Motivation

Motivation

interpolate

$$f(x) = \alpha_1 + \alpha_2 x^{100}$$

- Newton/Lagrange interpolation: 101 samples
- only 4 unknowns: α_1, α_2, x^0, x^{100}!
- how to solve it from 4 samples?

Motivation

- exponential analysis
- generalized eigenvalue problems
- computer algebra
- orthogonal polynomials
- signal processing
- moment problems
- nonlinear approximation theory
- many applications . . .

Motivation

$$x_s = s\Delta, \quad s = 0, 1, 2, \ldots$$

$$\sum_{i=1}^{n} \alpha_i x_s^{k_i} = f_s, \quad n \ll \max(k_i), \quad k_i \in \mathbb{N}$$

$$\sum_{i=1}^{n_1} \alpha_{i,1} \cos(\phi_{i,1} x_s) + \sum_{i=1}^{n_2} \alpha_{i,2} \sin(\phi_{i,2} x_s) = f_s$$

$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \quad \phi_i \in \mathbb{C}$$

Motivation

1. Univariate exponential sparse interpolation
   (Exercise)
2. Multivariate polynomial sparse interpolation
   (Exercise)
3. Connection with rational approximation theory
   (Exercise)
4. Applications unlimited

Basics: exponential

Basics: exponential

Figure: Gaspard Riche de Prony [1795]

Basics: exponential

interpolation problem:

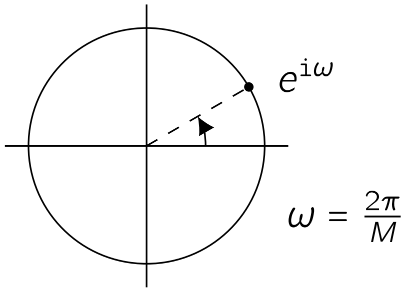
$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \qquad s = 0, \dots, 2n-1$$

$$x_s = s\frac{2\pi}{M}, \qquad \omega = 2\pi/M$$

$$|\Im(\phi_i)| < M/2, \qquad \Omega_i = \exp(\phi_i \omega),$$

$$f_s = \sum_{i=1}^{n} \alpha_i \Omega_i^s, \qquad s = 0, \dots, 2n-1$$

$$\begin{cases} \alpha_1 + \cdots + \alpha_n = f_0 \\ \alpha_1\Omega_1 + \cdots + \alpha_n\Omega_n = f_1 \\ \quad \vdots \\ \alpha_1\Omega_1^{2n-1} + \cdots + \alpha_n\Omega_n^{2n-1} = f_{2n-1} \end{cases}$$



Basics: exponential

finding Ω_i:

$$\prod_{i=1}^{n}(z-\Omega_i) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$$

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \alpha_i \Omega_i^s (\Omega_i^n + b_{n-1}\Omega_i^{n-1} + \cdots + b_0) \\ &= \sum_{i=1}^{n} \alpha_i \Omega_i^{n+s} + \sum_{j=0}^{n-1} b_j \left(\sum_{i=1}^{n} \alpha_i \Omega_i^{j+s} \right) \\ &= f_{s+n} + \sum_{j=0}^{n-1} b_j f_{s+j} \end{aligned}$$

Basics: exponential

$$\begin{pmatrix} f_0 & \dots & f_{n-1} \\ \vdots & \iddots & \vdots \\ f_{n-1} & \dots & f_{2n-2} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix} = - \begin{pmatrix} f_n \\ \vdots \\ f_{2n-1} \end{pmatrix}$$

Hankel matrix:

$$H_n^{(r)} = \begin{pmatrix} f_r & \dots & f_{r+n-1} \\ \vdots & \iddots & \vdots \\ f_{r+n-1} & \dots & f_{r+2n-2} \end{pmatrix}$$

Basics: exponential

Hadamard polynomial:

$$H_n^{(0)}(z) = \begin{vmatrix} f_0 & \dots & f_{n-1} & f_n \\ \vdots & \iddots & \vdots & \vdots \\ f_{n-1} & \dots & f_{2n-2} & f_{2n-1} \\ 1 & \dots & z^{n-1} & z^n \end{vmatrix}$$

$$\prod_{i=1}^{n}(z - \Omega_i) = \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$$

Basics: exponential

formally orthogonal polynomial:

$$\gamma : z^s \to f_s, \quad s = 0, 1, \ldots$$

$$\gamma : \exp(\phi_i x_s) = \Omega_i^s \to \sum_{i=1}^{n} \alpha_i \Omega_i^s = f_s$$

$$\gamma : z^i \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} \to 0, \quad i = 0, \ldots, n-1$$

$$\frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} \perp_\gamma z^i, \quad i = 0, \ldots, n-1$$

[Henrici, 1974]

