impulse in polar coordinates (here we make use of the same procedure from 5.6) we find the following equation of motion:

$$\frac{d^2u}{d\theta^2} + u\left[1 + \frac{2|E|\alpha_2}{L^2c^2}\right] - \frac{2\alpha_2^2u^3}{L^2c^2} = 0.$$

Because of the minus sign in the equation of energy $(|E| - \alpha_2/r^2)^2$ the original potential is repulsive (positive potential term because $\alpha_2 > 0$).

6.5 Numerical solutions of a relativistic particle in $V(r) = -\alpha_2/r^2$ potential

By making use of the derived energy for circular orbits as function of dimensionless angular momentum $\tilde{E} = (\tilde{L}^4 + 1)/2\tilde{L}^2$ and by inserting the dimensionless angular momentum we find the begin position of the particle making use of the found relation $\tilde{R} = \sqrt{2}\tilde{L}/\sqrt{\tilde{L}^4 - 1}$ and we decide to place the particle at a begin polar angle equal to $\theta_0 = 0°$. Note that we make use of the dimensionless equation of motion from Eq. (92) such that we find a dimensionless radius and consequently we have dimensionless Cartesian coordinates $\tilde{x}$ and $\tilde{y}$.

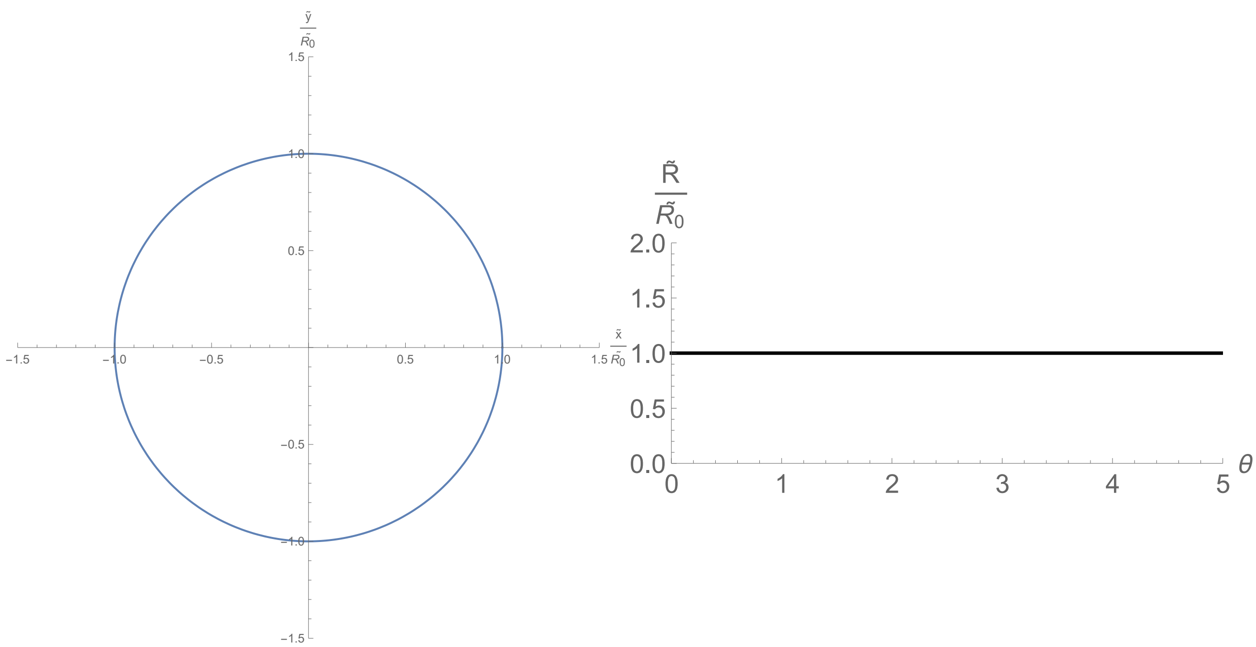


Figure 28: Stable circular orbit of a relativistic particle in $V(r) = -\alpha_2/r^2$ potential with parameters $\tilde{L} = 2$ and $\tilde{E} = 2.125$. The polar angle θ is taken in $\in [0, 2\pi]$ range. By making use of the found relation between dimensionless radius and dimensionless angular momentum we find that the begin dimensionless radius is given by $\tilde{R}_0 = 0.730297$.

Like explained in section 6.3 we expect that by taking a larger and larger angular momentum the particle will follow stable circular orbits around the origin of the potential if the particle is set in the right conditions. This can be graphically shown, see Fig. 29. Here we calculate different circular orbits by inserting a begin angular momentum and by using the right conditions to have a circular orbit, this means the energy which provides for circular orbits and the right begin position of the particle from the origin of the central force. This is very important because the potential in this case is very strong and consequently the singularity is large, this means that when the conditions for circular orbits are not satisfied because the particle for the given energy and angular momentum is not present in the right position, the particle will no longer follow a stable circular orbit but it will follow an unstable orbit such like falling into the origin.

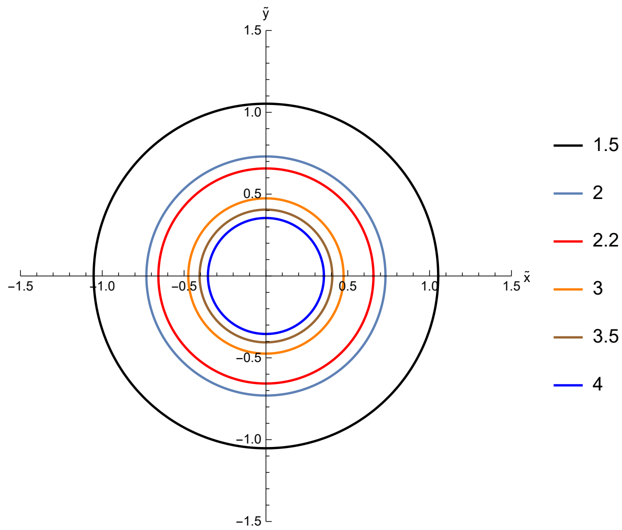


Figure 29: Stable circular orbits for different dimensionless angular momentum which are given by the values in the figure. For these orbits the polar angle is taken in $[0, 2\pi]$ range. From the lowest angular momentum to the highest, we give the dimensionless energy and begin position of the particle: $\tilde{E} = 1.34722$ and $\tilde{R}_0 = 1.05247$, $\tilde{E} = 2.125$ and $\tilde{R}_0 = 0.730297$, $\tilde{E} = 2.52331$ and $\tilde{R}_0 = 0.657$, $\tilde{E} = 4.55556$ and $\tilde{R}_0 = 0.474342$, $\tilde{E} = 6.16582$ and $\tilde{R}_0 = 0.405414$, $\tilde{E} = 8.03125$ and $\tilde{R}_0 = 0.354246$.

Further we can investigate what happens when we exceed the conditions which ensures us to have stable circular orbits. By taking the same parameters as in Fig. 28, $\tilde{L} = 2$ and $\tilde{R}_0 = 0.730297$ and the energy $\tilde{E} = -12.875$, we get stable repulsive orbits which can be physically interpreted as repulsive positrons:

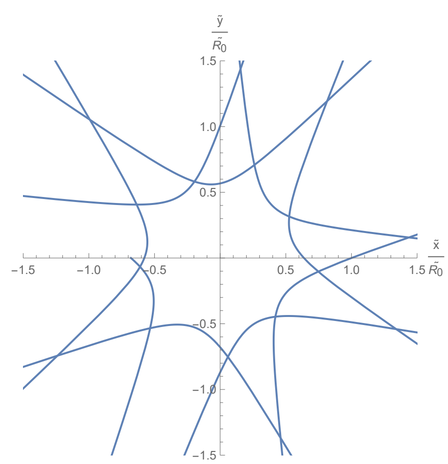


Figure 30: When the energy becomes large in negative sense we get repulsive particles. The polar angle is taken in $[0, 5\pi]$ range. The orbits shown in this figure are orbits where particles comes from infinity and repulses such that the trajectory changes. Because we take a large θ range the particle comes back and we get again a repulsion this means that the figure shows non interacting particles.

Until now we found a circular orbit when the conditions are satisfied and repulsive orbits when the energy becomes large in negative sense. But when we exceed the conditions for circular orbits its possible to get scattering states. This only depends on the begin position of the particle. When we set the particle far away from the origin of the attractive force, the particle, while flowing towards the origin gains some velocity which means that the particle begins to accelerate. The acceleration stops when the particle reaches a quasi balanced bound state which means that the particle is trapped in an almost circular orbit around the origin. The particle stays for some time in this state but because this is a temporary state the particle will escape from the system. Of course when we place the particle somewhere in space not too far from the origin and when there is not enough time to accelerate until the velocity of the particle is large enough to compensate the attractive force, the particle will fall into the origin. These are the unstable scattering orbits.

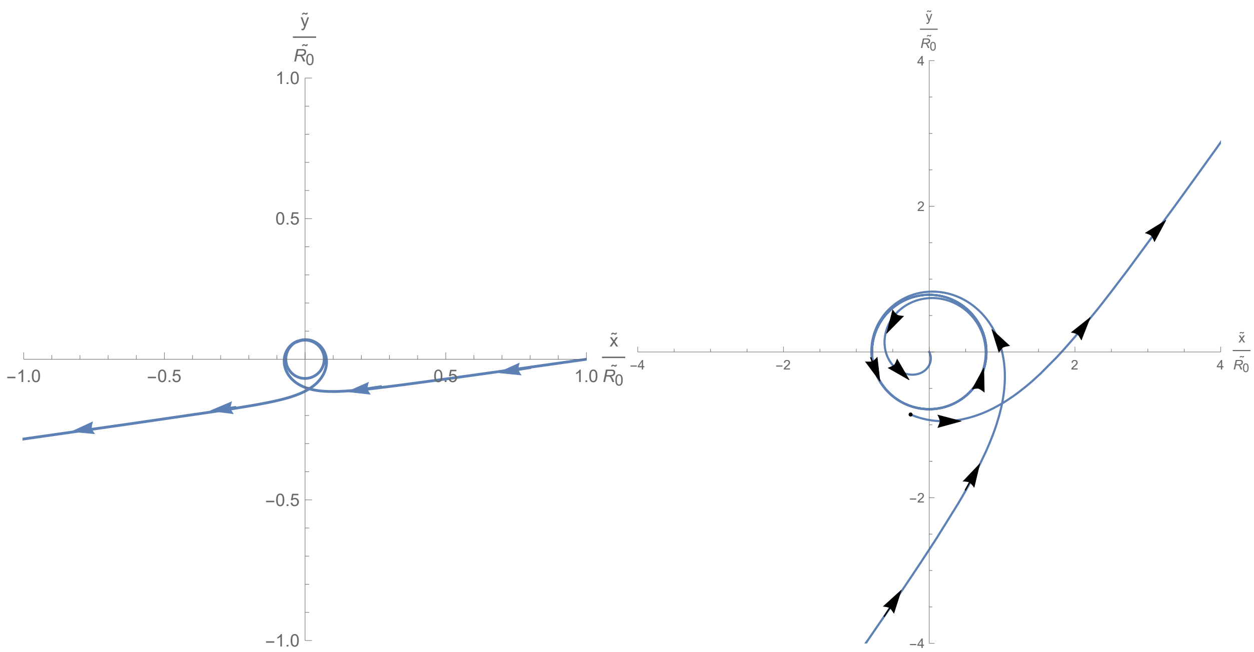


Figure 31: In this figure we get a stable scattering state on the left for the parameters $\tilde{E} = 2.125$ and $\tilde{L} = 2$. The begin position of the particle is set in $\tilde{R}_0 = 10.7303$ at a begin polar angle $\theta_0 = 2°$ and the polar angle is varied in the range in $[0, 5\pi]$. On the right we get an unstable scattering state for the parameters $\tilde{E} = 2.125$ and $\tilde{L} = 2$. The begin position of the particle is set in $\tilde{R}_0 = 0.930297$ and the polar angle is varied in the range $[0, 9.5\pi]$.

6.6 Phase diagram for a relativistic particle in $V(r) = -\alpha_2/r^2$ potential

In this section a dimensionless phase diagram is presented. Depending on the values of angular momentum and energy we get different trajectories. In figure Fig. 32 the numbers refer to characteristic orbits that are very similar to the orbits shown in the previous chapter.

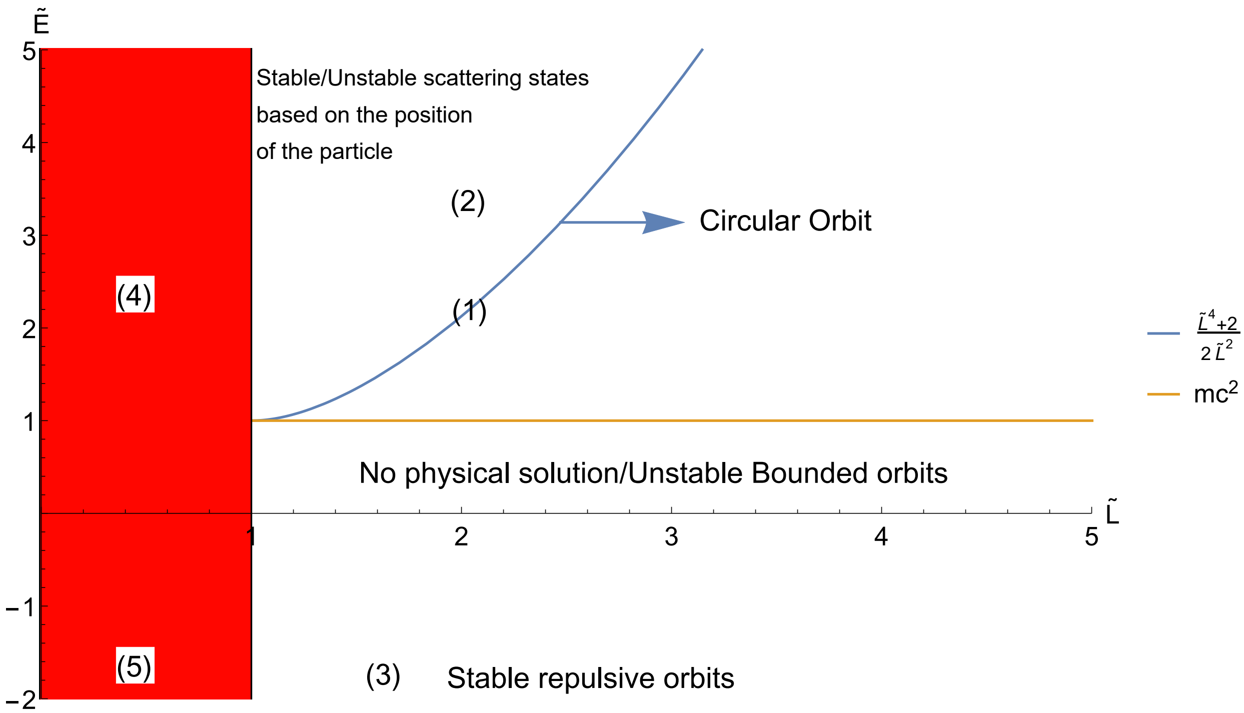


Figure 32: Phase diagram for a relativistic particle in $V(r) = -\alpha_2/r^2$ potential. With $\tilde{L} = L/\sqrt{2m\alpha_2}$ and $\tilde{E} = \frac{E}{mc^2}$. The blue curve is the found relation between dimensionless energy as function of dimensionless angular momentum in the special case of circular orbits found in Eq. (73).

Clear to see from Fig. 32 is that we get a stable region where the angular momentum is larger than the critical angular momentum $\tilde{L} > L_c = \sqrt{2m\alpha_2}$. On the blue curve we have stable circular orbits. Above the blue curve we get the scattering states which were discussed above and which are determined by the begin position of the particle. Below the blue curve we don't get physical solutions, the particle follows an unstable bounded orbit. Here the particle falls into the origin and afterwards escapes and at a certain time falls again into the origin. This is a characteristic of the potential because the potential is very strong in this case it doesn't allow elliptical orbits because the kinetic energy is not large enough to compensate the attractive force. The only way to have stable bounded orbits in this case are circular orbits where by making use of the right position, energy and angular momentum ensure that the centripetal force compensates the attractive force. When the energy becomes large in negative sense we always get stable repulsive orbits. In the non stable region where the angular momentum is less than the critical angular momentum $\tilde{L} < L_c = \sqrt{2m\alpha_2}$ we don't find stable solutions, where the particle falls into the origin as an unstable scattering state. Here the particle falls into the origin without performing circular motion. This typical orbit appears in the region where (4) is situated in Fig. 32. At the same time when the energy becomes negative we get stable repulsive orbits. This means that when the energy becomes negative we always get repulsive orbits. A typical orbit for (5) and (3) is given by Fig. 30. A typical orbit for (1) is a circular orbit as shown in Fig. 29. A typical orbit for (2) is shown in Fig. 31 where two orbits are possible based on the begin position of the particle.

7 Non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential

For a non-relativistic particle we can derive the equation of motion by using the lagrangian formalism. In this case we have:

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{\alpha_2}{r^2}.$$

Consequently we get the following equation of motion: $\dot{v} = \frac{2\alpha_2}{m}\frac{1}{r^3}$, with $v = \dot{r}$. Because we are interested in circular orbits we express the velocity of the particle in polar coordinates (r,θ) and at the same time we find the acceleration in polar coordinates, which is expressed as:[2][14]

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r + \underbrace{\frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\mathbf{e}_\theta}_{=\mathbf{a}_\theta}.$$

By multiplying the acceleration with the mass of the particle we get the equation of motion of the non-relativistic particle. In order to have circular orbits the central force (in this case $\mathbf{F}(r) = -2\alpha_2/r^3\mathbf{e}_r$ should be equal to the centripetal force ($2\alpha_2/r^3 = mv^2/r$). From this expression we derive the relationship between kinetic and potential energy $T = -U$. This problem looks like a relativistic particle in a Kepler/Coulomb potential problem which is discussed earlier. Because central forces leads to conserved systems we have a conserved angular momentum which is expressed as $d\mathbf{L}/dt = \mathbf{r} \times \mathbf{F} = 0$. Because central forces are proportional with the radial unit vector $\mathbf{e}_r$ we have $F_\theta = m\mathbf{a}_\theta = 0$ which leads to $\frac{d}{dt}(mr^2\frac{d\theta}{dt}) = \frac{dL}{dt} = 0$. Using this result we can derive a general equation of motion in the case of a general central force f(r): [2][14]

$$f(r) = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right].$$

From the previous results we know that $\dot{\theta} = L/mr^2$. Combining with previous expression we get: [14]

$$f(r) = m\left[\frac{d^2r}{dt^2} - \frac{L^2}{m^2r^3}\right]. \tag{96}$$

Because we are interested in circular orbits we have to find a relation between radius and polar angle. We have:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\left(\frac{d\theta}{dt}\right) = \frac{dr}{d\theta}\left(\frac{L}{mr^2}\right) = -\frac{L}{m}\frac{du}{d\theta}. \tag{97}$$

Where in the last equality we made use of $u = 1/r$ and $du = (-1/r^2)dr$. And finally we find the radial acceleration:

$$\ddot{r} = \frac{d\dot{r}}{dt} = -\frac{d}{dt}\left(\frac{L}{m}\frac{du}{d\theta}\right) = -\frac{L}{m}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -\frac{L^2}{m^2}\frac{d^2u}{d\theta^2}u^2. \tag{98}$$

Where in the last equality we made use of $\dot{\theta} = Lu^2/m$. Finally we find the equation of motion of a non-relativistic particle under influence of a general central force $f(r) = f(1/u)$ by combining Eq.(96) and $\ddot{r}$.[14]

$$f(1/u) = m\left[-\frac{L^2}{m^2}\frac{d^2u}{d\theta^2}u^2 - \frac{L^2u^3}{m^2}\right] \Rightarrow \frac{d^2u}{d\theta^2} + u + \frac{m}{L^2}\frac{f(1/u)}{u^2} = 0. \tag{99}$$

Because we want to describe the non-relativistic particle in $V(r) = -\alpha_2/r^2$ or $V(1/u) = -\alpha_2 u^2$ potential we have $F(1/u) = -\nabla V(1/u) = -2\alpha_2 u^3$ and the equation of motion becomes:

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{2\alpha_2 m}{L^2}\right)u = 0. \tag{100}$$

By using Mathematica we find the solutions of Eq. (100). We need conditions in order to find the general solution. Like we did before we insert as begin condition $u_0(\theta_0) = 1/r_0$ and as second begin condition we use again the fact that energy and angular momentum are conserved quantities and we derive again the energy expression in polar coordinates in the case of potential $V(r) = -\alpha_2/r^2$. The analysis is the same as in Appendix I where the equation of motion of a non-relativistic particle was derived. In this case we get as second condition:

$$E + \alpha_2 u^2 = \frac{1}{2m}\left(\frac{du}{d\theta}\right)^2 L^2 + \frac{1}{2m}u^2 L^2. \tag{101}$$

By inserting these conditions we get the general solution of a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential.

$$\begin{aligned} u(\theta) = \frac{e^{-\frac{\sqrt{2m\alpha_2 - L^2}}{L}(2\theta_0+\theta)}}{2r_0(L^2 - 2m\alpha_2)} \Bigg(&\pm r_0 e^{\frac{2\theta_0\sqrt{2\alpha_2 m - L^2}}{L}} \sqrt{\frac{(L^2 - 2m\alpha_2)e^{\frac{2\theta_0\sqrt{2m\alpha_2 - L^2}}{L}}(-2Emr_0^2 + L^2 - 2m\alpha_2)}{r_0^2}} \\ &\mp r_0 e^{\frac{2\theta\sqrt{2m\alpha_2 - L^2}}{L}} \sqrt{\frac{(L^2 - 2m\alpha_2)e^{\frac{2\theta_0\sqrt{2\alpha_2 m - L^2}}{L}}(-2Emr_0^2 + L^2 - 2\alpha_2 m)}{r_0^2}} + \\ L^2 e^{\frac{3\theta_0\sqrt{2\alpha_2 m - L^2}}{L}} - 2\alpha_2 m e^{\frac{3\theta_0\sqrt{2\alpha_2 m - L^2}}{L}} &+ L^2 e^{\frac{\theta_0\sqrt{2\alpha_2 m - L^2}}{L} + \frac{2\theta\sqrt{2\alpha_2 m - L^2}}{L}} - 2\alpha_2 m e^{\frac{\theta_0\sqrt{2\alpha_2 m - L^2}}{L} + \frac{2\theta\sqrt{2\alpha_2 m - L^2}}{L}} \Bigg). \end{aligned} \tag{102}$$

With:

1. L the given angular momentum
2. E the given energy of the particle
3. r_0 the initial position of the particle
4. θ_0 the initial polar angle between the x-axis and the position vector which binds the particle with the origin of the central force

Like explained in section 5.2.3 in the case of potential $V(r) = -\alpha_2/r^2$ we don't find stable circular orbits. So we don't find any relation between angular momentum and radius. What I do in this case, I find the initial conditions by using the found relation $L_0 = mr_0v_0$ for circular orbits. Because the angular momentum is a fixed quantity, we have to place the particle somewhere in space and in this way we got the begin velocity of the particle. Of course we should have the same differential equation by taking the non-relativistic limit ($c \to \infty$) in the equation of motion Eq. (88) of a relativistic particle. We show that by taking this limit we get the same differential equation as in Eq. (100). From Eq. (88) we have:

$$\frac{d^2u}{d\theta^2} + u\left(1 - \frac{2E\alpha_2}{L^2c^2}\right) - \frac{2\alpha_2^2 u^3}{L^2c^2} = 0. \tag{103}$$

By taking the limit $c \to \infty$ we have that the term in $\frac{2\alpha_2^2 u^3}{L^2 c^2}$ vanishes and goes to zero, with L the relativistic angular momentum. Important to notice is that by taking the non-relativistic limit, that the relativistic angular momentum approaches the non-relativistic angular momentum $L \to L_{nr}$ (look at definition of relativistic angular momentum in chapter 5.2.2). On the other hand we have that the energy becomes non-relativistic and we have $E_{rel} \to E_{nrel} \Rightarrow E_{rel} \to E_{nrel} = -\frac{mv^2}{2} + mc^2$. This fact leads to:

$$\lim_{\frac{1}{c}\to 0} \frac{2E_{rel}\alpha_2}{(Lc)^2} = \lim_{\frac{1}{c}\to 0} \frac{2\alpha_2}{(Lc)^2}(-\frac{mv^2}{2} + mc^2) = \frac{2m\alpha_2}{L_{nrel}^2}.$$

By looking at this results we indeed see that Eq. (100) is the non-relativistic version of the equation of motion of a relativistic particle in $V(r) = -\alpha_2/r^2$ potential. Finally, we express the equation of motion in Eq. (100) in dimensionless units:

$$\frac{d^2\tilde{u}}{d\theta^2} + \left(1 - \frac{1}{\tilde{L}^2}\right)\tilde{u} = 0. \tag{104}$$

with $\tilde{u} = 1/\tilde{r} = 1/(r/\sqrt{\alpha/mc^2})$ and $\tilde{L} = L/\sqrt{2m\alpha}$. The energy equation becomes (with $\tilde{E} = E/mc^2$):

$$\tilde{E} + \tilde{u}^2 = \left(\frac{d\tilde{u}}{d\theta}\right)^2 \tilde{L}^2 + \tilde{u}^2\tilde{L}^2. \tag{105}$$

7.1 Energy of a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential

The energy of a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential is expressed as:

$$E = \frac{1}{2}mv^2 - \frac{\alpha_2}{r^2} + mc^2. \tag{106}$$

We can make this expression dimensionless:

$$\tilde{E} = \frac{E}{mc^2} = \frac{1}{2}\tilde{v}^2 - \frac{1}{\tilde{r}^2} + 1. \tag{107}$$

With $\tilde{v} = v/c$, $\tilde{r} = r/\sqrt{\alpha_2/mc^2}$. We introduce the rest mass energy because we want to compare this results with the results of a relativistic particle. By solving the equation of motion for a circular orbit we found $L = mrv \Rightarrow \tilde{r}^2 = 2\tilde{L}^2/\tilde{v}^2$ with $\tilde{L} = L/\sqrt{2m\alpha_2}$. By eliminating the radius dependency in expression (107) we get:

$$\tilde{E} = \frac{1}{2}\tilde{v}^2\left(1 - \frac{1}{\tilde{L}^2}\right) + 1. \tag{108}$$

It is easy to see that if the angular momentum L approaches the critical angular momentum $L = \sqrt{2m\alpha_2}$ that the energy is constant because the kinetic and potential energy will compensate each other. If the angular momentum is larger than the critical angular momentum, the kinetic energy term will dominate and consequently we get scattering states. If the angular momentum is less than the critical angular momentum then the potential energy term will dominate and consequently we get bound states. Here we need to be careful. Because in this case we don't have stable orbits when we should have bounded orbits we get unstable orbits where the particle falls into the origin. In the following graphs I show the dimensionless energy as function of dimensionless velocity.

