literature (see Whittaker and Robinson [20] and Willers [21]). A related procedure is described in §9.7.

9.4. Exponential Approximation. In certain cases it is desired to determine an approximation of the form

$$f(x) \approx C_1 e^{a_1 x} + C_2 e^{a_2 x} + \cdots + C_n e^{a_n x}, \tag{9.4.1}$$

or, equivalently, of the form

$$f(x) \approx C_1 \mu_1^x + C_2 \mu_2^x + \cdots + C_n \mu_n^x, \tag{9.4.2}$$

where

$$\mu_k = e^{a_k}. \tag{9.4.3}$$

It is somewhat more convenient here to work with the second form (9.4.2). We suppose that a linear change of variables has been introduced, in advance, in such a way that values of $f(x)$ (exact or empirical) are specified at the N equally spaced points $x = 0, 1, 2, \ldots, N - 1$.

If (9.4.1) were to be an equality for these values of x, the equations

$$\begin{aligned} C_1 + C_2 + \cdots + C_n &= f_0, \\ C_1\mu_1 + C_2\mu_2 + \cdots + C_n\mu_n &= f_1, \\ C_1\mu_1^2 + C_2\mu_2^2 + \cdots + C_n\mu_n^2 &= f_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_1\mu_1^{N-1} + C_2\mu_2^{N-1} + \cdots + C_n\mu_n^{N-1} &= f_{N-1} \end{aligned} \tag{9.4.4}$$

would necessarily be satisfied, and the approximation (9.4.2) may be based on the result of satisfying these equations as nearly as possible. If the constants $\mu_1, \ldots, \mu_n$ were known (or preassigned), this set would comprise N linear equations in the n unknowns $C_1, \ldots, C_n$ and could be solved exactly if $N = n$ or approximately, by the least-squares method of §7.3, if $N > n$.

However, if the μ's are also to be determined, at least $2n$ equations are needed, and the difficulty consists in the fact that the equations are *nonlinear* in the μ's. This difficulty can be minimized by a method, similar to methods used in §8.14, next to be described.

Let $\mu_1, \ldots, \mu_n$ be the roots of the algebraic equation

$$\mu^n - \alpha_1\mu^{n-1} - \alpha_2\mu^{n-2} - \cdots - \alpha_{n-1}\mu - \alpha_n = 0, \tag{9.4.5}$$

so that the left-hand member of (9.4.5) is identified with the product $(\mu - \mu_1)(\mu - \mu_2) \cdots (\mu - \mu_n)$. In order to determine the coefficients $\alpha_1, \ldots, \alpha_n$, we multiply the first equation in (9.4.4) by α_n, the second equation by $\alpha_{n-1}, \ldots,$ the nth equation by α_1, and the $(n + 1)$th equation by -1, and add the results. If use is made of the fact that each μ satisfies (9.4.5), the result is seen to be of the form

$$f_n - \alpha_1 f_{n-1} - \cdots - \alpha_n f_0 = 0.$$

A set of $N - n - 1$ additional equations of similar type is obtained in the same way by starting instead successively with the second, third, . . . , $(N - n)$th equations. In this way we find that (9.4.4) and (9.4.5) imply the $N - n$ *linear* equations

$$\begin{aligned} f_{n-1}\alpha_1 + f_{n-2}\alpha_2 + \cdots + f_0\alpha_n &= f_n, \\ f_n\alpha_1 + f_{n-1}\alpha_2 + \cdots + f_1\alpha_n &= f_{n+1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{N-2}\alpha_1 + f_{N-3}\alpha_2 + \cdots + f_{N-n-1}\alpha_n &= f_{N-1}. \end{aligned} \qquad (9.4.6)$$

Since the ordinates f_k are known, this set generally can be solved directly for the n α's if $N = 2n$, or solved approximately, by the method of least squares, if $N > 2n$.

After the α's are determined, the n μ's are found as the roots of (9.4.5). They may be real or complex. The equations (9.4.4) then become linear equations in the n C's, with known coefficients. The C's can be determined, finally, from the first n of these equations or, preferably, by applying the least-squares technique to the entire set.

Thus the nonlinearity of the system is concentrated in the single algebraic equation (9.4.5). The technique described is known as *Prony's method.*

Obvious modifications are necessary when certain of the μ's (or a's) are prescribed and the remainder are to be determined. When such constraints are imposed, and are to be satisfied *exactly,* it is essential to satisfy them (by using them to eliminate unknowns from the set of equations to be solved) *before* applying the method of least squares.