Basics: exponential

roots of $\frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|}$ from GEP:

$$H_n^{(0)} = \begin{pmatrix} 1 & \dots & & 1 \\ \Omega_1 & \Omega_2 & \dots & \Omega_n \\ \vdots & & & \vdots \\ \Omega_1^{n-1} & \dots & & \Omega_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \Omega_1 & \dots & \Omega_1^{n-1} \\ \vdots & \Omega_2 & & \vdots \\ & \vdots & & \vdots \\ 1 & \Omega_n & \dots & \Omega_n^{n-1} \end{pmatrix}$$

$$= V_n^T D_\alpha V_n$$

$$H_n^{(1)} = V_n^T D_\alpha \begin{pmatrix} \Omega_1 & & \\ & \ddots & \\ & & \Omega_n \end{pmatrix} V_n$$

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Basics: exponential

$$\det\left(H_n^{(1)} - \lambda H_n^{(0)}\right) = \det\left(V_n^T D_\alpha \begin{pmatrix} \Omega_1 - \lambda & & \\ & \ddots & \\ & & \Omega_n - \lambda \end{pmatrix} V_n\right)$$
$$= 0 \text{ for } \lambda = \Omega_i, \quad i = 1, \dots, n$$

[Hua and Sarkar, 1990]

Basics: exponential

finding ϕ_i:

$$\exp(\phi_i) = \exp(\Re(\phi_i))e^{\mathrm{i}\Im(\phi_i)}$$

$$|\Im(\phi_i)| < \frac{M}{2}:$$

$$\begin{aligned}\arg(\Omega_i) &= \arg(\exp(\phi_i\omega)) \\ &= \Im(\phi_i)\,\frac{2\pi}{M} \in \,]-\pi, \pi[\end{aligned}$$

Basics: exponential

finding α_i:

$$\sum_{i=1}^{n} \alpha_i \Omega_i^{s+j} = f_{s+j}, \quad s = 0, \dots, n-1, \quad 0 \le j \le n$$

$$\begin{pmatrix} \Omega_1^j & \dots & \Omega_n^j \\ \vdots & & \vdots \\ \Omega_1^{j+n-1} & \dots & \Omega_n^{j+n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_j \\ \vdots \\ f_{j+n-1} \end{pmatrix}$$

remaining interpolation conditions are linearly dependent

Basics: exponential

finding n:

$$N < n : \left| H_N^{(r)} \right| \not\equiv 0, \qquad r = 0, 1, \ldots$$

$$N = n : \left| H_N^{(r)} \right| \neq 0 \qquad \text{if } \Omega_i \neq \Omega_j \text{ for } i \neq j$$

[Kaltofen and Lee, 2003]

$$N > n : \left| H_N^{(r)} \right| \equiv 0, \qquad r = 0, 1, \ldots$$

Example: exponential

$$\phi(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

$$\alpha_1 = 1 \qquad \phi_1 = 0$$
$$\alpha_2 = 2.4 \qquad \phi_2 = -5 + 19.97\mathtt{i}$$
$$\alpha_3 = -2.1 \qquad \phi_3 = 3 + 45\mathtt{i}$$
$$\alpha_4 = 0.2 \qquad \phi_4 = 5.3\mathtt{i}$$

evaluate at $x_s = s\frac{2\pi}{100}$, $\quad M = 100$, $\quad |\Im(\phi_i)| < 50$

sequence $f_0, \ldots, f_7, \ldots$ is linearly generated

Example: exponential

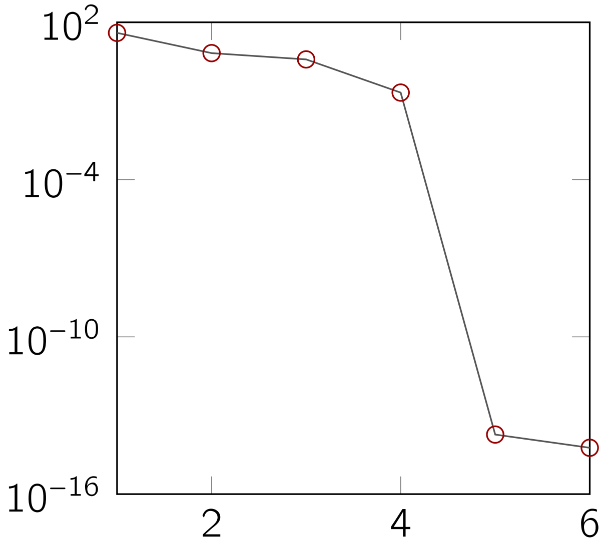

Figure: $H_N^{(0)}$ singular, $N = 6$

$$\frac{|\widetilde{\Omega}_j - \Omega_j|}{|\Omega_j|} \le 2 \times 10^{-12}, \qquad \frac{|\widetilde{\phi}_j - \phi_j|}{|\phi_j|} \le 2 \times 10^{-12}$$