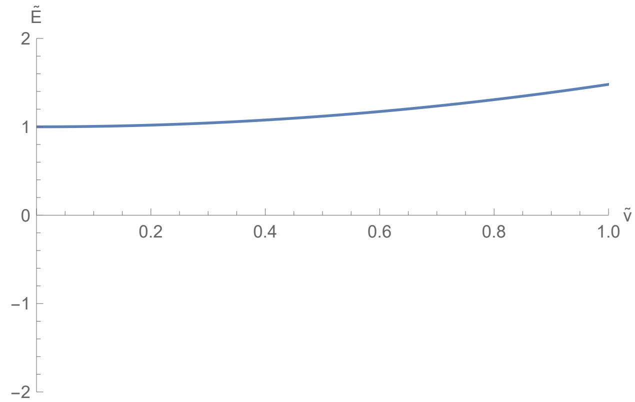


Figure 33: Dimensionless energy as function of velocity for $L \gg L_c = \sqrt{2m\alpha_2}$. This figure is made for $\tilde{L} = L/\sqrt{2m\alpha_2} = 5$.

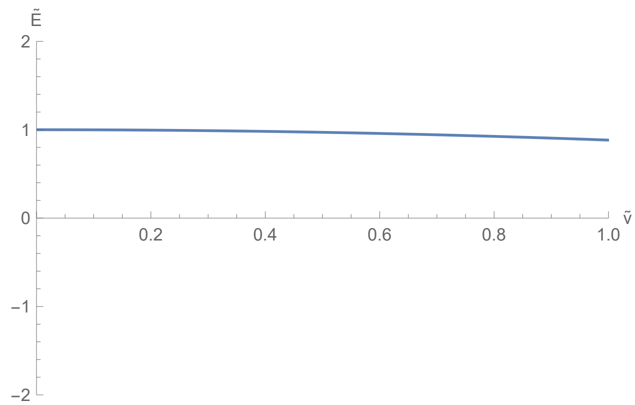


Figure 34: Dimensionless energy as function of velocity for $L < L_c = \sqrt{2m\alpha_2}$. This figure is made for $\tilde{L} = L/\sqrt{2m\alpha_2} = 0.9$.

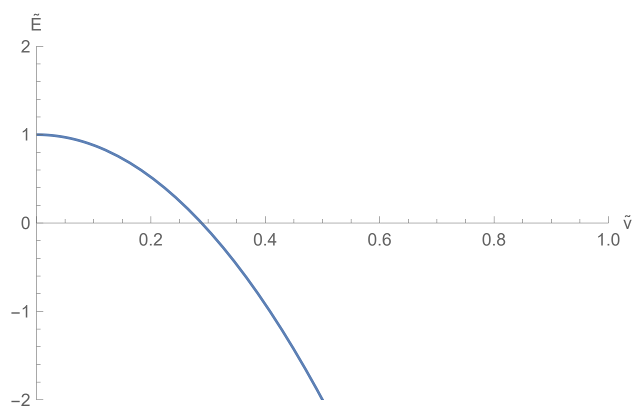


Figure 35: Dimensionless energy as function of velocity for $L \ll L_c = \sqrt{2m\alpha_2}$. This figure is made for $\tilde{L} = L/\sqrt{2m\alpha_2} = 0.2$.

Further, from the equation of motion for circular orbits we get:

$$\frac{mv^2}{r} = \frac{2\alpha_2}{r^3} \Rightarrow mv^2 = \frac{2\alpha_2}{r^2}. \tag{109}$$

We make this relation dimensionless by dividing both sides by the rest energy of the particle mc^2:

$$\tilde{v}^2 = \frac{2}{\tilde{r}^2}. \tag{110}$$

With: $\tilde{v} = v/c$ and $\tilde{r} = r/\sqrt{\alpha_2/mc^2}$. By inserting this relation in Eq.(107) we get:

$$\tilde{E} = \frac{1}{2}\tilde{v}^2 - \frac{1}{2}\tilde{v}^2 + 1. \tag{111}$$

Clear to see is that the energy is constant, because an increase in kinetic energy is compensated by a decrease in potential energy and the other way around. This behavior is almost the same as the behavior studied in chapter 5 where we studied a relativistic particle in $V(r) = -\alpha_1/r$ potential. In this case an increase/decrease in velocity corresponds with a decrease/increase in radius which leads to a full conversion between kinetic and potential energy $T = -U$. This case is bit different from the case in chapter 5 because:

1. In chapter 5 the energy is kept constant (angular momentum also see chapter 5) because of the presence of the γ-factor. This leads to the fact that the particle falls into the origin and at the same time the energy and angular momentum are conserved quantities.

2. In the case of a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential here we don't have a γ-factor but the potential and kinetic energy compensate each other because the relation found in Eq. (110) leads to unstable orbits which were predicted in section 5.2.3.

7.2 Orbits of a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential

I used the `Manipulate` function of Mathematica in order to make the following figures. What I basically do is, I put the particle somewhere in space $\tilde{r}_0$ with a certain begin angle θ_0, afterwards I adapt the energy and angular momentum. Note that here I made use of the dimensionless equation of motion Eq. (104)

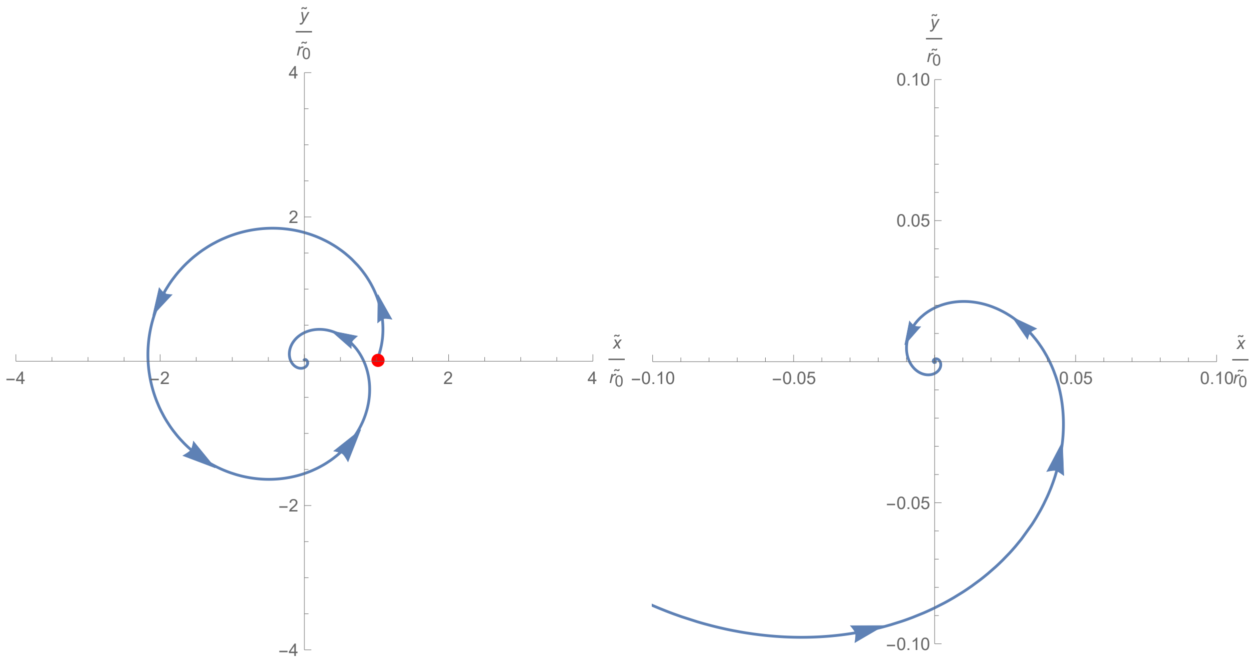


Figure 36: Solution of orbit equation Eq. (104). This plot is made by using the following parameters: $\tilde{L} = 0.9$, $\tilde{E} = -1$, $\theta_0 = 0.001°$ and $\tilde{r}_0 = 0.2$. The polar angle is varied in the range $\in [0, 10\pi]$. The right figure is an enlargement of the left figure which shows that the particle spirals into the origin of the force.

Clear to see from Fig. 36 is that by setting the particle around the origin (red dot), the particle gets trapped by the central force and spirals out from the origin and afterward at a certain moment it begins to spiral into it. This behavior is found when we investigated the behavior of a relativistic particle in $V(r) = -\alpha_1/r$ potential. In both cases when the angular momentum is less than the critical angular momentum, $L = \alpha_1/c$ for a relativistic particle in $V(r) = -\alpha_1/r$ potential and $L_c = \sqrt{2m\alpha_2}$ for a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential, the particle spirals into the origin of the central force which indeed leads to an unstable orbit. The fact that both cases exhibits the same behavior is not surprising. The biggest reason why they have the same behavior is found in the relation between kinetic and potential energy, $T = -U$. Notice that both relativistic and non-relativistic particles in these potentials show the same behavior but with a big difference. In the case of a relativistic particle in $V(r) = -\alpha_1/r$ potential there we got stable circular and elliptical (precessing ellipses) orbits while here is not the case anymore because we derived in 5.2.3 that for the potential $V(r) = -\alpha_2/r^2$ there cannot exist any stable bounded orbit. Further, by increasing the angular momentum and when it becomes larger than the critical angular momentum

$L_c = \sqrt{2\alpha_2 m}$, it is clear from the figures below that the particle will make loops around the origin of the central force but doesn't fall into the origin. It is clear a scattering state.

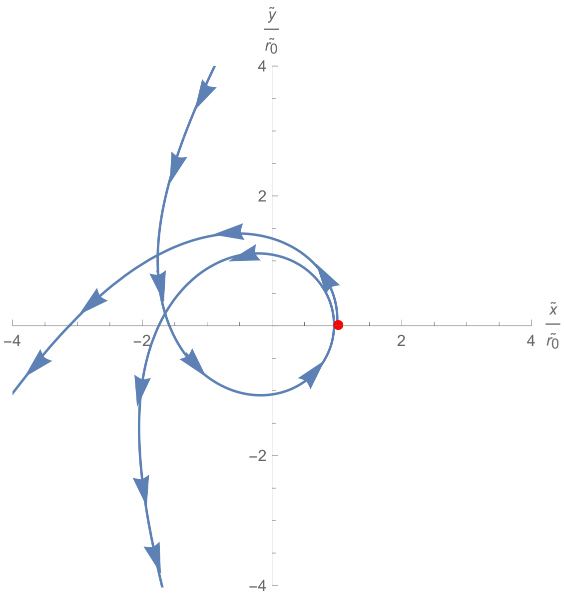


Figure 37: The particle starts in $\tilde{r} = 1$ (red dot) and escapes from the central force. Afterwards the particles comes back and performs a loop around the origin and escapes from the potential force. This plot is made by using the following parameters: $\tilde{L} = 1.05$, $\tilde{E} = 0.05$, $\theta_0 = 0°$ and $\tilde{r}_0 = 1.5$. The polar angle is varied in $\in [0\pi, 4.5\pi]$.

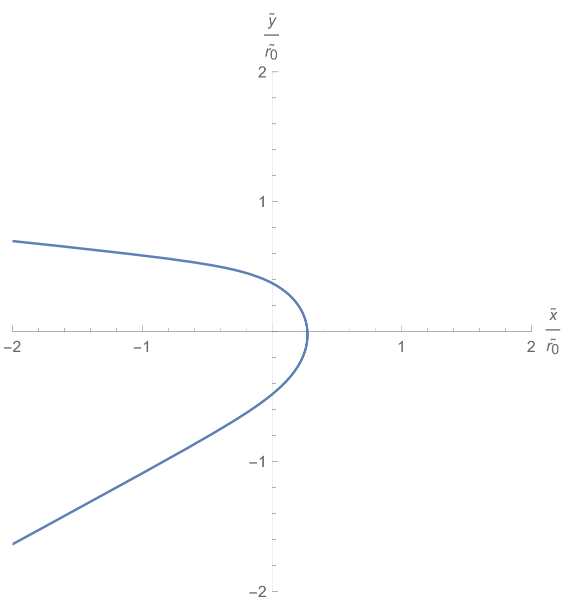


Figure 38: This plot is made by using the following parameters: $\tilde{L} = 1.2$, $\tilde{E} = 6$, $\theta_0 = 0°$ and $\tilde{r}_0 = 1$. The polar angle is varied in $\in [0.19\pi, 1.99\pi]$.

From the figures above we recognize for $\tilde{L} \gg 1$ Kepler hyperbolic/parabolic orbits which are also found for a relativistic particle in $V(r) = -\alpha_1/r$ potential. When the angular momentum is less or equal to the critical angular momentum it is easy to see from the figure below that the particle will spiral into the origin of the potential.

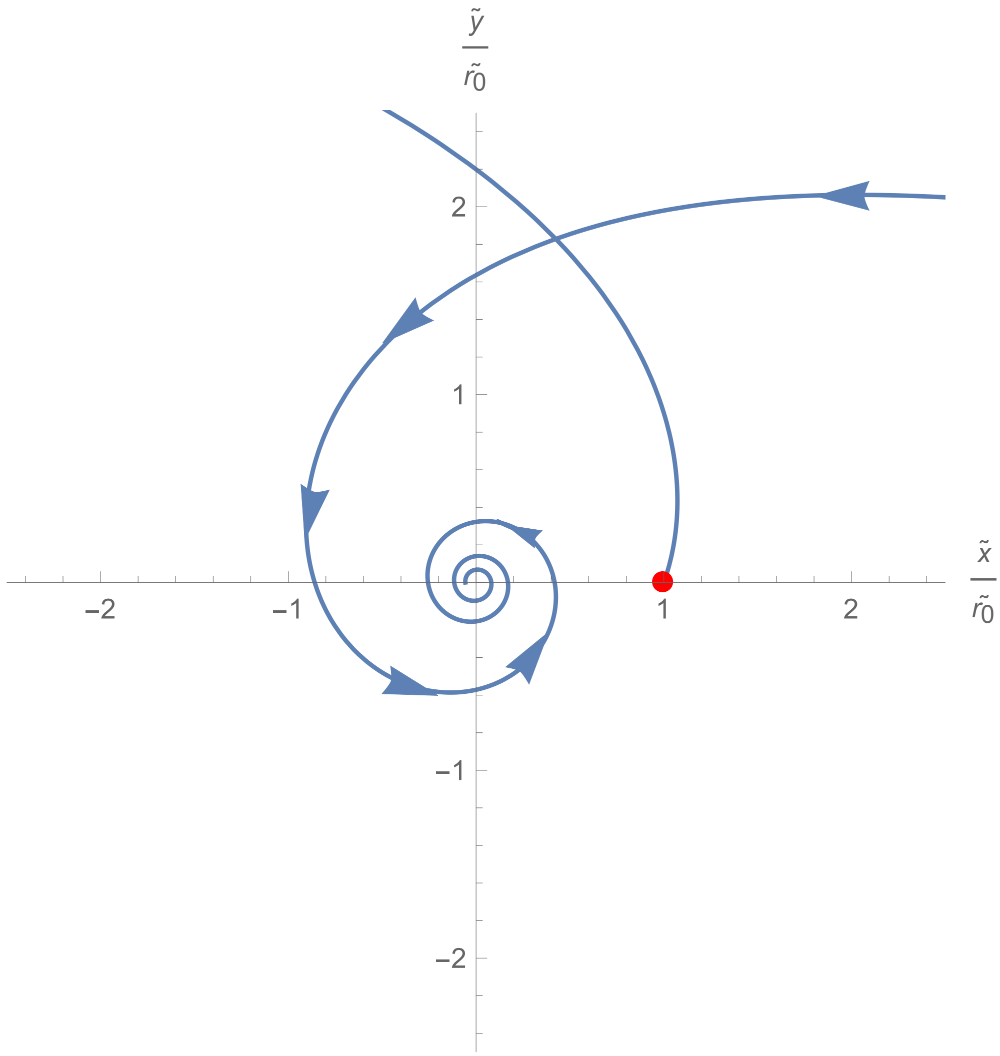


Figure 39: Solution of orbit equation Eq. (104). This plot is made by using the following parameters: $\tilde{L} = 0.994$, $\tilde{E} = 0.05$, $\theta_0 = 0°$ and $\tilde{r}_0 = 1.5$. The polar angle is varied in $\in [0, 8\pi]$. This graph is made for $\tilde{L} \leq 1$.

From Fig. 39 we see that the particle starts in $\tilde{r} = r/r_0 = 1$ (red dot) and escapes from the potential. Afterwards the particle comes back and gets captured by the potential force and will spiral into the origin of the potential. This behavior is found also for a relativistic particle in $V(r) = -\alpha_1/r$ potential. The reason for likeness in behavior between the two particles is explained later in chapter 9.

7.2.1 Radius as function of polar angle θ

In this section we will solve the equation of motion Eq.(104) numerically and stable/unstable orbits are studied as function of the polar angle θ. We know that an orbit becomes unstable if the angular momentum becomes less than the critical angular momentum ($\tilde{L} < 1$) and stable orbits are found when the angular momentum is larger than the critical value ($\tilde{L} > 1$).

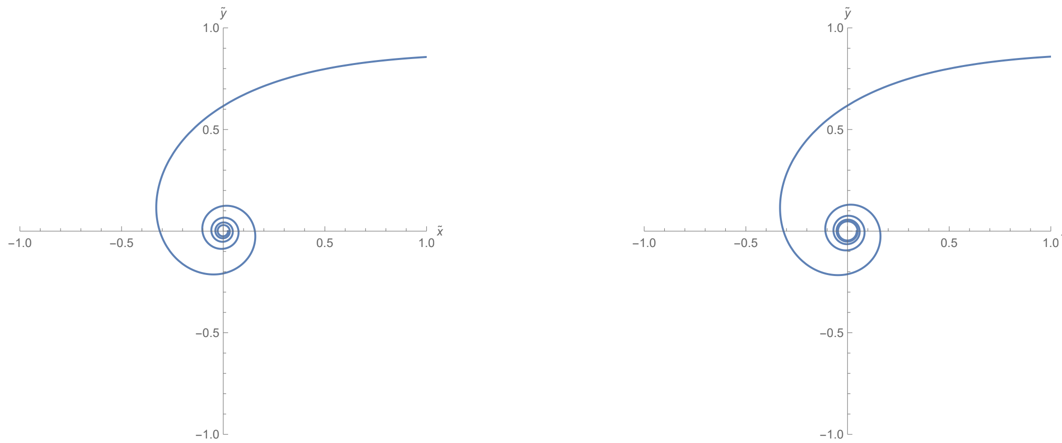


Figure 40: Solution of orbit equation Eq.(104) for parameters: $\tilde{E} = 1$, $\theta_0 = 0°$ and $\tilde{r}_0 = 20$. The polar angle is varied in $\in [0, 10\pi]$. In the left figure the angular momentum is less than the critical angular momentum $\tilde{L} = 0.999$ while in the right graph the angular momentum is slightly larger than the critical angular momentum $\tilde{L} = 1.001$.

From Fig. (40) we see that both figures are very similar but the evolution for a larger range of the polar angle is different. In Fig. (41) we see the radius as function of θ of the respective graphs.

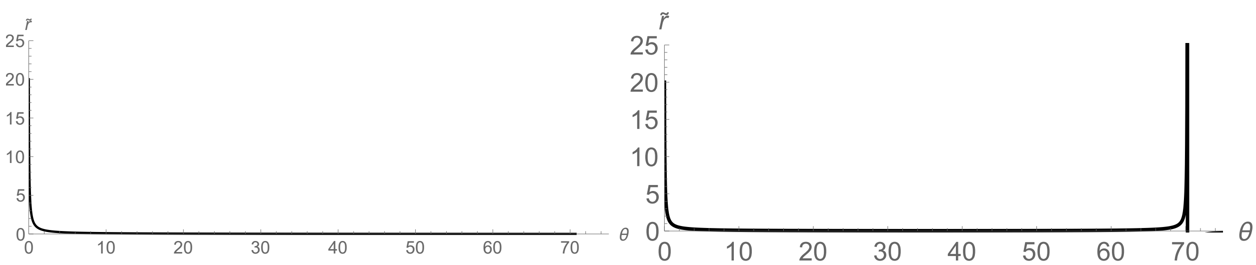


Figure 41: Dimensionless radius $\tilde{r}$ as function of the polar angle θ. The figures are to read from left to the right. The particle starts in $\tilde{r}_0 = 20$ and at a begin polar angle $\theta_0 = 0°$.

Clear to see from Fig. (41) is the fact that the particle in the case of $\tilde{L} < 1$ (left case in Fig. (40)) in a larger θ range, will fall into the origin of the central force. When $\tilde{L} > 1$ (right case in Fig. (40)) the radius will converge to a minimum radius $\tilde{r}_{min}$ and for a certain moment the particle will follow a 'quasi' bound circular orbit with radius $\tilde{r}_{min}$. Afterward the radius begins to increase and at a certain moment will diverge which means that the particle at some time will win some kinetic energy which provides for an escape from the attractive field. In Fig. (42) we see the evolution of the orbits in a larger polar angle interval.