The most common situation of this sort is that in which it is known that $f(x)$ tends to a finite limit (the value of which is generally unknown) as $x \to \infty$. The approximation

$$f(x) \approx C_0 + C_1e^{a_1x} + \cdots + C_ne^{a_nx} \qquad (9.4.7)$$

is then appropriate, where the a's are expected to have negative real parts. Since this approximation implies that

$$\Delta f(x) \approx C_1'e^{a_1x} + \cdots + C_n'e^{a_nx},$$

where the coefficient C_k' is an unknown constant which is simply related to the unknown C_k, the equations (9.4.6) may be modified, in this case, by replacing each f_k by the difference $\Delta f_k \equiv f_{k+1} - f_k$, after which the α's and μ's are determined as before. The equations (9.4.4) are then modified by the insertion of the unknown C_0 in each left-hand member. At least $N = 2n + 1$ independent data are needed for the determination.

If one or more of the μ's satisfying (9.4.5) are not real and positive, the corresponding values of the a's in (9.4.1) will not be real. In particular, if μ_k is real and negative, say $\mu_k = -\rho_k$, where ρ_k is positive, the term $u_k^x = (-\rho_k)^x$ is real only when x takes on the (integral) values for which data are prescribed, or values which differ from those values by integral multiples of the (unit) spacing. However, we may notice that $(-1)^x = \cos \pi x$ *for any such value of* x. Hence, if we replace $(-\rho_k)^x$ by $\rho_k^x \cos \pi x$ or, equivalently, by $e^{x \log \rho_k} \cos \pi x$, we so obtain a suitable interpolating function which is real for all real values of x.

More generally, if one value of μ is *complex*, and hence expressible in the polar form $\rho e^{i\beta}$, where ρ and β are real and ρ is *positive*, then the conjugate $\rho e^{-i\beta}$ must also be involved, since the coefficients in (9.4.5) are necessarily real. The corresponding part of (9.4.2) can then be written as

$$\rho^x(A_1 e^{i\beta x} + A_2 e^{-i\beta x})$$

where A_1 and A_2 are constants which must be conjugate complex in order that the expression be real when x is real. Hence, by writing $A_1 = (C_1 + iC_2)/2$ and $A_2 = (C_1 - iC_2)/2$, this part of the approximation can be expressed in the more convenient form

$$\rho^x(C_1 \cos \beta x + C_2 \sin \beta x) \equiv e^{x \log \rho}(C_1 \cos \beta x + C_2 \sin \beta x), \qquad (9.4.8)$$

after the μ's are determined from (9.4.5) and (9.4.6), but before equations corresponding to (9.4.4) are formed and solved for the coefficients of the approximating functions.

In order to illustrate both the technique and the existence of unfavorable situations, we consider the attempt to recover the equation of the function

$$f(x) = 2.32 - 1.08e^{-x} + 1.20e^{-2x} \qquad (9.4.9)$$

from the values of that function for $x = 0, 1, 2, 3$, and 4, under the hypothesis that the numerical coefficients in (9.4.9) are exact. These values are given, to four decimal places, in the following tabulation:

x	0	1	2	3	4
$f(x)$	2.4400	2.0851	2.1958	2.2692	2.3006

If the ordinates are arbitrarily rounded to *two* decimal places, the required differences of the rounded values are found to be -0.35, 0.11, 0.07, and 0.03, and Eqs. (9.4.6), with f_r replaced by Δf_r, become

$$\begin{aligned} 0.11\alpha_1 - 0.35\alpha_2 &= 0.07, \\ 0.07\alpha_1 + 0.11\alpha_2 &= 0.03, \end{aligned} \qquad (9.4.10)$$

from which there follows $\alpha_1 = \frac{91}{183} \doteq 0.497$ and $\alpha_2 = -\frac{8}{183} \doteq -0.0437$. Equation (9.4.5) then becomes

$$183\mu^2 - 91\mu + 8 = 0$$

and yields $\mu_1 \doteq 0.383$ and $\mu_2 \doteq 0.114$, to three places. Thus the required approximation is to be of the form

$$f(x) \approx C_0 + C_1(0.383)^x + C_2(0.114)^x = C_0 + C_1 e^{-0.96x} + C_2 e^{-2.18x}, \qquad (9.4.11)$$

after which the C's may be determined by fitting the data at three points, or by use of a least-squares procedure over the five points for which data are provided. More nearly accurate determinations of the decay factors would have resulted from a reduction of inherent errors in the data employed, or from the result of using additional data to supply additional equations, and solving the resultant set approximately by least-squares methods.