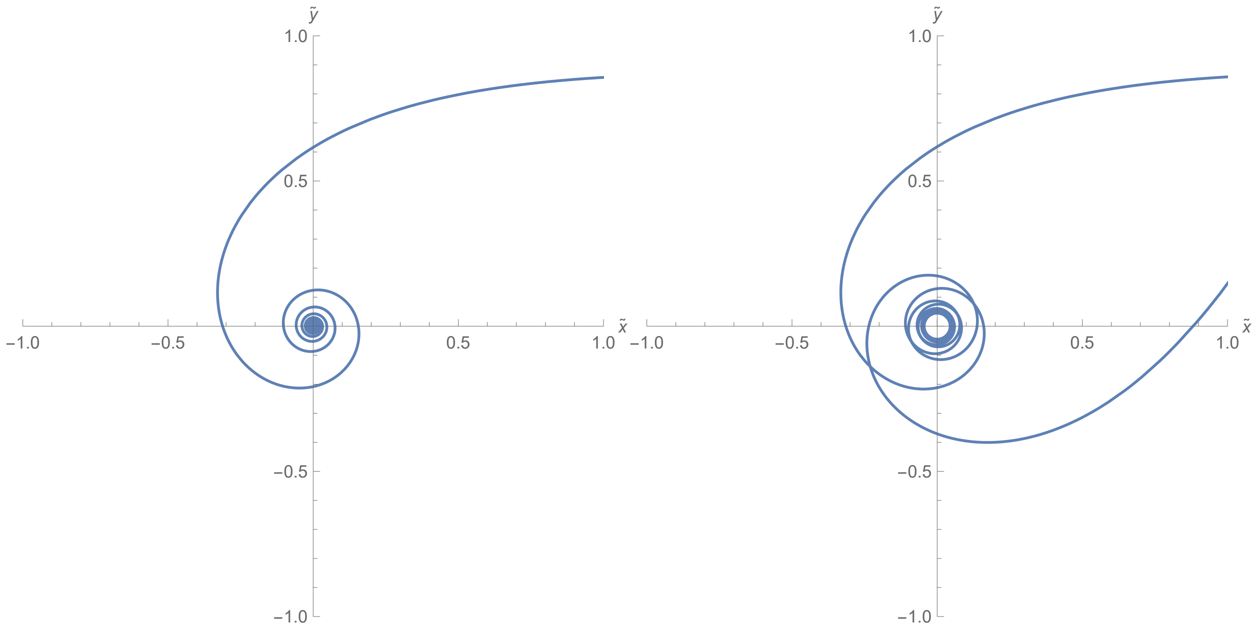


Figure 42: Solution of orbit equation Eq.(104) for parameters: $\tilde{E} = 1$, $\theta_0 = 0°$ and $\tilde{r}_0 = 20$. The polar angle is varied in $\in [0, 22.5\pi]$. In the left figure the angular momentum is less than the critical angular momentum $\tilde{L} = 0.999$ while in the right graph the angular momentum is slightly larger than the critical angular momentum $\tilde{L} = 1.001$.

Further, in Fig. 43 an enlargement of the right figure of Fig. 41 is shown while Fig. 44 shows an enlargement of the left figure of Fig. 41.

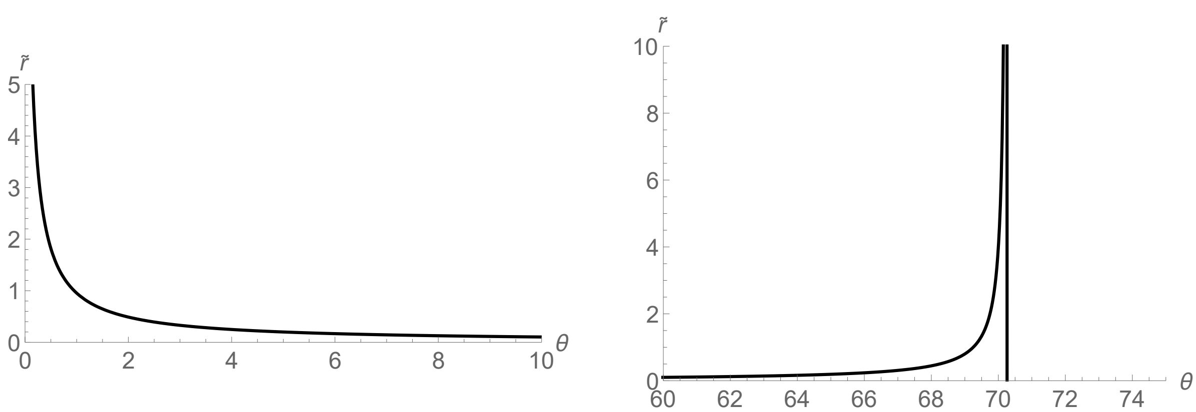


Figure 43: Enlargement of the right figure of Fig. 41. In the left figure we clearly see that the particle approaches the origin of the potential reaching a minimum distance from the center. Afterward we see from the right figure that the particle escapes from the potential.

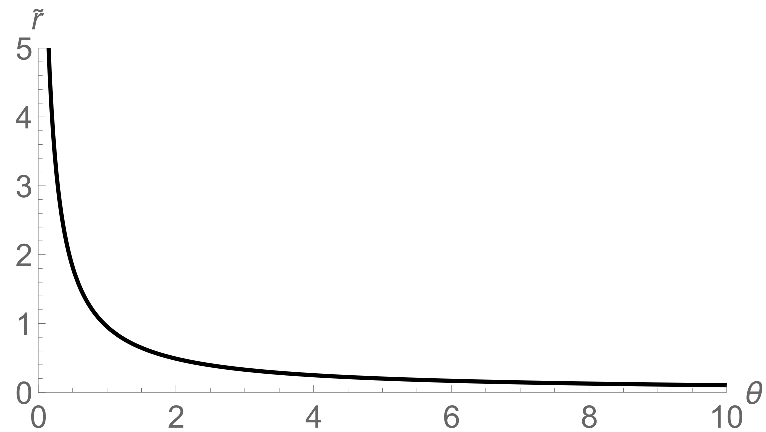


Figure 44: Enlargement of the left figure of Fig. 41. Here we clearly see that the particle approaches closer and closer to the origin of the potential and at a certain time the particle falls into it.

7.3 Phase diagram for a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential

In Fig. 45 a phase diagram for a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential is presented. Because in section 7.1 we didn't find a relation between dimensionless energy and angular momentum we cannot draw a curve in the phase diagram. From the previous figures and calculations we know that below the critical angular momentum $\tilde{L} < 1$ we get unstable orbits which provides for atomic collapse. These are situated in the red region of Fig. 45. Above the critical angular momentum $\tilde{L} > 1$ the particle performs stable scattering states which in the phase diagram are to be found in the orange region. Note that bounded stable orbits are not possible in both cases.

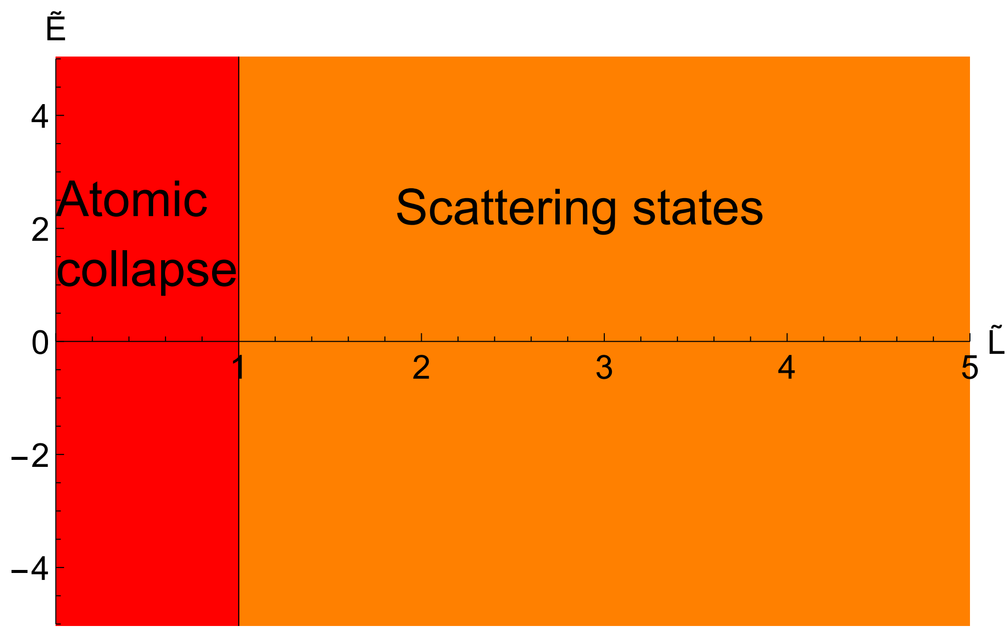


Figure 45: Phase diagram for a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential.

8 Influence on the orbits by removing singularity in $r = 0$

Now that we have the orbits for a relativistic and non-relativistic particle in $V(r) = -\alpha_1/r$ and $V(r) = -\alpha_2/r^2$ potentials we can investigate the consequence on the orbits by removing the singularity in $r = 0$. In the next figure we can visualize how the Kepler/Coulomb potential is affected by cutting off the potential as function of the radius.

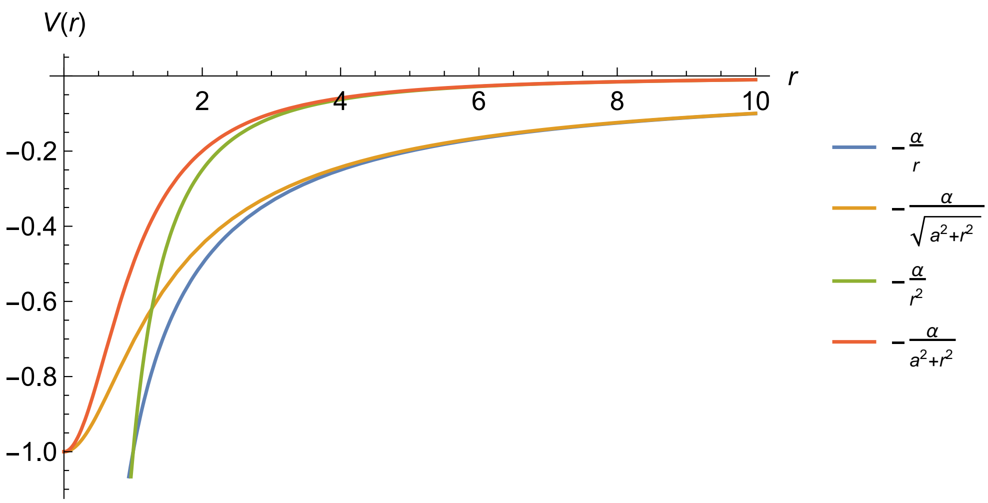


Figure 46: These curve's represents the Kepler/Coulomb potentials for different cases. Blue: $V(r) = -\alpha_1/r$,Green: $V(r) = -\alpha_2/r^2$, Orange: $V(r) = -\alpha_1/\sqrt{r^2 + a^2}$, Red: $V(r) = -\alpha_2/(r^2 + a^2)$. This plot is made for $\alpha_1 = 1kgm^3/s^2$ for $V(r) = -\alpha_1/r$ and $V(r) = -\alpha_1/\sqrt{r^2 + a^2}$ and $\alpha_2 = 1kgm^4/s^2$ for $V(r) = -\alpha_2/(r^2 + a^2)$ and $V(r) = -\alpha_2/r^2$. In all of this cases we take $a = 1m$. Note that the radius is given in meters and the potentials in joule (J).

By introducing the parameter a the potential converges to a constant value for $r \to 0$. Now we can distinguish two cases when $r \to 0$, one when a becomes very small near to zero and the other when a becomes very big. When a is near zero it is easy to visualize that the potential becomes very big in the negative sense. Of course when a is zero the potential becomes $-\infty$. This is a special case that is discussed in 9. Again because the system is under influence of a central force we have that energy ($E = T + V$) and angular momentum are conserved quantities. When potential becomes $-\infty$ it means that we have a full conversion from kinetic to potential energy and consequently the particle fall's into the center of the central force. So, this cannot be the case when $a > 0$, here we expect that the particle cannot fall into the center of the central force. Finally when a becomes big then the potential becomes very small near zero, and of course when a becomes ∞ potential energy becomes zero which leads to a free particle. Because of these reasoning we can deduce some facts:

1. For parameter a small, the potential energy will dominate in the energy expression which leads to negative energy and consequently we get bound states.

2. For parameter a large, the potential energy is very weak and consequently the kinetic energy takes control in the expression of energy and in this case we get scattering states.

Note that the unit of parameter a in all cases is meters.

8.1 Non-relativistic particle in a $V(r) = -\alpha_1/\sqrt{r^2+a^2}$ potential

By using the energy equation with $V(r) = -\alpha_1/\sqrt{r^2+a^2} \Rightarrow V(1/u) = -\alpha_1 u/\sqrt{1+a^2u^2}$ potential we derive the equation of motion of a non-relativistic particle in a $V(r) = -\alpha_1/\sqrt{r^2+a^2}$ potential. The procedure is the same as we made in Appendix I. Look at Appendix I Eq. (141).

$$E + \frac{u}{\sqrt{1+a^2u^2}} = \frac{1}{2m}\left(\frac{du}{d\theta}\right)^2 L^2 + \frac{1}{2m}u^2L^2. \tag{112}$$

By taking the derivative of Eq. (112) with respect to θ and afterward dividing by $du/d\theta$ we find the equation of motion:

$$\frac{d^2u}{d\theta^2} + u - \frac{\alpha_1 m}{L^2}\frac{1}{\sqrt{(1+a^2u^2)}} + \frac{\alpha_1 m}{L^2}\frac{a^2}{\sqrt{(1+a^2u^2)^3}} = 0. \tag{113}$$

With $u = 1/r$. In order to compare our results we resize the parameters such that $u(\theta)$ scales as mc^2/α_1 like we did in chapter 5.7 and 5.8 by introducing $\tilde{L} = Lc/\alpha_1$ and $\tilde{E} = E/mc^2$. Of course by inserting $a = 0\alpha_1/mc^2$ we get again the equation of motion of a non-relativistic particle in $V(r) = -\alpha_1/r$ potential given in Eq. (145) (Look at Appendix I). Now we can solve this equation of motion numerically and investigate the influence on the orbits by regulating the singularity of the potential V(r) as function of parameter a.

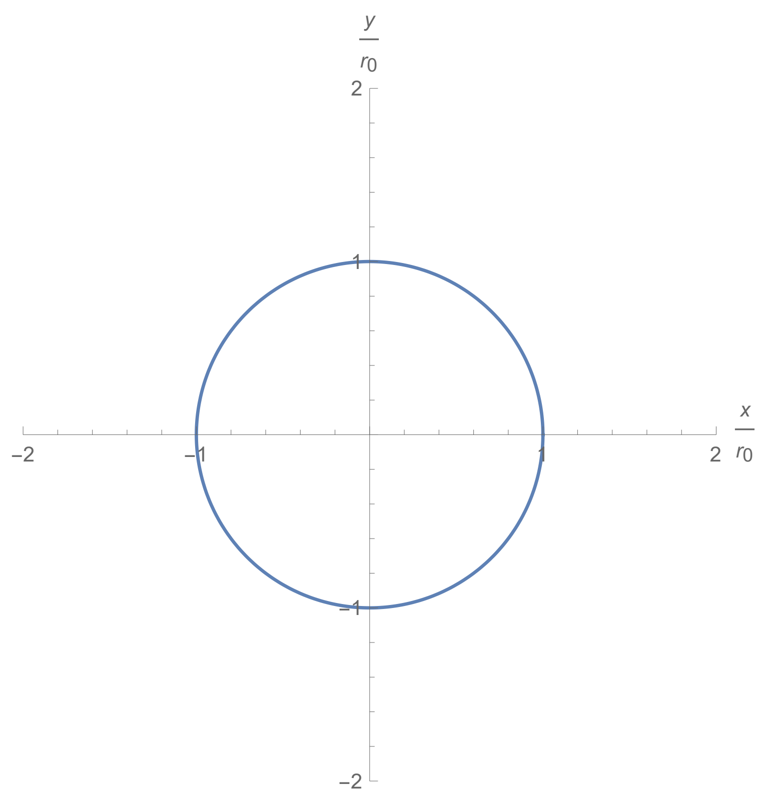


Figure 47: This plot is made for $a = 0\alpha_1/mc^2$, $\tilde{L} = 1$, $\tilde{E} = -(1/2\tilde{L}^2)$=-0.5 and the polar angle is varied in the range $\theta \in [0, 2\pi]$. The begin position of the particle is $r_0 = 1\alpha_1/mc^2$ in $\theta_0 = 0°$.

By keeping the same parameters as in Fig.(47) and increasing parameter a we find that the particle feels less potential which leads to a free particle, look at Fig.48.

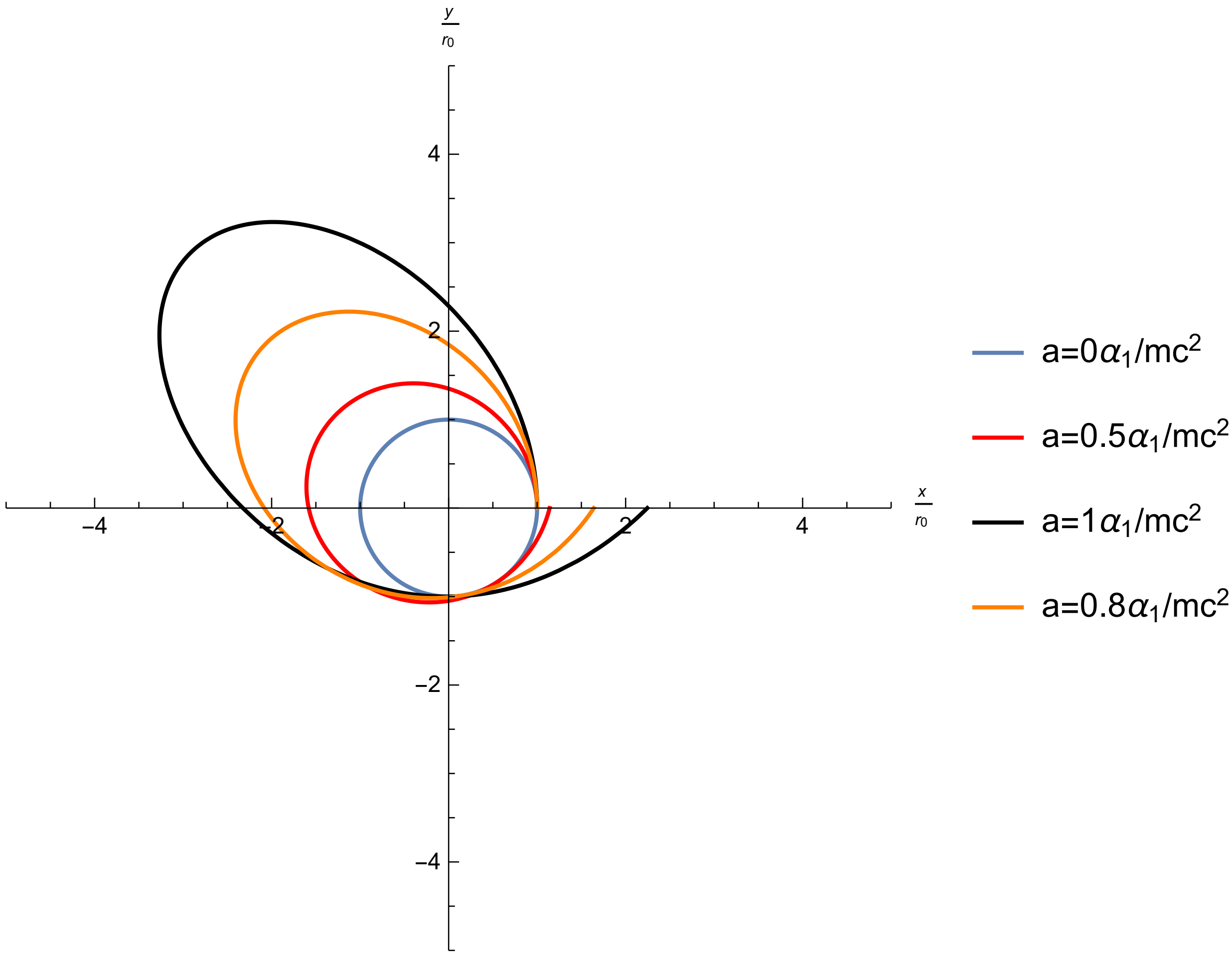


Figure 48: This plot is made for different a while all parameters $\tilde{L} = 1$,$\tilde{E} = -(1/2\tilde{L}^2)$=-0.5, $r_0 = 1\alpha_1/mc^2$,$\theta_0 = 0°$ are kept. The polar angle is varied in $\theta \in [0, 2\pi]$ for each case.

From Fig. 48 it is easy to note that by the increasing parameter a the particle gains some kinetic energy because the potential becomes weaker and this leads, for weak a, to precessing ellipses around the origin of the central force. This behavior was never found for a non-relativistic particle, where only circular and elliptical orbits were allowed in the case of a bounded orbit. This means that by removing the singularity in the potential we get precessing motion in the case of a bounded orbit. Of course by taking a very big the solution becomes a straight line which is the solution of a free particle.

8.2 Non-relativistic particle in a $V(r) = -\alpha_2/(r^2 + a^2)$ potential

The equation of motion of a non-relativistic particle in $V(r) = -\alpha_2/(r^2+a^2)$ potential can be found in two different manners. The first by using the general equation of motion found in Eq. (99) the equation of motion of a non-relativistic particle in $V(r) = -\alpha_2/(r^2 + a^2)$ potential can be found by using the relation between force and potential energy $\mathbf{F} = -\nabla V(r)$. The second by deriving it from the energy equation. Note that we make use of the r-inverse variable $u = 1/$. This means that the potential becomes:

$$V(r) = \frac{-\alpha}{r^2 + a^2} \Rightarrow \tilde{V}(1/\tilde{u}) = \frac{-\tilde{u}}{1 + \tilde{a}^2\tilde{u}^2}. \tag{114}$$

With $\tilde{V}(\tilde{r})$ the dimensionless form of the adapted potential, $\tilde{r} = r/\sqrt{\alpha_2/mc^2}$, $\tilde{a} = a/\sqrt{\alpha_2/mc^2}$ and $\tilde{u} = 1/\tilde{r}$. Now we replace the potential $\tilde{V}(1/\tilde{u})$ in the energy equation (105):

$$\tilde{E} + \frac{\tilde{u}}{1 + \tilde{a}^2\tilde{u}^2} = \left(\frac{d\tilde{u}}{d\theta}\right)^2 \tilde{L}^2 + \tilde{u}^2\tilde{L}^2. \tag{115}$$

By taking the derivative with respect to θ and by dividing by $d\tilde{u}/d\theta$ we get the equation of motion of a non-relativistic particle in $\tilde{V}(1/\tilde{u})$ potential:

$$\frac{d^2\tilde{u}}{d\theta^2} + \tilde{u} - \frac{\tilde{u}}{\tilde{L}^2}\frac{1}{(1 + \tilde{a}^2\tilde{u}^2)} + \frac{\tilde{u}^2\tilde{a}^2}{\tilde{L}^2}\frac{1}{(1 + \tilde{a}^2\tilde{u}^2)^2} = 0. \tag{116}$$

Note that the same equation of motion could be derived by using the general equation of motion in Eq. (99). Of course by inserting $\tilde{a} = 0$ we get again the equation of motion of a non-relativistic particle in $V(r) = -\alpha_2/r^2$ potential given in Eq. (104). Now we can solve this equation of motion numerically and investigate the influence on the orbits by regulating the potential $\tilde{V}(1/\tilde{u})$ as function of the parameter $\tilde{a}$.

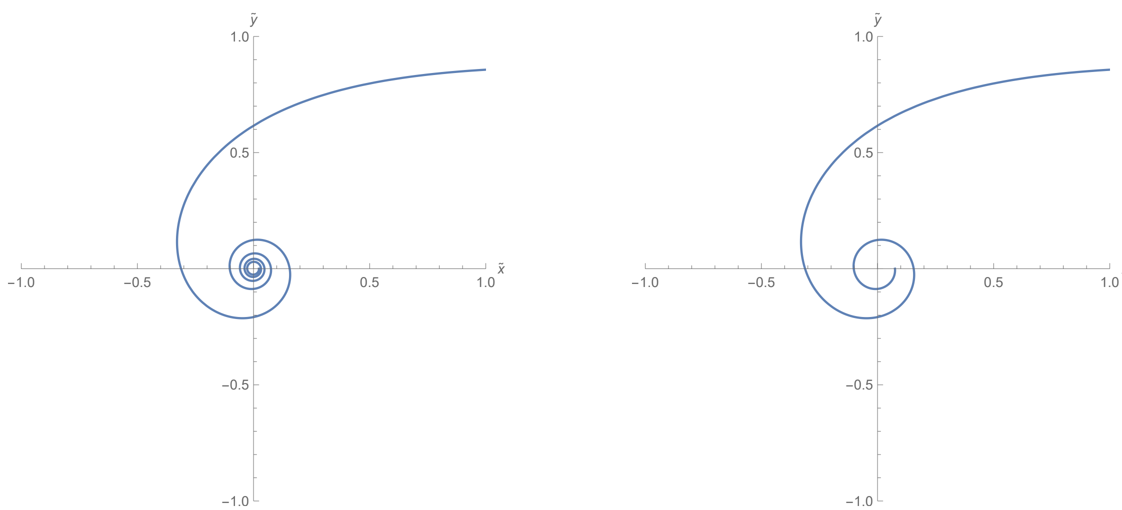


Figure 49: Solution of orbit equation Eq. (116) for parameters: $\tilde{E} = 1$, $\theta_0 = 0°$, $\tilde{r}_0 = 20$,$L = 0.999$ and $\tilde{a}$. The polar angle is varied in $\in [0, 10\pi]$ for the left figure, while $\in [0, 4\pi]$ for the figure on the right. The left and right figure shows the same orbit but I took a smaller polar angle range in order to compare the influence on the orbit by increasing parameter $\tilde{a}$. It is easy to notice that by taking parameter $\tilde{a} = 0$ we get again the same orbit as shown in Fig. 40.

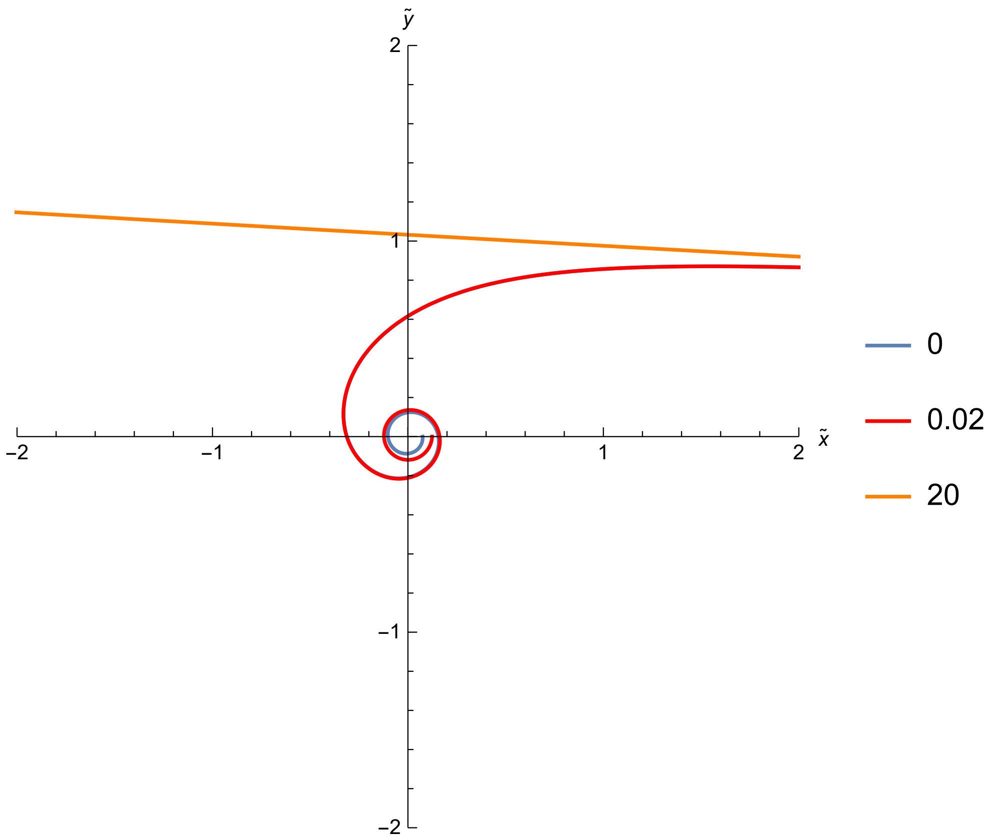


Figure 50: Solution of orbit equation Eq. (116) for parameters: $\tilde{E} = 1$, $\theta_0 = 0°$, $\tilde{r}_0 = 20$,$L = 0.999$ and $\tilde{a}$. The polar angle is varied in $\in [0, 4\pi]$. The numbers refers to the magnitude of $\tilde{a}$.

It is easy to notice that by increasing parameter $\tilde{a}$ the particle feels less attractive potential which means that the particle at a certain moment acts as a free particle like shown by the orange straight line in Fig.50.

8.3 Relativistic particle in a $V(r) = -\alpha_1/\sqrt{r^2+a^2}$ potential

By using potential $V(r) = -\alpha_1/\sqrt{r^2+a^2}$ and making use of the same calculation as in chapter 5.6, we get the equation of motion of a relativistic particle in $V(r) = -\alpha_1/\sqrt{r^2+a^2}$ potential. (see Appendix J)

$$\frac{d^2u}{d\theta^2}+u-\frac{E\alpha_1}{L^2c^2}\frac{1}{\sqrt{1+(au)^2}}+\frac{E\alpha_1}{L^2c^2}\frac{u^2a^2}{\sqrt{(1+(au)^2)^3}}-\frac{\alpha_1^2}{L^2c^2}\frac{u}{(1+(au)^2)}+\frac{\alpha_1^2}{L^2c^2}\frac{u^3a^2}{(1+(au)^2)^2}=0. \quad (117)$$

With $u = 1/r$. Again, in order to compare our results we resize the parameters such that $u(\theta)$ scales as mc^2/α_1 like we did in chapter 5.7 and 5.8 by introducing $\tilde{L} = Lc/\alpha_1$ and $\tilde{E} = E/mc^2$. It is easy to visualize that by inserting $a = 0\alpha_1/mc^2$ we get again the equation of motion of a relativistic particle in $V(r) = -\alpha_1/r$ potential, given in Eq.(50):

$$\frac{d^2u}{d^2\theta}+\left[1-\left(\frac{\alpha_1}{Lc}\right)^2\right]u-\frac{E\alpha_1}{(Lc)^2}=0. \quad (118)$$

Now we can solve the equation of motion Eq.(117) numerically and investigate the influence on the orbits by changing the potential V(r) as function of the parameter a. By making use of the same parameters used to make Fig. 17 and by inserting $a = 0\alpha_1/mc^2$ we obtain of course the same orbit found in Fig. 17:

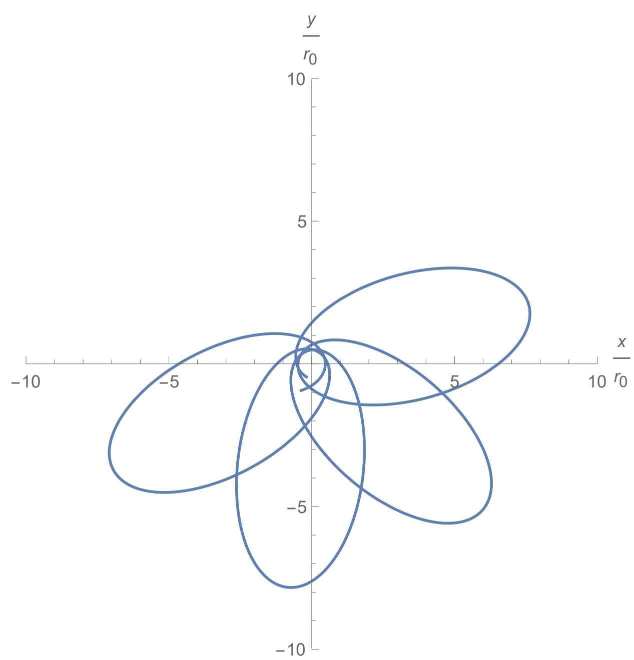


Figure 51: Solution of orbit equation Eq. (117) for parameters: $\tilde{L} = 2, \tilde{E} = \sqrt{1-(1/\tilde{L})^2}+0.09999 = 0.96603$, $\theta_0 \approx -1.948995°$, $a = 0\alpha_1/mc^2$ and $r_0 = 3.4641\alpha_1/mc^2$. The polar angle is varied in $\in [-2\pi, 8\pi]$.

Note that in Fig. 17 the polar angle was taken in $\in [0, 10\pi]$ and in Fig. 51 $\in [-2\pi, 8\pi]$. If I start in $\theta = 0$ in the numerical solution the particle starts to move on the orbit later which intuitive looks like that the orbits are not the same. But I looked better to the orbits and I shifted my begin conditions by 2π such that in order to have the same orbit as analytic we need to solve the begin conditions in $2\pi - \theta_0$ and no longer $0 - \theta_0$ like we did for the analytic solution. Now, we take one loop of the orbit shown in Fig. 51 and we investigate what happens with the orbit by increasing parameter a.

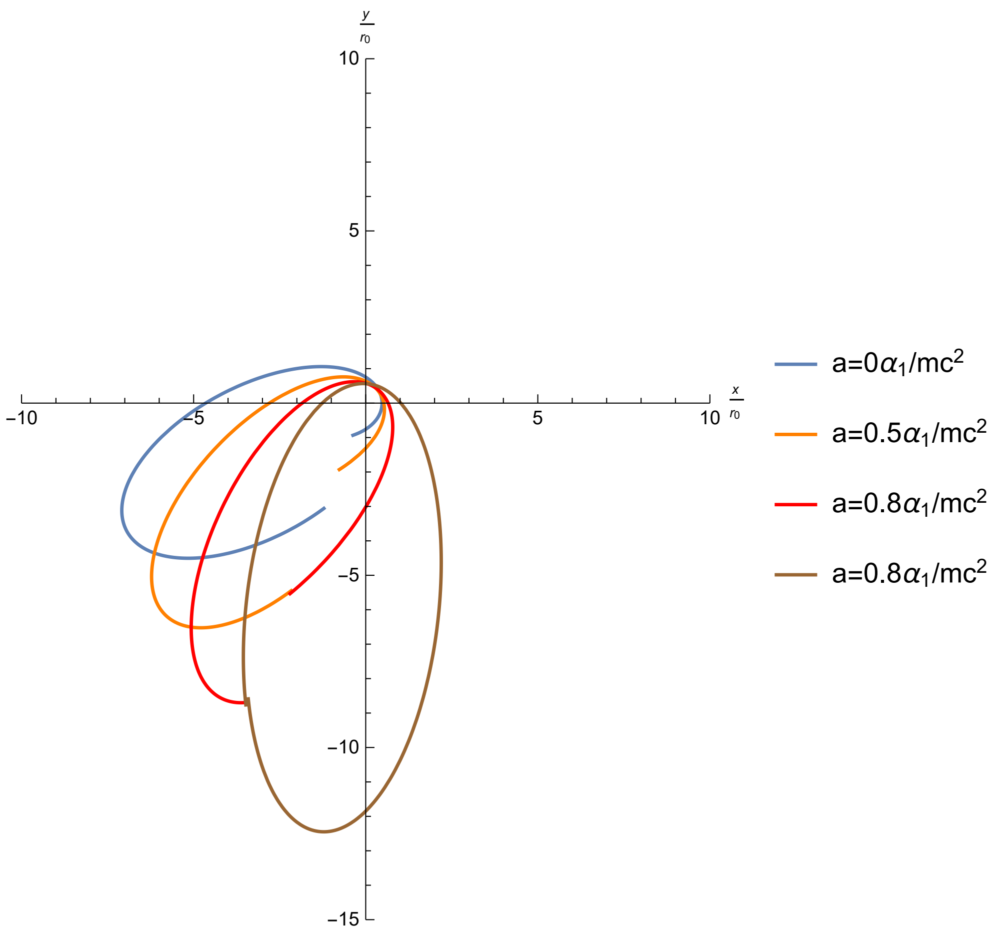
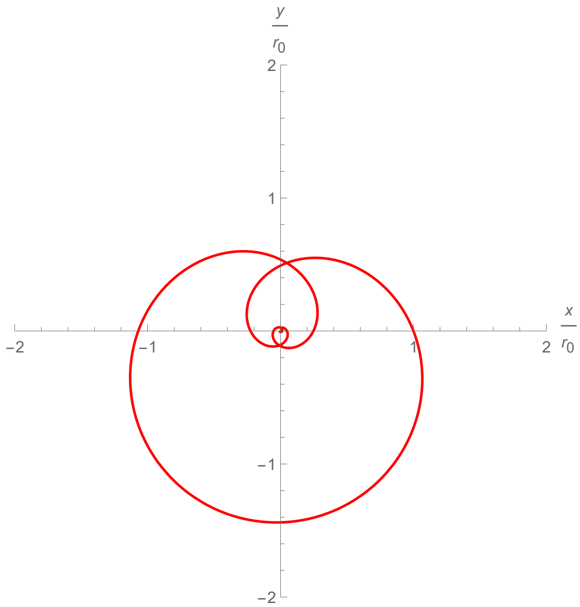
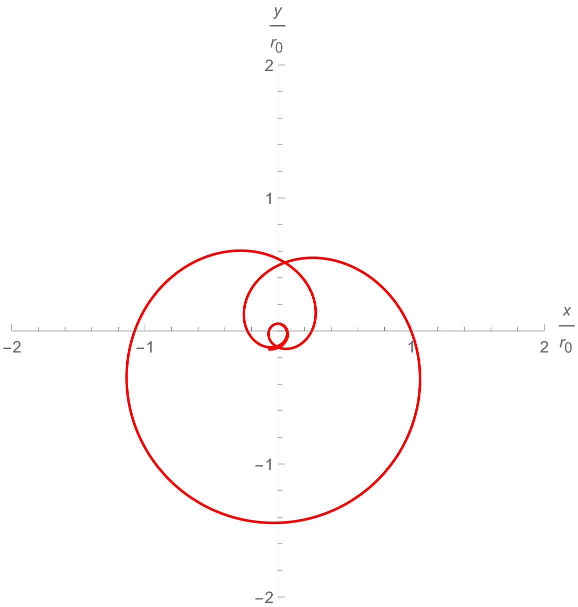


Figure 52: Solutions of orbit equation Eq. (117) for parameters: $\tilde{L} = 2$,$\tilde{E} = \sqrt{1 - (1/\tilde{L})^2} + 0.09999 = 0.96603$, $\theta_0 \approx -1.948995°$. The polar angle is varied in $\in [-2\pi, 0\pi]$.

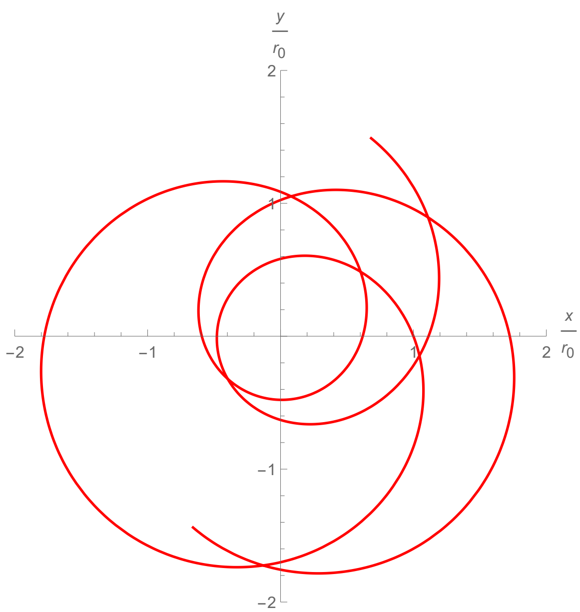
It is easy to notice that by increasing parameter a the particle feels less attractive potential which means that the particle at a certain moment acts as a free particle like shown by the black straight line in Fig. 52. Further, we can investigate what happens with the unstable orbits where the particle falls into the origin of the central force by removing the singularity in $r = 0$. The fact that before when $a = 0$ we had $T = -U$ tells us that the particle can fall into the origin while angular momentum and energy are conserved quantities. Now if we remove the singularity this cannot happen anymore because this relation is broken, and will not provide for a collapse anymore. Look at Fig. 53.



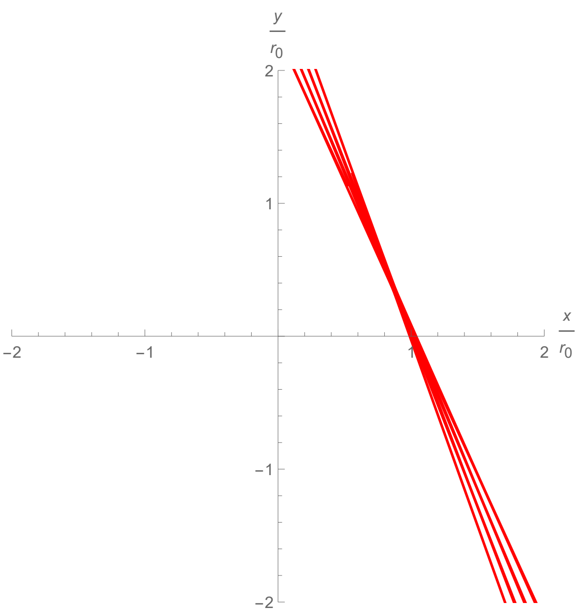
(a) $a = 0\alpha_1/mc^2$.



(b) $a = 0.01\alpha_1/mc^2$.



(c) $a = 0.09\alpha_1/mc^2$.



(d) $a = 5\alpha_1/mc^2$.

Figure 53: Numerical solution of orbit equation Eq.(117) with parameters: $\tilde{L} = 0.9, \tilde{E} = 0.1$, $\theta_0 \approx -2°$. The polar angle is taken for all figures in $\in [-2\pi, 5\pi]$. In figure (a) we see again a typical unstable orbit where the particle set in position $r_0 = 0.382165\alpha_1/mc^2$, will spiral out from the origin of the central force and afterwards spirals into the origin. Clear to see is that by removing the singularity the particle will no longer fall into the origin and leads to precessing motion around the origin. At a certain time the potential becomes so small such that the particle becomes almost a free particle.

8.4 Relativistic particle in a $V(r) = -\alpha_2/(r^2 + a^2)$ potential

From the dimensionless energy equation in Eq. (91) we derive the dimensionless equation of motion for a relativistic particle in $V(r) = -\alpha_2/r^2$. The dimensionless energy equation is expressed as:

$$\left(\tilde{E} + \tilde{u}^2\right)^2 = 2\tilde{L}^2\left(\frac{d\tilde{u}}{d\theta}\right)^2 + 2\tilde{L}^2\tilde{u}^2 + 1. \tag{119}$$

Now we remove the singularity in $r = 0$ such that the potential becomes:

$$V(r) = -\frac{\alpha_2}{r^2 + a^2} \Rightarrow \tilde{V}(r) = -\frac{1}{\tilde{r}^2 + \tilde{a}^2}. \tag{120}$$

With $\tilde{V}(\tilde{r})$ the dimensionless form of the adapted potential, $\tilde{r} = r/\sqrt{\alpha_2/mc^2}$ and $\tilde{a} = a/\sqrt{\alpha_2/mc^2}$. With this potential we find the new equation of motion where the singularity in $r = 0$ is removed.

$$\left(\tilde{E} + \frac{1}{\tilde{r}^2 + \tilde{a}^2}\right)^2 = 2\left(\frac{1}{\tilde{r}^2}\frac{d\tilde{r}}{d\theta}\right)^2\tilde{L}^2 + \frac{2}{\tilde{r}^2}\tilde{L}^2 + 1. \tag{121}$$

Using the substitution $\tilde{u} = 1/\tilde{r}$ we find:

$$\left(\tilde{E} + \frac{\tilde{u}}{1 + \tilde{a}^2\tilde{r}}\right)^2 = 2\left(\frac{d\tilde{u}}{d\theta}\right)^2\tilde{L}^2 + 2\tilde{u}^2\tilde{L}^2 + 1. \tag{122}$$

By taking the derivative of Eq. (122) with respect to θ and by dividing the expression by $d\tilde{u}/d\theta$ we find the equation of motion of a relativistic particle in $\tilde{V}(r)$ potential:

$$\frac{d^2\tilde{u}}{d\theta^2}\frac{1}{\tilde{u}} + 1 - \frac{1}{\tilde{L}^2}\left(\tilde{E} + \frac{\tilde{u}^2}{1 + \tilde{a}^2\tilde{u}^2}\right)\left(\frac{1}{1 + \tilde{a}^2\tilde{u}^2} - \frac{\tilde{u}^2\tilde{a}^2}{(1 + \tilde{a}^2\tilde{u}^2)^2}\right) = 0. \tag{123}$$

Alternatively we make the same calculation and we find the respective equation of motion of Eq. (88) for the adapted potential:

$$\frac{d^2u}{d\theta^2}\frac{1}{u} + 1 - \frac{2E\alpha_2}{L^2c^2}\frac{1}{(1 + (au)^2)} + \frac{2u^2\alpha_2(\alpha_2 - Ea^2)}{L^2c^2(1 + (au)^2)^2} - \frac{2\alpha_2^2u^4a^2}{(1 + a^2u^2)^3L^2c^2} = 0. \tag{124}$$

With $u = 1/r$. Of course when we take $a = 0\sqrt{\alpha_2/mc^2}$ in Eq. (124) we get the same equation of motion as Eq. (88) which is expressed as:

$$\frac{d^2u}{d\theta^2} + u\left(1 - \frac{2E\alpha_2}{L^2c^2}\right) - \frac{2\alpha_2^2u^3}{L^2c^2} = 0. \tag{125}$$

This is also valid for Eq. (123) where by taking $\tilde{a} = 0$ we get again the equation of motion found in Eq. (92) which is expressed as:

$$\frac{d^2\tilde{u}}{d\theta^2} + \tilde{u}\left(1 - \frac{\tilde{E}}{\tilde{L}^2}\right) - \frac{\tilde{u}^3}{\tilde{L}^2} = 0. \tag{126}$$

Now we can solve the equation of motion Eq. (123) numerically and investigate the influence on the orbits by changing the potential $\tilde{V}(\tilde{r})$ as function of the parameter $\tilde{a}$. By making use of the same parameters used to make Fig. 28 and by inserting $\tilde{a} = 0$ we obtain of course the same orbit found in Fig. 28:

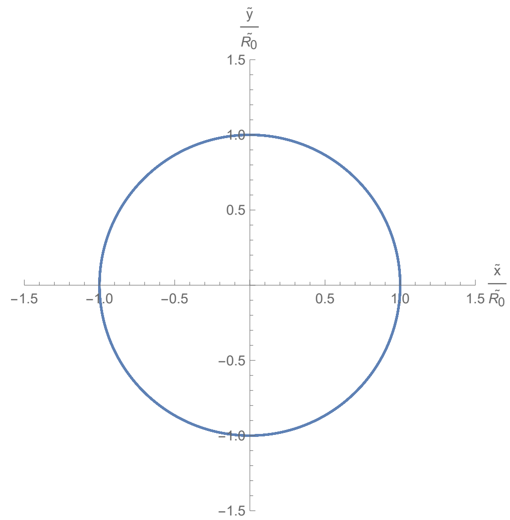


Figure 54: Solution of orbit equation Eq. (123) with parameters $\tilde{L} = 2$, $\theta_0 = 0°$, $\tilde{R}_0 = 0.730297$, $\tilde{a} = 0$ and $\tilde{E} = 2.125$. The polar angle θ is varied in range $\in [0, 20\pi]$.