Suppose, however, that the values $f(1) \doteq 2.0851$ and $f(2) \doteq 2.1958$ were incorrectly rounded to 2.08 and 2.19, respectively. We notice that the round-off errors so introduced are only slightly greater than those effected by the correct rounding, and we may consider these additional errors as representative of observational errors which could result if the data were empirical. The four relevant differences are then -0.36, 0.11, 0.08, and 0.03, and the equations replacing (9.4.10) become

$$\begin{aligned} 0.11\alpha_1 - 0.36\alpha_2 &= 0.08, \\ 0.08\alpha_1 + 0.11\alpha_2 &= 0.03, \end{aligned} \qquad (9.4.12)$$

from which there follows $\alpha_1 = \frac{196}{409} \doteq 0.479$ and $\alpha_2 = -\frac{31}{409} \doteq -0.0758$. The equation which determines approximations to μ_1 and μ_2 is then $409\mu^2 - 196\mu + 31 = 0$, which yields the *complex* roots $\mu_{1,2} \doteq 0.240 \pm 0.361i$. Since, accordingly, $\mu_{1,2} \doteq e^{-1.29 \pm 0.515i}$, the form replacing (9.4.11) here becomes

$$f(x) \approx C_0 + e^{-1.29x}(C_1 \cos 0.515x + C_2 \sin 0.515x), \qquad (9.4.13)$$

from which the C's may be determined by collocation or by least squares.

Whereas it is found that the coefficients in (9.4.11) and (9.4.13) can be determined in such a way that they *both* provide good approximations to the true function (9.4.9) for $0 \leqq x \leqq 4$ and, indeed, depart only slightly from $f(x)$ for all $x \geqq 0$, the latter approximation is oscillatory, while the true function and the former approximation are not. The slight additional errors introduced into the given data here lead to completely incorrect information concerning the *decay factors*.

While this example was selected deliberately to illustrate a particularly unfavorable situation, this type of "instability" is of common occurrence when it is necessary to determine the approximating *coordinate functions* themselves, in addition to the constants of combination to be associated with them. In such cases, it is particularly desirable that an error analysis be made.

Since here the true values of μ_1 and μ_2 are $e^{-1} \doteq 0.368$ and $e^{-2} \doteq 0.135$, the true values of α_1 and α_2 are $e^{-1} + e^{-2} \doteq 0.503$ and $-e^{-3} \doteq -0.0498$. Thus, in the second calculation, errors of magnitude smaller than 0.006 in the data employed lead to errors of about 0.024 and 0.026 in the calculation of α_1 and α_2, respectively, and these errors, in turn, lead to complex approximations of the real μ_1 and μ_2. The *possibility* of appreciably larger errors than those actually encountered in the calculation of α_1 and α_2 from *either* of the sets (9.4.10) and (9.4.12), assuming the coefficients to be correct to the places given, could have been predicted by an analysis of those sets.† Once such estimates are obtained, the maximum (or RMS) values of the errors $\delta\mu_1$ and $\delta\mu_2$ in the roots of $\mu^2 - \alpha_1\mu - \alpha_2 = 0$ may be estimated, by use of the differential relation

$$(2\mu - \alpha_1)\, d\mu = \mu\, d\alpha_1 + d\alpha_2,$$

as

$$|\delta\mu_1|_{\max} \approx \frac{1 + |\mu_1|}{|\mu_2 - \mu_1|}\,|\delta\alpha|_{\max}, \qquad |\delta\mu_2|_{\max} \approx \frac{1 + |\mu_2|}{|\mu_2 - \mu_1|}\,|\delta\alpha|_{\max} \tag{9.4.14}$$

or

$$(\delta\mu_1)_{\text{RMS}} \approx \frac{\sqrt{1 + \mu_1^2}}{|\mu_2 - \mu_1|}\,(\delta\alpha)_{\text{RMS}}, \qquad (\delta\mu_2)_{\text{RMS}} \approx \frac{\sqrt{1 + \mu_2^2}}{|\mu_2 - \mu_1|}\,(\delta\alpha)_{\text{RMS}}, \tag{9.4.15}$$

with μ_1 and μ_2 replaced by their calculated values, if those calculated values are real, and if the errors are small. The reality of μ_1 and μ_2 depends upon the positivity of $\alpha_1^2 + 4\alpha_2$ and is in doubt if $|\alpha_1^2 + 4\alpha_2| < 2|2 + \alpha_1|\,|\delta\alpha|_{\max}$, when α_1 and α_2 are estimated by their calculated values. Similar considerations apply to the more involved cases in which more coordinate functions are employed.

9.5. Determination of Constituent Periodicities. It frequently happens that an empirical function $f(x)$ is known to be expressible as a linear combination of two or more periodic terms whose periods are unknown and are not necessarily commensurable, and the approximate determination of these periods from empirical data is often of considerable importance.

If m distinct periods, denoted by $2\pi/\omega_1, \ldots, 2\pi/\omega_m$, are known (or assumed) to be present, then $f(x)$ correspondingly can be assumed to be approximated by an expression of the form

$$f(x) \approx A_1 \cos \omega_1 x + B_1 \sin \omega_1 x + \cdots + A_m \cos \omega_m x + B_m \sin \omega_m x. \tag{9.5.1}$$

But such an approximation is a special case of (9.4.1), in which $n = 2m$

† The analysis of RMS errors relevant to the normal equations obtained in a least-squares procedure is described in §7.3 [see (7.3.36)]. For the corresponding analysis of maximum errors when least-squares methods are not used, as in the present case, see §10.6