Now we take the same parameters as in Fig. (54) and we regulate the singularity of the potential.

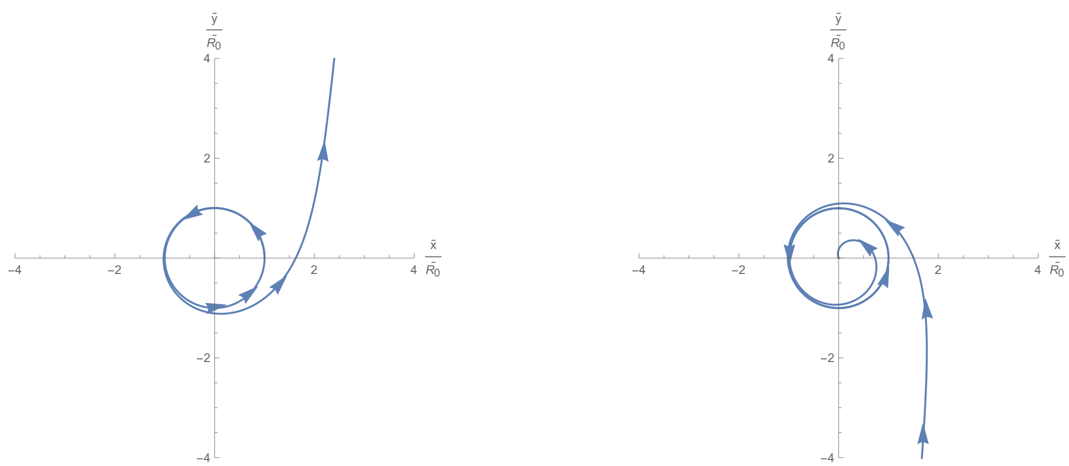
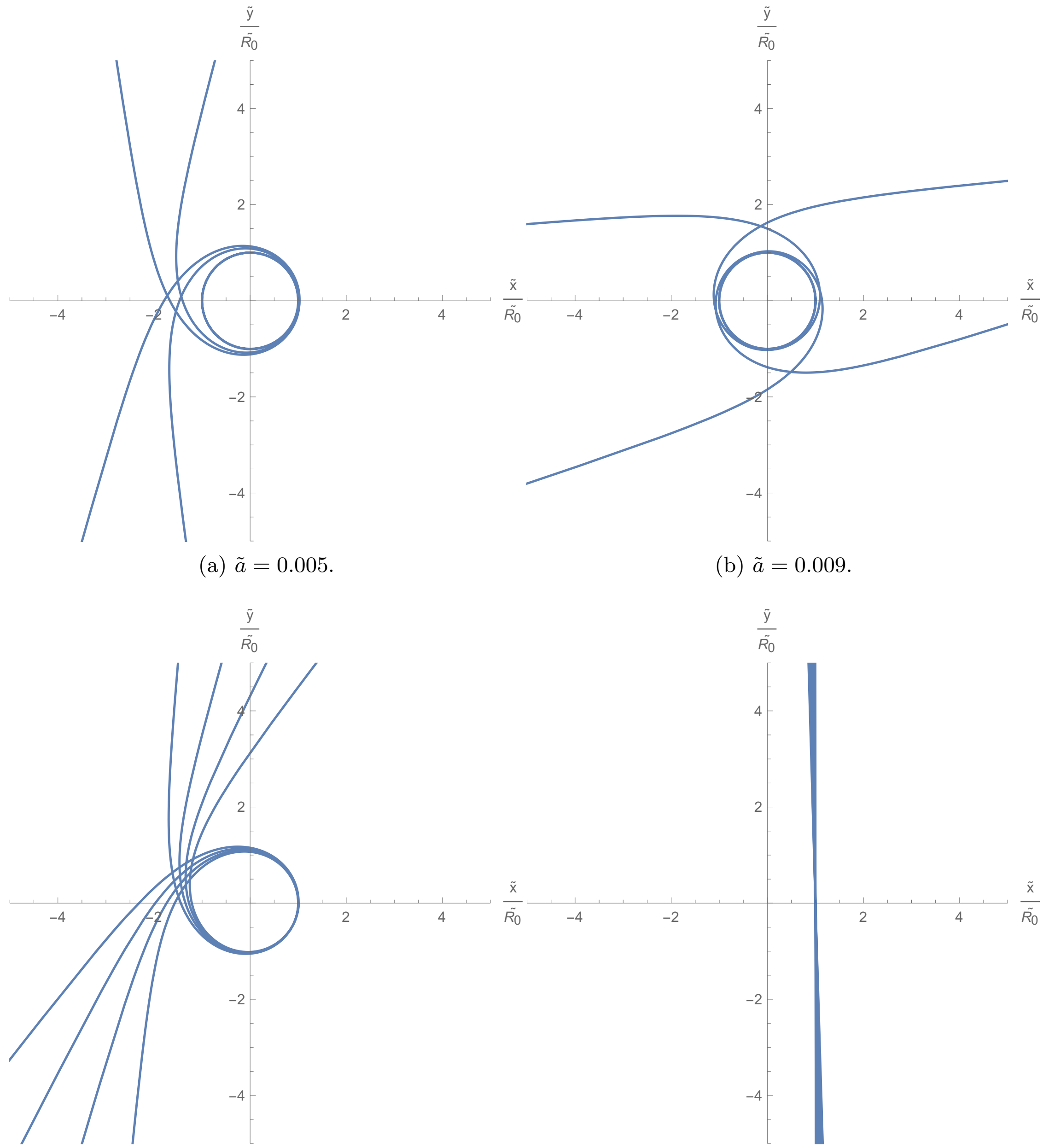


Figure 55: In both cases the singularity is removed by $\tilde{a} = 0.001$. In the left figure the polar angle is varied in the range $\in [0, 4.5\pi]$. In the right figure the polar angle is varied in the range $\in [4.5\pi, 12\pi]$.

In the left figure of Fig. 55 we observe that the particle starts at $\tilde{R} = 1$ with begin polar angle $\theta_0 = 0°$. From this point on the particle performs a few circles around the origin and afterwards escapes from the attractive potential. In the right figure of Fig. 55 we varies the polar angle in a larger range. From here we see that the particle comes back and performs a few loops around the origin before falling into it. This means by regulating a bit the singularity of the potential, a circular orbit becomes an unstable scattering state where the particle falls into the origin of the potential.

But when we regulate the singularity by taking a large $\tilde{a}$ we find that the particle performs a stable scattering state as shown in Fig. 56. Note that we make use of the same parameters as used in the previous figures.



(a) $\tilde{a} = 0.005$.

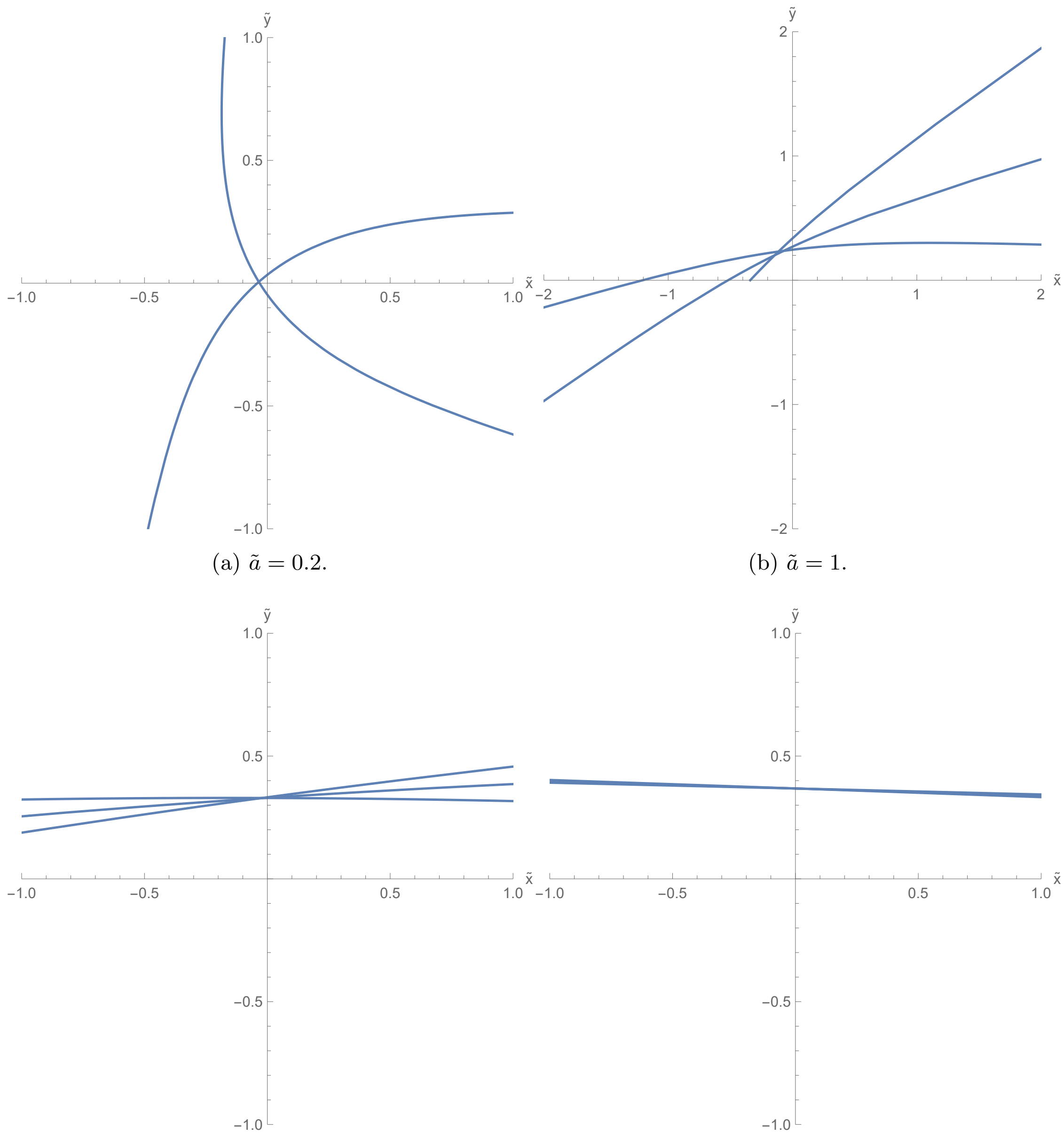
(b) $\tilde{a} = 0.009$.

(c) $\tilde{a} = 0.01$.

(d) $\tilde{a} = 5$.

Figure 56: In all cases the polar angle is varied in the range $\in [0, 12\pi]$.

From Fig. 56 we see that an increase in $\tilde{a}$ leads to stable scattering states. From Fig. 56 (d) we see that the particle becomes a free particle, this almost doesn't feel the attraction from the potential. Further, by increasing $\tilde{a}$ we get stable scattering states where before only unstable orbits were found ($\tilde{L} < 1$) as shown in Fig. 57.



(a) $\tilde{a} = 0.2$.

(b) $\tilde{a} = 1$.

(c) $\tilde{a} = 2$.

(d) $\tilde{a} = 1$.

Figure 57: In all figures we made use of the parameters: $\tilde{L} = 0.5$, $\theta_0 = 0$, $\tilde{E} = 2.125$ and $\tilde{R} = 10$. In all cases the polar angle is varied in the range $\in [0, 3\pi]$.

From figure Fig. 57 we clearly see that an increase in $\tilde{a}$ leads to stable scattering states and at a certain moment the particle doesn't feel the attractive potential anymore which means that the particle becomes free.

9 Relation between kinetic and potential energy for a particle in $V(r) = -\alpha_n/r^n$ potential with $n = 1, 2$

In Table 4 we give the relation between kinetic and potential energy for the different cases.

Particle	T	$V_1(r)$	$V_2(r)$	$T \sim V_1(r)$	$T \sim V_2(r)$
Relativistic	$T \sim p \sim \frac{1}{r}$	$\sim \frac{1}{r}$	$\sim \frac{1}{r^2}$	$\frac{1}{r} \sim \frac{1}{r}$	$\frac{1}{r} \sim \frac{1}{r^2}$
Non-relativistic	$T \sim p^2 \sim \frac{1}{r^2}$	$\sim \frac{1}{r}$	$\sim \frac{1}{r^2}$	$\frac{1}{r^2} \sim \frac{1}{r}$	$\frac{1}{r^2} \sim \frac{1}{r^2}$

Table 4: This table gives a summary of the found relations between kinetic (T) and potential (V_n) energy for the different cases.

Because the kinetic energy of the system is proportional with the linear momentum and at the same time the linear momentum is inversely proportional with the length scale of the system we can clarify some likeness in behavior between the two cases.

A non-relativistic particle in $V_2(r)$ potential shows the same behavior as a relativistic particle in $V_1(r)$ potential. This is because the relation between kinetic and potential energy in both cases is equal to $T = -U$. This means that kinetic/potential energy can be completely converted into potential/kinetic energy and there is no dominant term in the energy equation which forces the system to some characteristic behavior. In this case we have that the potential and kinetic energy term scales in the same way as the length scale of the system.

A relativistic particle in $V_2(r)$ potential shows the same behavior as a non-relativistic particle in $V_1(r)$ potential. In both cases the particle cannot spiral into the origin while energy is conserved and in both cases the particle can follow a circular orbit. The reason why both particles have the same behavior is found in the relation between kinetic and potential energy, here we find $T = -2U$ for a relativistic particle in $V_2(r)$ potential and $2T = -U$ for a non-relativistic particle in $V_1(r)$ potential. In both cases kinetic and potential energy scales in a different manner with respect to the length scale of the system which provides for a dominating term in the energy equation which forces the system to some characteristic behavior.

10 Frustrated state

Until now we investigated the case where a potential is situated in the origin of the coordinate system. Here we split this potential into two potentials and investigate how the orbits of a relativistic and non-relativistic particle are affected. The split potential, has two origins of attractive force. Note that here the system is no longer spherical symmetric and as explained in the previous chapters the angular momentum is a conserved quantity only if the system is spherically symmetric, this means that in such a system the angular momentum is no longer a conserved quantity because the circular symmetry is broken. This fact is a consequence of the presence of the two potentials which results in different forces on the particle an external torque which is mathematically expressed as $\vec{\tau} = d\mathbf{L}/dt = \mathbf{r} \times \mathbf{F}$. In the case of spherically symmetric system $\vec{\tau} = d\mathbf{L}/dt = 0$ as shown in previous chapters. We will see that the angular momentum as function of θ shows oscillations in the presence of the two-charge potential. Further it is easy to show that the energy is still conserved in this case:

$$E = \frac{1}{2} m\dot{\mathbf{r}}^2 + \Phi(r) \Rightarrow \dot{E} = m\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \frac{d\Phi(r)}{dr}\frac{dr}{dt}. \tag{127}$$

Because the relation between force and potential energy $\mathbf{F} = -\nabla V$ and $\mathbf{F} = m\ddot{\mathbf{r}}$ we have that:

$$\dot{E} = \dot{\mathbf{r}} \cdot \mathbf{F} - \mathbf{F} \cdot \dot{\mathbf{r}} = 0. \tag{128}$$

Which shows that the total energy is conserved.
Now we can visualize the original and the two-charge potential by making a contour plot. The original potential $V_1(\mathbf{r}) = -\alpha_1/r$ leads to an attractive central force situated in the origin of the coordinate system as shown in Fig. 58.

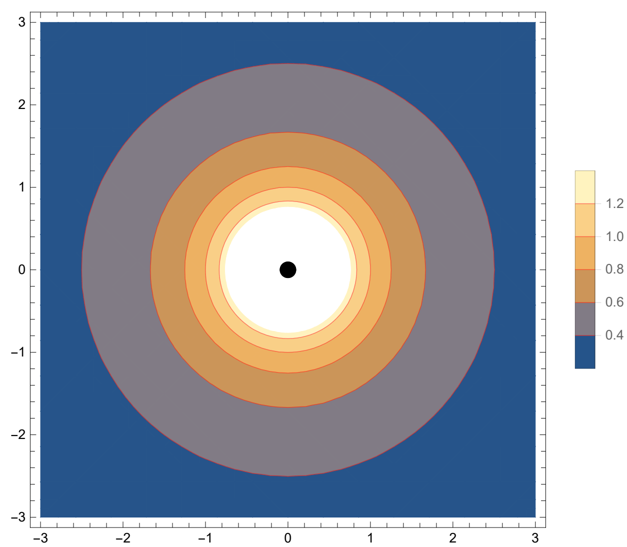


Figure 58: Contour plot of $V_1^*(\tilde{\mathbf{r}}) = -1/\tilde{r} = -1/\sqrt{\tilde{x}^2 + \tilde{y}^2}$ potential.

Note that we made use of dimensionless expressions in Fig. 58 and we call $V_1^*(\tilde{\mathbf{r}})$ the dimensionless potential. From Fig. 58 we see the two charges which are placed on top of each other in the origin ($\tilde{x} = 0$, $\tilde{y} = 0$) and the potential is spherically symmetric. The red circles give the equipotential surfaces which are spherically symmetric in this case.

Now, we can derive the two-charge potential in polar coordinates. We choose to put the origin of the two potentials along the x-axis around the origin.

$$\tilde{V}_1(\mathbf{r}) = -\frac{\alpha_1}{r} = -\frac{\alpha_1/2}{\sqrt{(x-x_0)^2+y^2}} - \frac{\alpha_1/2}{\sqrt{(x+x_0)^2+y^2}}. \tag{129}$$

Because in polar coordinates we have $x = r\cos(\theta_0)$ and $y = r\sin(\theta_0)$ we rewrite Eq.(129) up to:

$$\tilde{V}_1(\mathbf{r}) = -\frac{\alpha_1}{r} = -\frac{\alpha_1/2}{\sqrt{(rcos(\theta_0)-x_0)^2+y^2}} - \frac{\alpha_1/2}{\sqrt{(rcos(\theta_0)+x_0)^2+(rsin(\theta_0))^2}}. \tag{130}$$

$$\Leftrightarrow \tilde{V}_1(\mathbf{r}) = -\frac{\alpha_1}{r} = -\frac{\alpha_1/2}{r\sqrt{1-\frac{2x_0cos(\theta_0)}{r}+\frac{x_0^2}{r^2}}} - \frac{\alpha_1/2}{r\sqrt{1+\frac{2x_0cos(\theta_0)}{r}+\frac{x_0^2}{r^2}}}. \tag{131}$$

In Fig. (59) we show the dimensionless two-charge potential $\tilde{V}_1^*$ with two origins of central forces placed in $\tilde{x}_0 = \pm 0.5$.

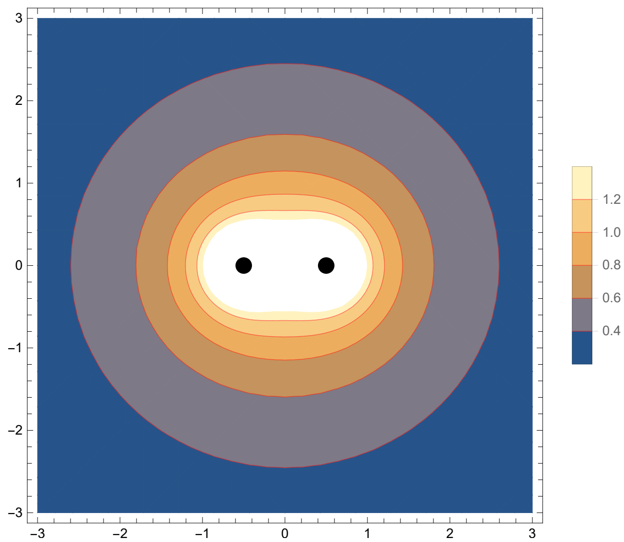


Figure 59: Contour plot of $\tilde{V}_1^*(\tilde{\mathbf{r}}) = -1/(2\sqrt{(\tilde{x}-\tilde{x}_0)^2+\tilde{y}^2}) - 1/(2\sqrt{(\tilde{x}+\tilde{x}_0)^2+\tilde{y}^2})$ potential with $\tilde{x}_0 = 0.5$.

Clear to see from Fig. 59 that the equipotential surfaces are no longer spherically symmetric because the equipotential surfaces from both charges interfere with each other. In the next sections we will see what happens with the orbits of a particle in such system. Of course we can make the same calculation for potential $V_2(\tilde{\mathbf{r}}) = -\alpha_2/r^2$. In this case the two-charge potential becomes:

$$\tilde{V_2}(\mathbf{r}) = -\frac{\alpha_2}{r^2} = -\frac{\alpha_2/2}{\left(r\sqrt{1-\frac{2x_0cos(\theta_0)}{r}+\frac{x_0^2}{r^2}}\right)^2} - \frac{\alpha_2/2}{\left(r\sqrt{1+\frac{2x_0cos(\theta_0)}{r}+\frac{x_0^2}{r^2}}\right)^2}. \tag{132}$$

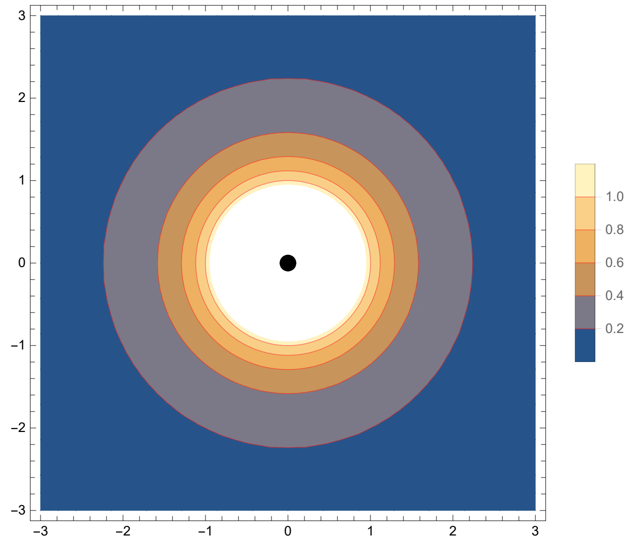


Figure 60: Contour plot of $V_2^*(\tilde{\mathbf{r}}) = -1/\tilde{r}^2 = -1/(\tilde{x}^2 + \tilde{y}^2)$ potential.

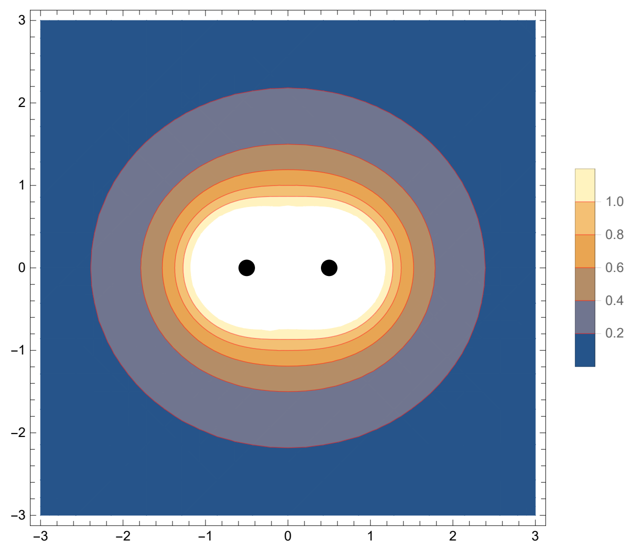


Figure 61: Contour plot of $\tilde{V}_2^*(\tilde{\mathbf{r}}) = -1/(2((\tilde{x} - \tilde{x}_0)^2 + \tilde{y}^2)) - 1/(2((\tilde{x} + \tilde{x}_0)^2 + \tilde{y}^2))$ potential with $\tilde{x}_0 = 0.5$.

In Fig. 60 and Fig. 61 we show the potential $\tilde{V}_2^*$ and the two-charge potential $\tilde{V}_2^*$ where the two charge are placed along the x axis. Again, the system becomes spherically asymmetric. Note that in this case the potential is stronger and more singular than in the previous cases.

10.1 Non-relativistic particle

10.1.1 Non-relativistic particle in $\tilde{V}_1(\mathbf{r})$ potential

We can verify that we get the same results as before if we put both potentials in the same place $x_0 = 0\alpha_1/mc^2$ in the origin of the coordinate system. In order to find the same orbits as before, we reproduce Fig. 12 by taking the same parameters.

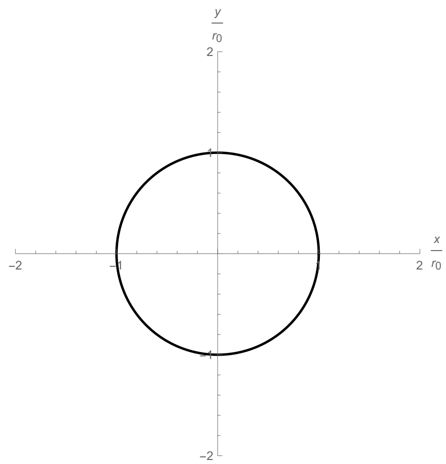


Figure 62: By putting the potentials on each other and by making use of the same parameters as in Fig. 12, $\tilde{L}^2_{nr} = 1$, $\tilde{E} = -1/(2\tilde{L}^2_{nr}) = -0.5$ and $\theta_0 = 0°$ we see that we get again the same orbit as before. The polar angle is varied in the range $[0, 2\pi]$.

Now by making use of the two-charge potential given in Eq. (131) we can investigate the influence on the orbits of a non-relativistic particle by putting two potentials near each other. We will make use of dimensionless units.

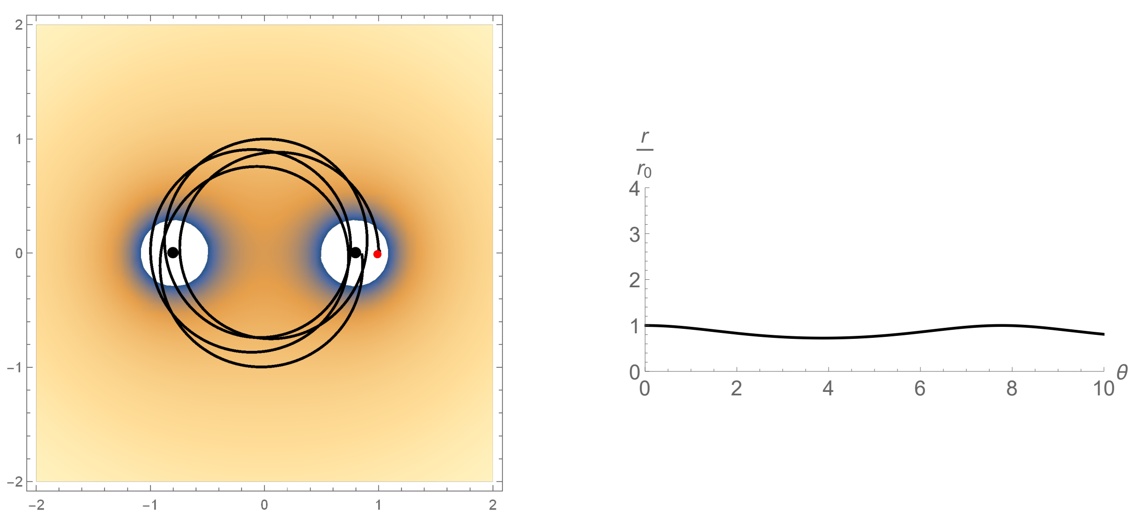


Figure 63: Solution of orbit equation for $\tilde{L}_{nr} = 2$, $\tilde{E} = -0.135412$, $\theta_0 = 0.005\pi$ and the particle starts at $r_0 = \tilde{L}^2_{nr}\alpha_1/mc^2 = 4\alpha_1/mc^2$ (red dot). The origin of the potentials is situated in $x_0 = \pm 0.8\alpha_1/mc^2$. The polar angle is varied in the range $[0, 8\pi]$.

From Fig. 63 we see how the orbit is affected in the presence of two potentials. The initial position of the particle is indicated by the red dot in the left figure of Fig. 63. The particle performs precessing ellipses around the potentials with the origin of these as alternately focus. Note that in the non-relativistic case we never observed precessing ellipses. From the right figure of Fig. 63 we see indeed that the particle performs an elliptical orbit and oscillates around 1 where we use dimensionless units with $\tilde{r} = r/r_0$. Now we can take a look at the scattering states.

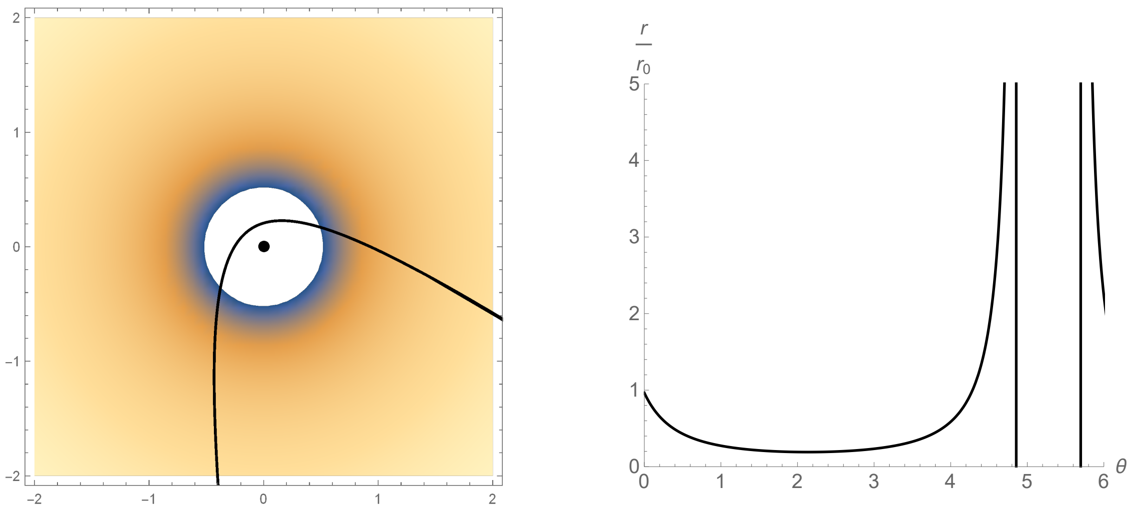


Figure 64: Solution of orbit equation for $\tilde{L}_{nr} = 2$, $\tilde{E} = 0.025$, $\theta_0 = 0.005\pi$ and the particle starts at $\tilde{r}_0 = \tilde{L}_{nr}^2 \alpha_1/mc^2 + 6\alpha_1/mc^2 = 10\alpha_1/mc^2$. The origin of the potentials are situated along the x-axis at $x_0 = 0\alpha_1/mc^2$. The polar angle is varied in $[0, 2\pi]$.

From the left figure of Fig. 64 we see a scattering state which is described by a parabolic orbit with the origin of the central force as focus. Because we plot the orbit in dimensionless units, the particle begins the orbit in $r_0 = 10\alpha_1/mc^2$ at a begin polar angle $\theta_0 = 0.005\pi$. From the right figure of Fig. 64 we see the radius as function of the polar angle. Clear to see is that the particle is placed in $\tilde{r} = r/r_0 = 1$ and afterwards reaches a minimal radius which means that the particle is very close to the origin of the potential. Afterwards the particle escapes and goes to infinity. Next we investigate what happens in the presence of two potentials. We take the same parameters as in Fig. 64 but we decide to put the potentials in $x_0 = \pm 0.8\alpha_1/mc^2$ and to change the begin position of the particle in $r_0 = 10$. In the left figure of Fig. 65 we see the orbits in the case of two-charge potential. In this figure two orbits are shown. The particle's motion begins in $\tilde{r} = 1$ and the begin position is indicated by the red dot. The first orbit is followed by the red arrows, when the particle escapes from the system it comes back and the second orbit is indicated by the black arrows. The particle moves between the two potentials and at a certain moment reaches a minimum radius which is the closest point of the orbit to the origin of the coordinate system. On the right figure of Fig. 65 the radius as function of the polar angle is shown.

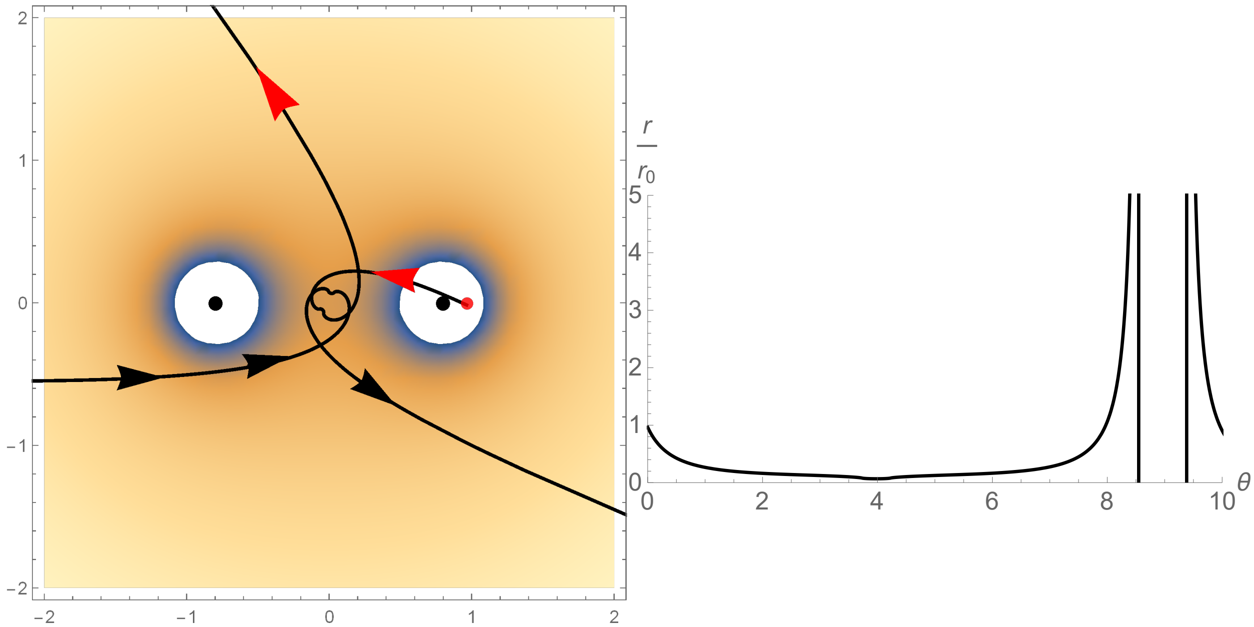
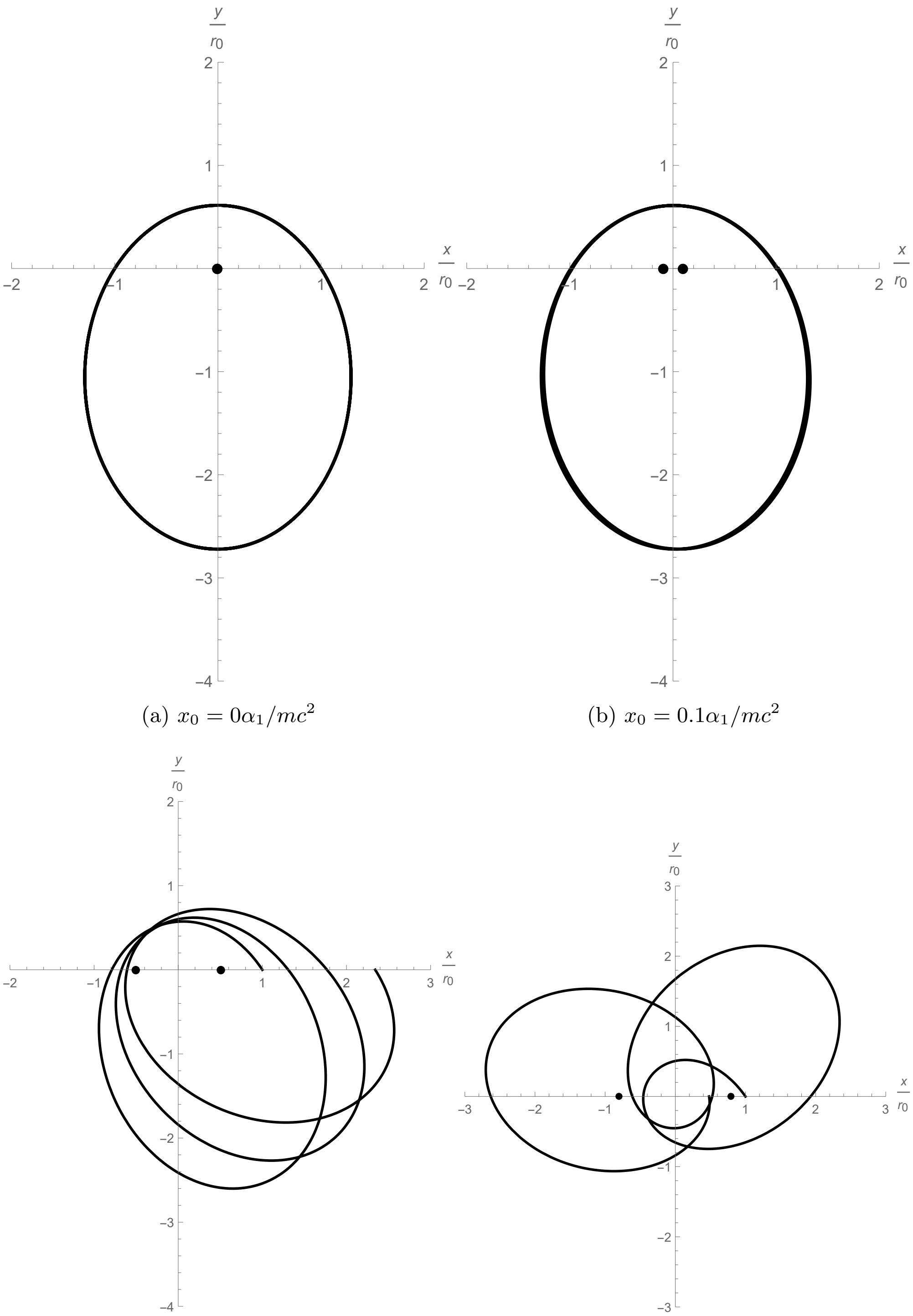


Figure 65: Solution of orbit equation for $\tilde{L}_{nr} = 2$, $\tilde{E} = 0.025$, $\theta_0 = 0.005\pi$ and the particle starts at $r_0 = \tilde{L}^2_{nr}\alpha_1/mc^2 + 6\alpha_1/mc^2 = 10\alpha_1/mc^2$ (red dot). The origin of the potentials are situated along the x-axis at $x_0 = 0.8$ (black dots). The polar angle is varied over the range $[0, 6\pi]$.

We have seen in Fig. 63 that the two-charge potential leads to precessing ellipses around the two origins of the forces. We show another example where such pattern is found again. In Fig. 66 we investigate the influence on an elliptical orbit. We decide to start at a begin position $r_0 = 4\alpha_1/mc^2$ at a begin polar angle $\theta_0 = 0°$. In Fig. 66 (a) we show the elliptical orbit in the case when the two potentials are on top of each other in the origin of the coordinate system. But when we increase a bit the distance between the two potentials as shown in Fig. 66 (b) we note that the particle begins to follow precessing ellipses which are very weak pressing motions. When the distance increases we note in Fig. 66 (c) and (d) that the particle begins to make more clearly precessing ellipses which in (d) if we varies the polar angle in a larger range, the orbit takes a rosette structure.



(a) $x_0 = 0\alpha_1/mc^2$ (b) $x_0 = 0.1\alpha_1/mc^2$

(c) $x_0 = 0.5\alpha_1/mc^2$ (d) $x_0 = 0.8\alpha_1/mc^2$

Figure 66: Solution of orbit equation for $\tilde{L}_{nr} = 2$, $\tilde{E} = -0.075$, $\theta_0 = 0°$ and the particle starts with $r_0 = \tilde{L}_{nr}^2 \alpha_1/mc^2 = 4\alpha_1/mc^2$. The polar angle is varied in $[0, 6\pi]$.

In the last part of this section we investigate what happens when we place the particle between the two potentials as begin condition.

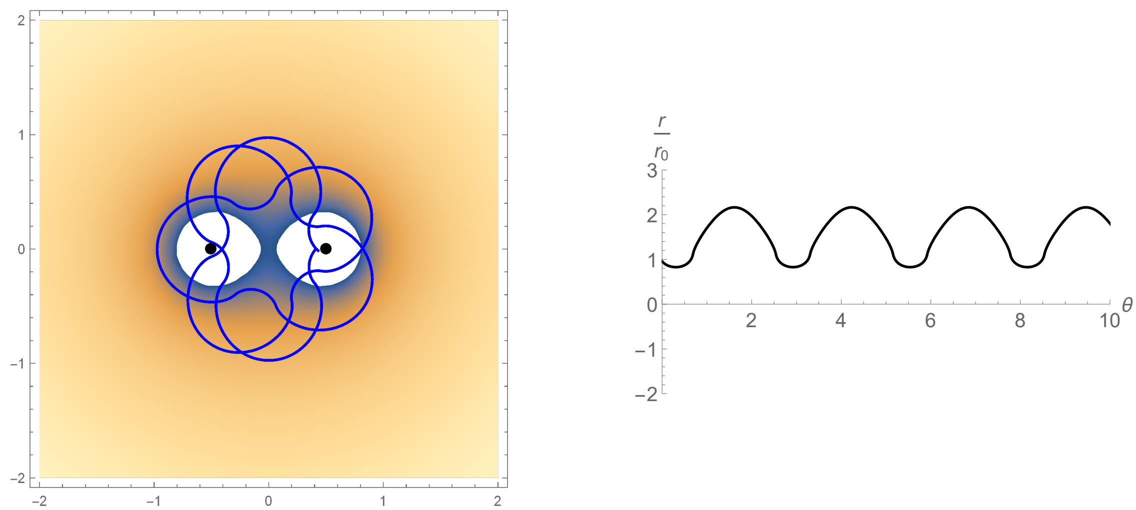


Figure 67: Solution of orbit equation for $\tilde{L} = 1$ kgm^2/s, $\theta_0 = 0.01\pi$ and $\tilde{E} = -0.86612$. The origin of the central forces are situated along the x-axis in $x_0 = \pm 0.5\alpha_1/mc^2$ and are given by the black dots. The begin radius is given by $r_0 = 0.45\alpha_1/mc^2$. The polar angle is varied in $[0\pi, 6\pi]$.

We clearly see in the left figure of Fig. 67 that the particle is strongly bound by the two potentials and circle infinite around them. The right figure of Fig. 67 shows the radius as function of the polar angle θ. We see that the radius indeed oscillates and will never diverge which means that the particle will stay bounded to the potentials.

10.1.2 Non-relativistic particle in $\tilde{V_2}^*(\tilde{r})$ potential

In this section we will visualize what happens with the orbits of a non-relativistic particle in the presence of the two-charge potential. We will show some solutions in the case where both potentials are on top of each other and afterwards we will split the potentials along the x axis and we will investigate what happens with the orbits. The equation of motion in this case becomes:

$$\frac{\partial^2 \tilde{u}}{\partial \theta^2} + \tilde{u} + \frac{1}{\tilde{L}^2}\left(\frac{\tilde{u}^2\left(2\tilde{x}_0{}^2\tilde{u} + 2\tilde{x}_0\cos(\theta_0)\right)}{2\left(\tilde{x}_0{}^2\tilde{u}^2 + 2x_0\cos(\theta_0)\tilde{u}+1\right)^2} + \frac{\tilde{u}^2\left(2\tilde{x}_0{}^2\tilde{u} - 2\tilde{x}_0\cos(\theta_0)\right)}{2\left(\tilde{x}_0{}^2\tilde{u}^2 - 2\tilde{x}_0\cos(\theta_0)\tilde{u}+1\right)^2} - \frac{\tilde{u}}{\tilde{x}_0{}^2\tilde{u}^2 + 2\tilde{x}_0\cos(\theta_0)\tilde{u}+1} - \frac{\tilde{u}}{\tilde{x}_0{}^2\tilde{u}^2 - 2\tilde{x}_0\cos(\theta_0)\tilde{u}+1}\right) = 0. \tag{133}$$

And the energy equation becomes:

$$\tilde{L}^2\tilde{u}'^2 + \tilde{L}^2\tilde{u}^2 = \tilde{E} + \frac{\tilde{u}^2}{2\left(\tilde{x}_0^2\tilde{u}^2 + 2\tilde{x}_0\cos(\theta_0)\tilde{u}+1\right)} + \frac{\tilde{u}^2}{2\left(\tilde{x}_0^2\tilde{u}^2 - 2\tilde{x}_0\cos(\theta_0)\tilde{u}+1\right)}. \tag{134}$$

Of course the only way to solve Eq. (133) is numerically.

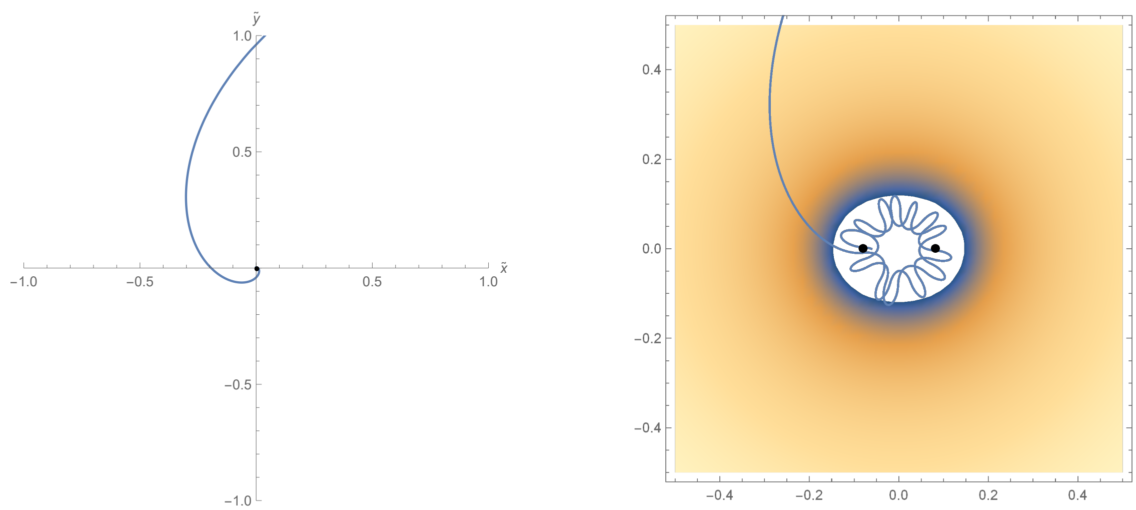


Figure 68: Solution of orbit equation Eq. (133), $\tilde{L} = 0.8$, $\tilde{E} = 0.1$, $\theta_0 = 0.00001\pi$ and $\tilde{r}_0 = 20$. In the left figure the origins of the central forces are situated along the x-axis and they are on top of each other in $\tilde{x}_0 = \pm 0$ (black dot). On the right we show for the same parameters the orbit, but here the potentials are split $\tilde{x}_0 = \pm 0$ (black dots). The polar angle is varied in the range $[0\pi, 5\pi]$ for both cases.

Clear to see from the left figure of Fig. 68 that the particle is trapped by the potential and will spiral into the origin. The fact that the particle spirals into the origin can be graphically shown by plotting the radius as function of θ, see Fig. 69. Further, from the right figure of Fig. 68 is too see that the particle approaching to the two-charge potential gets trapped and will infinitely circle around the two charges, this means that by splitting the potentials the orbit becomes bound. In Fig. 70 we illustrate the radius as function of θ in the presence of the two-charge potential.

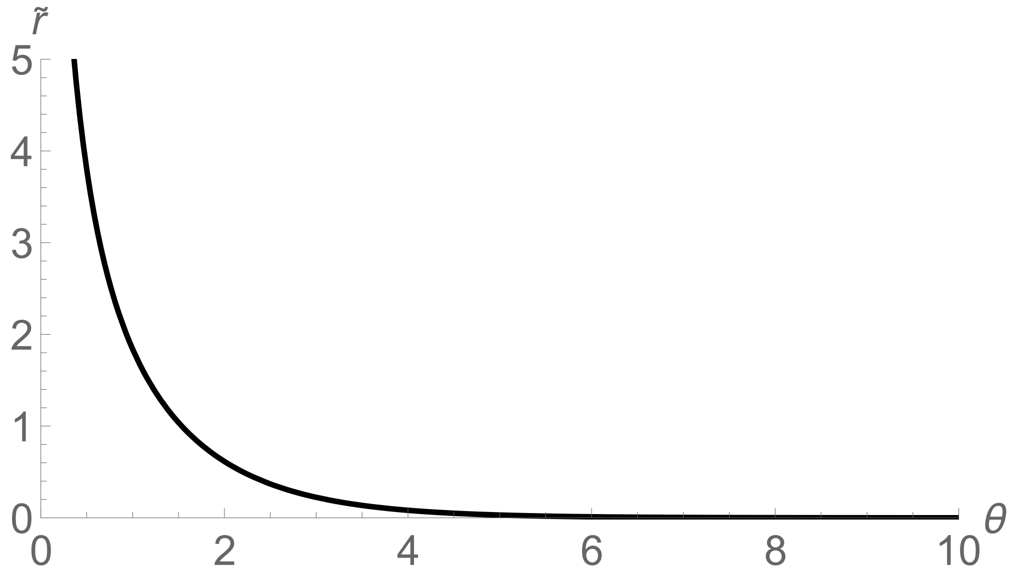


Figure 69: Radius as function of polar angle where both potentials are on top of each other.

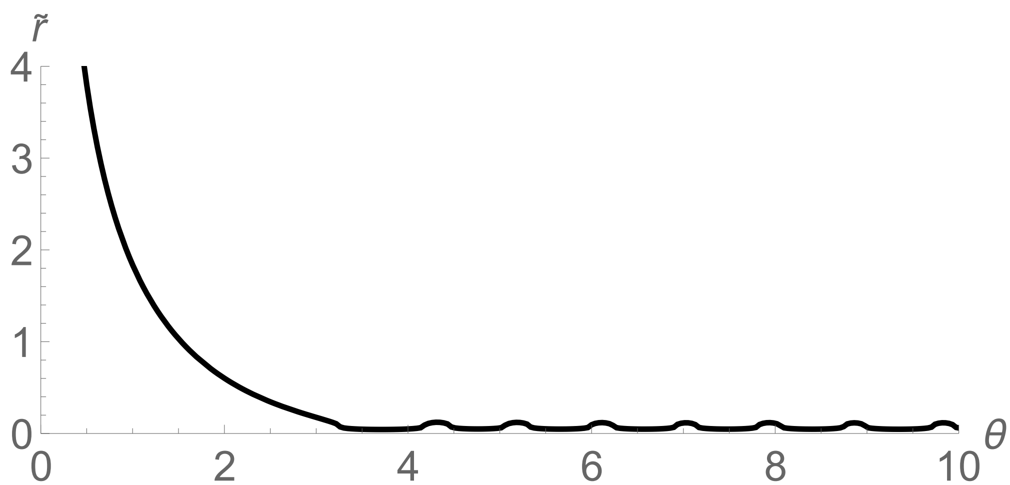


Figure 70: Radius as function of polar angle in presence of the two-charge potential.

It is clear from Fig. 70 that the particle gets trapped and performs a bounded orbit because of the oscillations of the radius.

10.2 Relativistic particle in $\tilde{V}_1(\mathbf{r})$ potential

We can verify that we get the same results as before if we put both potentials on top of each other, i.e. $x_0 = 0\alpha_1/mc^2$ in the origin of the coordinate system. In order to find the same orbits as before, we reproduce Fig. 51 by taking the same parameters.

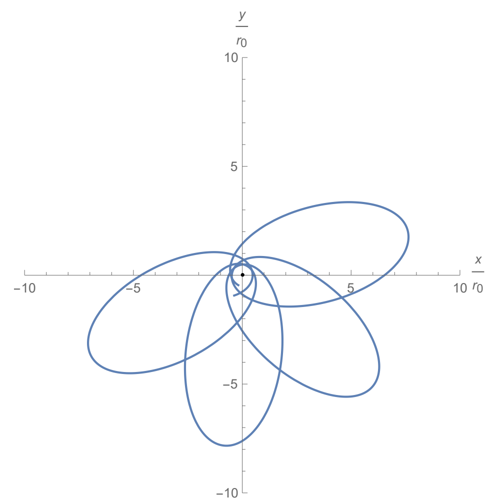


Figure 71: Solution of orbit equation in the case of two-charge potential from Eq.(131) for parameters: $\tilde{L} = 2, E = \sqrt{1 - (1/\tilde{L})^2} + 0.09999 = 0.966025$, $\theta_0 \approx -1.94899°$ and $x_0 = 0\alpha_1/mc^2$. The polar angle is varied in $\in [-2\pi, 8\pi]$.

By using the same parameters as in Fig. 71 we can investigate and observe what happens if we split the two potentials along the x axis.

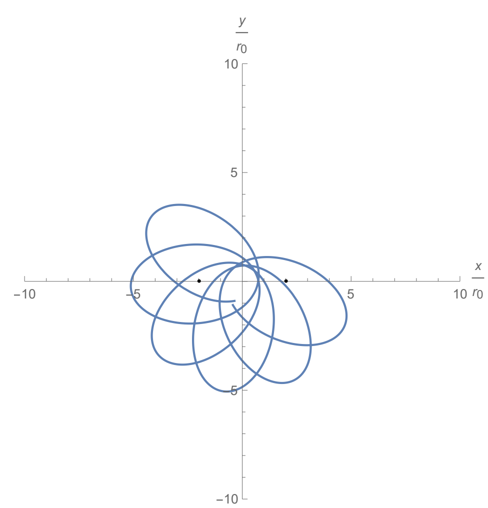


Figure 72: Solution of orbit equation in the case of two-charge potential from Eq.(131) for parameters: $\tilde{L} = 2, E = \sqrt{1 - (1/\tilde{L})^2} + 0.09999 = 0.966025$, $\theta_0 \approx -1.94899°$ and $x_0 = 2\alpha_1/mc^2$. The polar angle is varied in $\in [-2\pi, 8\pi]$.

Clear to see from Fig. 72 is that the particle is more bounded in comparison with Fig. 71, this is because the particle feels the two-charge potential and physically is attracted in more directions which provides for more changes in the trajectory. We can graphically see that the particle follows a more bounded orbit by showing the radius as function of the polar angle θ.

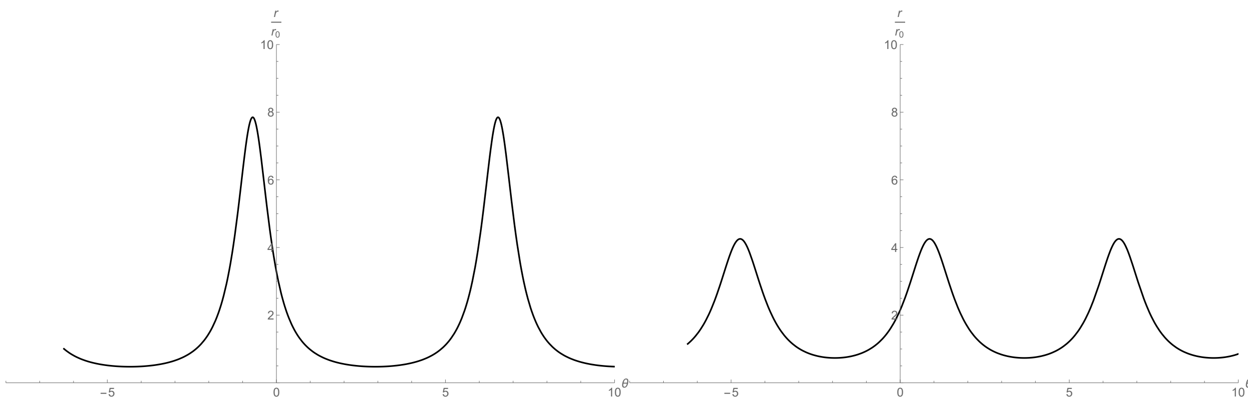


Figure 73: Radius as function of the polar angle θ.

The left figure of Fig. 73 shows the radius as function of the polar angle for the orbit in Fig. 71 while on the right is for Fig. 72. Clear to see as a consequence of the two-charge potential is that the radius becomes smaller which means that the radius is more bounded by the potentials. Finally we show graphically in Fig. 74 that the angular momentum in the case of a two-charge potential is no longer a conserved quantity and exhibits oscillations.

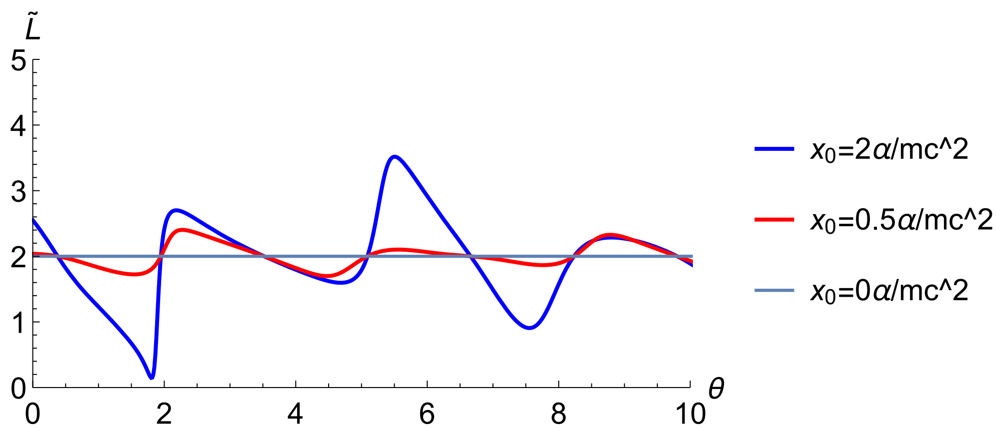


Figure 74: Angular momentum as function of the polar angle θ for different distances between the two potentials.

Further we can visualize what happens with the unstable orbits where the particle spirals into the origin. We clearly see from Fig. 75, Fig. 76 and Fig. 77 that the particle gets trapped by the two-charge potential and will circle around the charges. This behavior is also found in the previous case for a non-relativistic particle.

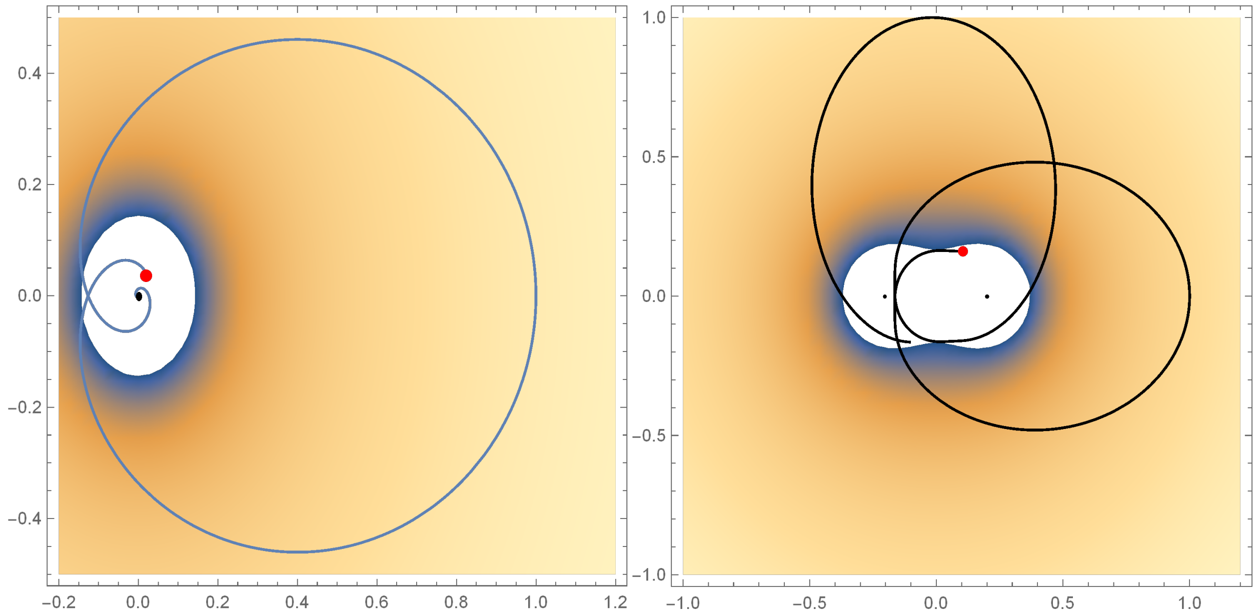


Figure 75: Solution of orbit equation for parameters: $\tilde{L} = 0.9$,$\tilde{E} = 0.1$, $\theta_0 \approx 1°$. In the left figure the potentials are on top of each other in $x_0 = 0\alpha_1/mc^2$. In the right figure the potentials are in $x_0 = \pm 0.2\alpha_1/mc^2$. In both cases the polar angle is varied in $\in [0, 5\pi]$. The red dot indicates the begin position of the particle.

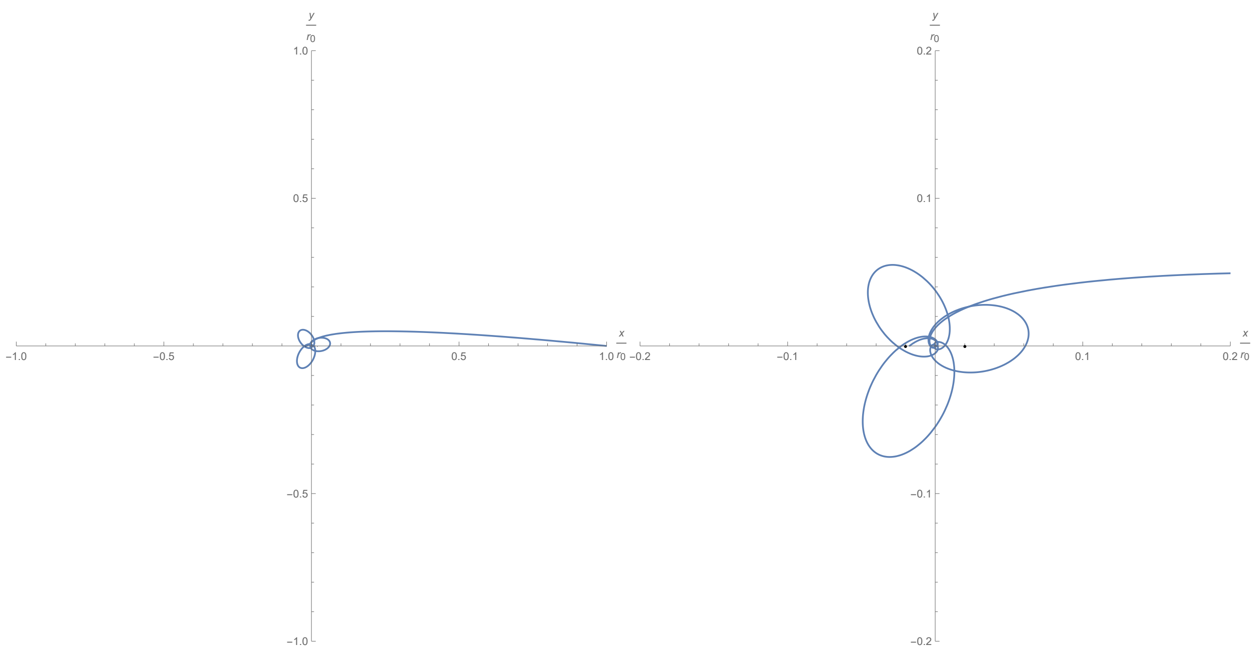


Figure 76: Solution of orbit equation in presence of two-charge potential in Eq. (131) for parameters: $\tilde{L} = 0.9$,$\tilde{E} = 1$, $\theta_0 \approx -0.0127259°$ and $x_0 = \pm 0.02\alpha_1/mc^2$. The polar angle is varied in the range $\in [0, 15\pi]$.

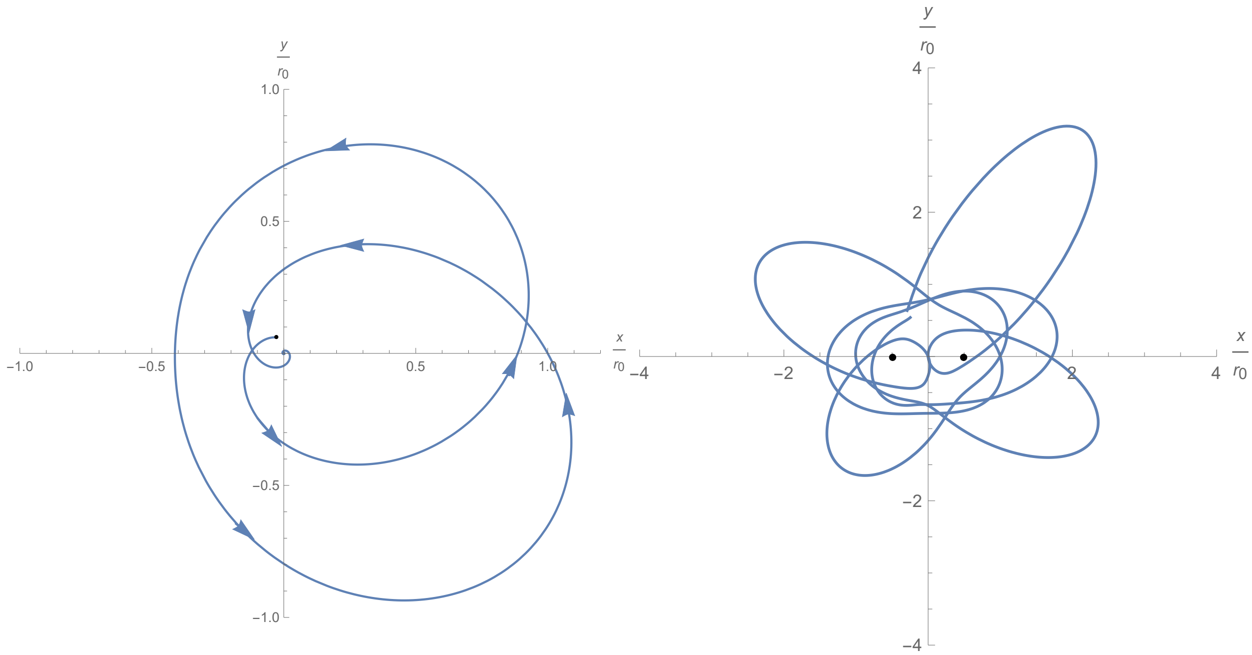


Figure 77: Solution of orbit equation in presence of two-charge potential in Eq. (131) for parameters: $\tilde{L} = 0.9$, $\tilde{E} = 0.1$, $\theta_0 \approx 2°$. The polar angle is varied in both cases in the range $\in [-2\pi, 8\pi]$. On the left the potentials are on top of each other with $x_0 = 0\alpha_1/mc^2$ and the start position is given by the black dot. The arrows indicate the direction of the orbit. In this case the particle spirals into the origin. On the right we take the same parameters but we split the potential along the x axis in $x_0 = \pm 0.5\alpha_1/mc^2$.

In the left figure of Fig. 77 we see the unstable orbit where an atomic collapse is observed. In the right figure of Fig. 77 we split the potential along the x axis in $x_0 = 0.5\alpha_1/mc^2$. We clearly see that the particle is trapped in the two-charge potential system and circles around both potentials. This is because both potentials have the same strength and there is no reason why the particle should be more attracted by one of the two charges. Here we clearly see that the particle is trapped by the system of the two potentials and will move infinitely between them.

11 Conclusions and outlook

In this master thesis I investigated the possible orbits of a relativistic and non-relativistic particle in Kepler/Coulomb-type potentials and studied the phenomena of atom collapse using classical physics.

We found that in a Kepler or Coulomb potential a relativistic and non-relativistic particle exhibits different behavior. In the non-relativistic case we have shown that the particle can have circular, elliptical, parabolic/ hyperbolic Kepler orbits. In the case of bound states the particle follows the same circular/elliptical orbit and shows no precessing motion. Here we always have stable orbits unless the magnitude of the angular momentum becomes zero. By introducing the relativistic nature of the particle we have shown that the angular momentum of the particle plays an important role and depending on its magnitude we find stable and unstable orbits. The stable orbits are found when the magnitude of the angular momentum is larger than the critical angular momentum which depending on the magnitude of the energy we find scattering and bound states. As scattering states we find for angular momentum much larger than the critical angular momentum a Kepler orbit while approaching the critical angular momentum we find that the particle circles around the origin and escapes. As bound states we find circular and elliptical orbits. The latter shows precessing motion. As unstable orbit we find that the particle spirals into the center of the potential causing an atomic collapse. From the graph which shows the relation between dimensionless energy and dimensionless velocity we show that the energy is conserved while the relativistic particle spirals into the origin. This cannot be the case for a non-relativistic particle which needs infinite energy to provide for atomic collapse.

We have shown that by increasing the power of the potential up to $n = 2$ a non-relativistic particle cannot perform a stable bound orbit. In this case we found that kinetic and potential energy compensate each other and there is no way to find relations between energy and velocity such that these can be graphically shown. We observed that the distinction between stable and unstable orbits is fully determined by the angular momentum and the critical angular momentum in this case is different from the previous cases and is equal to $L_c = \sqrt{2m\alpha_2}$. Below this critical value we observe unstable orbits causing atomic collapse and above this threshold we get stable orbits as parabolic/ hyperbolic orbits. By introducing the relativistic nature of the particle we have shown that circular orbits are possible. Here we found two critical angular momenta one is given by $L_c^{(1)} = \sqrt{2m\alpha_2}$ and provide for a divergence in radius which leads to a free particle, the other $L_c^{(2)}$ which provides for atomic collapse ($\tilde{R} = 0$) when the magnitude of the angular momentum becomes infinite. In this case the kinetic and potential energy doesn't compensate each other and consequently we are capable to find a relation between energy and velocity. We studied the two limits, one when $\tilde{v} \ll c$ (non-relativistic region) kinetic and potential energy compensate each other and the other limit where $\tilde{v} \to c$. From this last limit we found interesting results where we concluded that a relativistic particle in $V(r) = -\alpha_2/r^2$ potential behaves as a non-relativistic particle in $V(r) = -\alpha_1/r$ potential in the sense that spiraling into the center of the potential is not physically possible because infinite energy is needed. As possible orbits we found stable repulsive orbits when the energy is large in negative sense, a circular orbit when the conditions are satisfied, a stable scattering state where the particle approaching the origin performs circles around the origin of the potential and afterwards escapes. Unstable orbits reminiscent of atomic collapse are found where the particle suddenly falls on the origin of the potential or depending on the begin position of the particle, this one gets trapped by the potential and performs a few circles before falling into the origin.

By removing the singularity in $r = 0$ in each case we have that the particle gains some kinetic energy which results in scattering states.

In the presence of the two-charge potential we found for each case the same patterns. When we place the particle between the two potentials the particle performs an infinite circular motion around the two potentials. When we have scattering states the trajectory deforms and the particle escapes from the potentials.

For a non-relativistic particle in case of the two-charge potential we found bound orbit with a precessing elliptic motion with one of the two charges as alternately focal point. Also are found for a non-relativistic particle scattering states where the orbit is being deformed because of the two-charge potential.

These last two behaviors are also found for a relativistic particle in $V \sim 1/r$ potential. The only difference in observed trajectories is that in the relativistic case the unstable orbits become bounded orbits in the presence of two-charge potential.

One could investigate how atomic collapse depends on the distance between the core of the two potentials. In my thesis the only way to find atomic collapse in the presence of two potentials is to make one of them much stronger. But one could make an accurate analysis in order to find a critical angular momentum where the particle falls into the core of one potential and afterwards making a phase diagram in order to classify the different trajectories.

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Appendices

A Relativistic energy

The relativistic kinetic energy is given by:

$$T = mc^2\gamma - mc^2.$$

By expanding the γ factor for non-relativistic limit case, we find $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots$ This way the relativistic kinetic energy becomes the kinetic energy of a non-relativistic particle $T \approx mc^2(1 + \frac{1}{2}\frac{v^2}{c^2}) - mc^2 \approx \frac{1}{2}mv^2$. The total energy is rewritten in this way:

$$E = \epsilon + mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{\alpha_1}{r}.$$

Such that ϵ is the energy difference from the particle rest energy.

B Dimension analysis

In the following table we find a review of the found quantities:

Particle	Energy	Angular momentum	velocity	radius
Relativistic	$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\alpha_1}{\mathbf{r}}$	$L_{rel} = \frac{rmv}{\sqrt{1-\frac{v^2}{c^2}}}$	$v = \frac{\alpha_1}{L_{rel}}$	$r = \frac{L^2}{\alpha_1 m}\sqrt{1 - (\frac{\alpha_1}{cL})^2}$
Non-relativistic	$\frac{1}{2}mv^2 - \frac{\alpha_1}{\mathbf{r}} + mc^2$	$L_{nr} = rmv$	$v = \frac{\alpha_1}{L_{nr}}$	$r = \frac{L_{nr}^2}{m\alpha_1}$

Table 5: In this table there is a summary of quantities for a relativistic and non-relativistic particle in a $V(r) = -\alpha_1/r$ potential.

Looking at the expressions in this table we find:

Parameter	Unit
c	[m/s]
v	[m/s]
L	[kgm^2/s]
α_1	[kgm^3/s^2]
E	[mc^2]
r	[m]
m	[kg]

Table 6: Dimension analysis for a relativistic particle in $V(r) = -\alpha_1/r$ potential.

By introducing dimensionless parameters $\tilde{E} = E/mc^2$,$\tilde{R} = r/(\alpha_1/mc^2)$,$\tilde{v} = v/c$ and $\tilde{L} = Lc/\alpha_1$ we can rewrite the expressions such:

Particle	Energy	r vs v	$\tilde{R}$ vs $\tilde{v}$	$\tilde{E}$ vs $\tilde{v}$
Relativistic	$\tilde{E} = \frac{1}{\sqrt{1-\tilde{v}^2}} - \frac{1}{\tilde{R}}$	$r_{rel} = \frac{L^2}{m\alpha_1} \cdot \sqrt{1 - \frac{v^2}{c^2}}$	$\tilde{R}_{rel} = \frac{1}{\tilde{v}^2}\sqrt{1-\tilde{v}^2}$	$\tilde{E} = \sqrt{1-\tilde{v}^2}$
Non-relativistic	$\tilde{E} = \frac{1}{2}\tilde{v}^2 - \frac{1}{\tilde{R}} + 1$	$r_{nr} = L_{nr}/mv$	$\tilde{R}_{nr} = 1/\tilde{v}^2$	$\tilde{E} = 1 - \frac{1}{2}\tilde{v}^2$

Table 7: In this table there is a summary of dimensionless quantities for a relativistic and non-relativistic particle in a $V(r) = -\alpha_1/r$ potential.

Particle	dimensionless radius
Relativistic	$\tilde{R} = \tilde{L}^2\sqrt{1 - \frac{1}{\tilde{L}^2}}$
Non-relativistic	$\tilde{R} = \tilde{L}^2$

Table 8: In this table we find the relation between radius and angular momentum in dimensionless units for a relativistic and non-relativistic particle in a $V(r) = -\alpha_1/r$ potential.

C Velocity and angular momentum in polar coordinates

Take $\mathbf{r} = r\widehat{r}$, with $\widehat{r}$ the unit polar vector.

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\widehat{r} + r\frac{d\widehat{r}}{dt} = \dot{r}\widehat{r} + r\frac{d\widehat{r}}{dt}.$$

From [2]:

$$\widehat{r} = \widehat{x}\cos(\theta) + \widehat{y}\sin(\theta); \widehat{\theta} = -\widehat{x}\sin(\theta) + \widehat{y}\cos(\theta).$$

We have:

$$\frac{d\widehat{r}}{dt} = -\widehat{x}\sin(\theta)\dot{\theta} + \widehat{y}\cos(\theta)\dot{\theta} = \dot{\theta}(-\widehat{x}\sin(\theta) + \widehat{y}\cos(\theta)) = \dot{\theta}\widehat{\theta}.$$

The unit vectors of the Cartesian frame doesn't change in time, this means that the derivatives of the vectors $\widehat{x}$,$\widehat{y}$ with respect to time are zero. Finally we can compute:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\widehat{r} + r\dot{\theta}\widehat{\theta}.$$

Now we can compute the angular momentum for a relativistic particle in polar coordinates. We combine the definition $\mathbf{L} = \mathbf{r} \times \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}$ with $\mathbf{r} = r\widehat{r}$ and the found velocity. Since the radial unit vector $\widehat{r}$ and the angular unit vector $\widehat{\theta}$ are perpendicular we can simplify the expression of angular momentum by dropping out the term $\dot{r}\widehat{r}$ in the velocity, this is because $\mathbf{r} \times \dot{r}\widehat{r} = 0$. So we find $\mathbf{r} \times \mathbf{v} = r\widehat{r} \times r\dot{\theta}\widehat{\theta} = r^2\dot{\theta}|\widehat{r}||\widehat{\theta}|\sin(\pi/2) = r^2\dot{\theta}$. Now we have to calculate v^2:

$$v^2 = (\mathbf{v})^2 = (\dot{r}\widehat{r} + r\dot{\theta}\widehat{\theta})^2 = (\dot{r}\widehat{r})^2 + (r\dot{\theta}\widehat{\theta})^2 + 2\dot{r}r\dot{\theta}\widehat{r} \cdot \widehat{\theta} = \dot{r}^2 + r^2\dot{\theta}^2.$$

Here we made use of the fact that the magnitude of the unit vectors are one and the dot product of the unit vectors is equal to zero because $\widehat{r} \cdot \widehat{\theta} = |\widehat{r}||\widehat{\theta}|\cos(\frac{\pi}{2}) = 0$. Finally we find the angular momentum of a relativistic particle in polar coordinates.

$$\mathbf{L} = \mathbf{r} \times \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow L = \frac{mr^2\dot{\theta}}{\sqrt{1-\frac{(\dot{r}^2+r^2\dot{\theta}^2)}{c^2}}}.$$

D Angular velocity

$$L^2 = \frac{m^2r^4\dot{\theta}^2}{1-\frac{(\dot{r}^2+r^2\dot{\theta}^2)}{c^2}} \Leftrightarrow L^2\left(1-\frac{(\dot{r}^2+r^2\dot{\theta}^2)}{c^2}\right) = m^2r^4\dot{\theta}^2 \Leftrightarrow L^2c^2 - L^2\dot{r}^2 - L^2r^2\dot{\theta}^2 = m^2r^4\dot{\theta}^2c^2.$$

$$\Leftrightarrow L^2c^2 - L^2\dot{r}^2 - L^2r^2\dot{\theta}^2 = m^2r^4\dot{\theta}^2c^2.$$

$$\Rightarrow \dot{\theta} = \sqrt{\frac{L^2c^2-L^2\dot{r}^2}{L^2r^2+m^2r^4c^2}}.$$

E Energy in polar coordinates

$$E = \frac{mc^2}{\sqrt{1-\frac{\dot{r}^2+r^2\left(\frac{L^2c^2-L^2\dot{r}^2}{L^2r^2+m^2r^4c^2}\right)}{c^2}}} - \frac{\alpha_1}{r}.$$

By working out the square root we have:

$$\sqrt{1-\frac{\dot{r}^2+r^2\left(\frac{L^2c^2-L^2\dot{r}^2}{L^2r^2+m^2r^4c^2}\right)}{c^2}} = \sqrt{1-\frac{\dot{r}^2}{c^2}-\frac{r^2\left(\frac{L^2c^2-L^2\dot{r}^2}{L^2r^2+m^2r^4c^2}\right)}{c^2}} = \sqrt{1-\frac{\dot{r}^2}{c^2}-\left(\frac{L^2c^2-L^2\dot{r}^2}{c^2(L^2+m^2r^2c^2)}\right)}.$$

$$\Leftrightarrow \sqrt{1-\frac{\dot{r}^2}{c^2}-L^2\left(\frac{1-(\frac{\dot{r}}{c})^2}{L^2+m^2r^2c^2}\right)}.$$

Consequently we find:

$$E = \frac{mc^2}{\sqrt{1-(\dot{r}/c)^2-L^2(1-(\dot{r}/c)^2)/(L^2+m^2r^2c^2)}} - \frac{\alpha_1}{r}.$$

F Energy of a relativistic particle

$$(\mathbf{p})^2 = \left(\frac{m\dot{r}\widehat{r}}{\sqrt{1-\frac{v^2}{c^2}}} + \widehat{\theta}\frac{L}{r}\right)^2 = \frac{m^2\dot{r}^2}{1-\frac{v^2}{c^2}} + \frac{L^2}{r^2}. \qquad (135)$$

Here we made use of the fact that the magnitude of the unit vectors is equal to 1 ($|\widehat{\theta}|^2 = 1$,$|\widehat{r}|^2 = 1$) and that both unit vectors are perpendicular with each other ($\widehat{r} \cdot \widehat{\theta} \sim cos(\pi/2) = 0$). By looking at $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\alpha_1}{r}$ and $E = \sqrt{p^2c^2 + m^2c^4} - \alpha_1/r$ it is easy to see that we need to proof :

$$\sqrt{p^2c^2 + m^2c^4} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

We work out the left hand side:

$$\sqrt{p^2c^2 + m^2c^4} = \sqrt{\frac{m^2\dot{r}^2c^2}{1 - \frac{v^2}{c^2}} + \frac{L^2c^2}{r^2} + m^2c^4} = \sqrt{\frac{m^2\dot{r}^2c^2}{1 - \frac{v^2}{c^2}} + \frac{m^2r^4\dot{\theta}^2c^2}{r^2(1 - \frac{v^2}{c^2})} + m^2c^4}.$$

$$\Leftrightarrow \sqrt{\frac{m^2v^2c^2}{1 - \frac{v^2}{c^2}} + m^2c^4} = \sqrt{\frac{m^2v^2c^2}{1 - \frac{v^2}{c^2}} + \frac{m^2c^4(1 - \frac{v^2}{c^2})}{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{m^2v^2c^2 + m^2c^4 - m^2c^2v^2}{1 - \frac{v^2}{c^2}}}.$$

$$\Leftrightarrow \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Where in the third equality we made use of the relativistic angular momentum in Eq.(27) and in the fourth equality we made use of the velocity in polar coordinates squared. (see 5.4)

G Coefficient differential equation for a relativistic particle in $V(r) = -\alpha_1/r$ potential

General solutions	
$L > \alpha_1/c$	$u(\theta) = (c_1 + c_2) \cdot cos\left(\sqrt{1 - (\frac{\alpha_1}{Lc})^2}(\theta - \theta_0)\right) + \frac{E\alpha_1}{L^2c^2 - \alpha_1^2}$
$L < \alpha_1/c$	$u(\theta) = (c_1 + c_2) \cdot cosh\left(\sqrt{(\frac{\alpha_1}{Lc})^2 - 1}(\theta - \theta_0)\right) + \frac{E\alpha_1}{L^2c^2 - \alpha_1^2}$

We will derive the coefficient in the case of $L > \alpha_1/c$, for the case $L < \alpha_1/c$ the calculation is the same but we need to take into account the relations $cosh^2(x) - sinh^2(x) = 1$ and $(cosh(x))' = sinh(x)$. By putting the general solution for $L > \alpha_1/c$ into

$$\left(E + \alpha_1 u\right)^2 = \left(\frac{du}{d\theta}\right)^2 L^2c^2 + u^2L^2c^2 + m^2c^4$$

We get:

$$\left(E + \alpha_1 u\right)^2 = E^2 + 2E\alpha_1\left((c_1 + c_2)cos\left(\sqrt{1 - (\frac{\alpha_1}{Lc})^2}(\theta - \theta_0)\right) + \frac{E\alpha_1}{L^2c^2 - \alpha_1^2}\right) + \left((c_1 + c_2)cos\left(\sqrt{1 - (\frac{\alpha_1}{Lc})^2}(\theta - \theta_0)\right)\right) + \frac{E\alpha_1}{L^2c^2 - \alpha_1^2}\right)^2.$$

By working out the second power we get:

$$\left(E+\alpha_1 u\right)^2 = E^2 + 2E\alpha_1(c_1+c_2)cos\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right) + 2\frac{E^2\alpha_1^2}{L^2c^2-\alpha_1^2} +$$

$$(c_1+c_2)^2 cos^2\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right) + \frac{E^2\alpha_1^2\alpha_1^2}{(L^2c^2-\alpha_1^2)^2} + \frac{2(c_1+c_2)cos\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right)E\alpha_1\alpha_1^2}{L^2c^2-\alpha_1^2}.$$

On the other hand:

$$\left(\frac{du}{d\theta}\right)^2 L^2c^2 + u^2L^2c^2 + m^2c^4 = \left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(c_1+c_2)sin\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right)\right)^2 L^2c^2 +$$

$$\left((c_1+c_2)cos\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right)^2 + \frac{E\alpha_1}{L^2c^2-\alpha_1^2}\right)^2 L^2c^2 + m^2c^4.$$

By working out the second power and by taking into account the relation $cos^2(x)+sin^2(x)=1$, the right hand side of the equation becomes:

$$\left(\frac{du}{d\theta}\right)^2 L^2c^2 + u^2L^2c^2 + m^2c^4 = (c_1+c_2)^2\left(L^2c^2 - \alpha_1^2 sin^2\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right)\right) +$$

$$\frac{E^2\alpha_1^2L^2c^2}{(L^2c^2-\alpha_1^2)^2} + \frac{2(c_1+c_2)cos\left(\sqrt{1-(\frac{\alpha_1}{Lc})^2}(\theta-\theta_0)\right)E\alpha_1L^2c^2}{L^2c^2-\alpha_1^2} + m^2c^4.$$

By combining right and left hand side we get:

$$E^2L^2c^2 - m^2c^4(L^2c^2-\alpha_1^2) = (c_1+c)^2[(L^2c^2-\alpha_1^2)^2].$$

$$\Rightarrow (c_1+c_2) = \sqrt{\frac{E^2L^2c^2 - m^2c^4(L^2c^2-\alpha_1^2)}{(L^2c^2-\alpha_1^2)^2}}.$$

H Dimension analysis for solutions of a relativistic particle in $V(r) = -\alpha_1/r$

From Appendix B we make use of Table 2. In expression (57) and (58) we have almost the same prefactor which differs just by a minus sign, so the unit analysis is the same.

$$\sqrt{\frac{E^2L^2c^2 - m^2c^4(L^2c^2-\alpha_1^2)}{(L^2c^2-\alpha_1^2)^2}} \sim \left[\sqrt{\frac{kg^4m^{10}/s^8 - kg^4m^{10}/s^8}{(kg^2m^6/s^4 - kg^2m^6/s^4)^2}}\right] \sim \left[\sqrt{\frac{1}{m^2}}\right] \sim \left[\frac{1}{m}\right].$$

The particular solution(s) is/are given by $E\alpha_1/(L^2c^2-\alpha_1^2)$ (for (58) differs by a minus sign). By using Tabel 2 we find:

$$\frac{E\alpha_1}{L^2c^2-\alpha_1^2} \sim \left[\frac{kg^2m^5/s^4}{kg^2m^6/s^4}\right] \sim \left[\frac{1}{m}\right].$$

Because the solutions are $u(\theta) = 1/r$ we have that the radius is given in meters. The same analysis can be done in the case of $L = L_c = \alpha_1/c$ which the solution is given by (53). There the particular solution is given by $(m^2c^4 - E^2)/(2E\alpha_1)$.

$$\frac{m^2c^4 - E^2}{2E\alpha_1} \sim \left[\frac{kg^2m^4/s^4 - kg^2m^4/s^4}{kg^2m^5/s^4}\right] \sim \left[\frac{1}{m}\right].$$

I non-relativistic particle in $V(r) = -\alpha_1/r$ potential

We can find the orbit equations $r(\theta)$ by expressing the non-relativistic momentum ($c \to \infty$) in polar coordinates:

$$\mathbf{p} = \widehat{r}p_r + \widehat{\theta}p_\theta = m(\dot{r}\widehat{r} + \widehat{\theta}r\dot{\theta}) = m\dot{r}\widehat{r} + m\widehat{\theta}r\dot{\theta} = m\dot{r}\widehat{r} + \widehat{\theta}\frac{L}{r}. \tag{136}$$

The last equality is find by making use of $L = mr^2\dot{\theta}$ (see chapter 7). Because we will need later the following expression, we work out p_r/p_θ.

$$\frac{p_r}{p_\theta} = \frac{m\dot{r}}{mr\dot{\theta}} = \frac{1}{r}\frac{\dot{r}}{\dot{\theta}} = \frac{1}{r}\frac{dr}{d\theta}. \tag{137}$$

By using $p_\theta = L/r$ found in expression (136) we finally find:

$$p_r = \frac{L}{r^2}\frac{dr}{d\theta}. \tag{138}$$

Now we can rewrite the non-relativistic energy expression $E = \frac{p^2}{2m} - \alpha_1/r$ up to:

$$E + \frac{\alpha_1}{r} = \frac{p^2}{2m} = \frac{1}{2m}\underbrace{m^2\dot{r}^2}_{=(p_r)^2} + \frac{L^2}{2mr^2}. \tag{139}$$

Combining (138) with (139) gives the following differential equation:

$$E + \frac{\alpha_1}{r} = \frac{1}{2m}\left(\frac{L}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{1}{2m}\frac{L^2}{r^2}. \tag{140}$$

By introducing the r-inverse variable $u = 1/r \Rightarrow du/d\theta = -(1/r^2)dr/d\theta$ we can rewrite the differential equation in (140) up to:

$$E + \alpha_1 u = \frac{1}{2m}\left(\frac{du}{d\theta}\right)^2 L^2 + \frac{1}{2m}u^2L^2. \tag{141}$$

If we differentiate (141) with respect to θ we find:

$$\alpha_1\frac{du}{d\theta} = \frac{1}{m}L^2u\frac{du}{d\theta} + \frac{1}{m}L^2\frac{d^3u}{d^3\theta}. \tag{142}$$

By dividing (142) by $du/d\theta$ and rewriting the equation, we find a second order linear differential equation which describes the orbits of a non-relativistic particle in $V(r) = -\alpha_1/r$ potential.

$$\frac{d^2u}{d^2\theta} + u - \frac{m\alpha_1}{L^2} = 0. \tag{143}$$

By using methematica we find the general solution of (144):

$$u(\theta) = c_1 cos(\theta - \theta_0) + \frac{m\alpha_1}{L^2}. \tag{144}$$

With c_1 an integration constant which we easly find by inserting the solution $u(\theta)$ in (141). We find that the integration constant is given by $c_1 = \frac{m\alpha_1}{L_{nr}^2}\sqrt{1 + \frac{2L_{nr}^2 E_{nr}}{m\alpha_1^2}}$ (same calculation as Appendix G). In order to compare our results we can find the solutions of a non-relativistic particle in $V(r) = -\alpha_1/r$ potential. Of course we know the solutions in this case they are Kepler (parabolic,hyperbolic,elliptic,circular) orbits. In this case the energy is expressed as:

$$E = \frac{1}{2}mv^2 - \frac{\alpha_1}{r} + mc^2 \Rightarrow \tilde{E} = \frac{E}{mc^2} = \frac{1}{2}\tilde{v}^2 - \frac{1}{\tilde{r}} + 1.$$

With $\tilde{v} = v/c$, $\tilde{r} = r/(\alpha_1/mc^2)$. By using the equation of motion in order to have a circular orbit we get the relation between kinetic and potential energy, in this case we get $2T = -U$. By using the derived general equation of motion in Eq.(99) we derive the equation of motion in the case of $V(r) = -\alpha_1/r = -\alpha_1 u \Rightarrow F(1/u) = -\nabla V(1/u) = -\alpha_1 u^2$:

$$\frac{d^2u}{d\theta^2} + u - \frac{m\alpha_1}{L^2} = 0. \tag{145}$$

Of course we should have the same differential equation by taking non-relativistic limit ($c \to \infty$) in the equation of motion Eq.(50) of a relativistic particle in $V(r) = -\alpha_1/r$ potential. We can show that by taking this limit we get the same differential equation as in Eq.(145). From Eq.(50) we have:

$$\frac{d^2u}{d^2\theta} + \left[1 - \left(\frac{\alpha_1}{Lc}\right)^2\right]u - \frac{E\alpha_1}{(Lc)^2} = 0. \tag{146}$$

By taking the limit $c \to \infty$ we have that the term in square brackets $\frac{\alpha_1}{Lc}$ vanishes and goes to zero, with L the relativistic angular momentum. Important to notice is that by taking the non-relativistic limit, that the relativistic angular momentum approaches the non-relativistic angular momentum $L \to L_{nr}$ (look at definition of relativistic angular momentum in chapter 5.2.2). On the other hand we have that the energy becomes non-relativistic and we have $E_{rel} \to E_{nrel} \Rightarrow E_{rel} \to E_{nrel} = -\frac{mv^2}{2} + mc^2$. This fact leads to:

$$\lim_{\frac{1}{c}\to 0} \frac{E_{rel}\alpha_1}{(Lc)^2} = \lim_{\frac{1}{c}\to 0} \frac{\alpha_1}{(Lc)^2}(-\frac{mv^2}{2} + mc^2) = \frac{m\alpha_1}{L_{nrel}^2}.$$

By looking at this results we indeed have that Eq.(145) the non-relativistic version is of the equation of motion of a relativistic particle in $V(r) = -\alpha_1/r$ potential. The solution of Eq.(145) is already studied in chapter 5.7 and 5.8.

J Relativistic particle in $V(r) = -\alpha_1/\sqrt{r^2 + a^2}$ potential

To begin we rewrite the potential as function of $u = 1/r$, we get $V(1/u) = -\alpha_1 u/\sqrt{1 + a^2u^2}$. We make use of the same calculation as in chapter 5.6 but we replaced the potential by the found expression in function of u.

$$\left(E + \frac{\alpha_1 u}{\sqrt{1 + a^2u^2}}\right)^2 = \left(\frac{du}{d\theta}\right)^2 L^2c^2 + u^2L^2c^2 + m^2c^4. \tag{147}$$

If we differentiate this expression with respect to θ we get:

$$\frac{2\alpha_1 E}{\sqrt{1+a^2u^2}}\frac{du}{d\theta}-\frac{2\alpha_1 E a^2u^2}{\sqrt{(1+a^2u^2)^3}}\frac{du}{d\theta}+\frac{2\alpha_1^2 u}{(1+a^2u^2)}\frac{du}{d\theta}-\frac{2\alpha_1^2 a^2u^3}{(1+a^2u^2)^2}\frac{du}{d\theta}=2L^2c^2u\frac{du}{d\theta}+2L^2c^2\frac{d^3u}{d^3\theta}. \quad (148)$$

By dividing the previous expression by $(1/2L^2c^2)du/d\theta$ and by rearranging the expressions we get the equation of motion:

$$\frac{d^2u}{d\theta^2}+u-\frac{E\alpha_1}{L^2c^2}\frac{1}{\sqrt{1+(au)^2}}+\frac{E\alpha_1}{L^2c^2}\frac{u^2a^2}{\sqrt{(1+(au)^2)^3}}-\frac{\alpha_1^2}{L^2c^2}\frac{u}{(1+(au)^2)}+\frac{\alpha_1^2}{L^2c^2}\frac{u^3a^2}{(1+(au)^2)^2}=0. \quad (149)$$