Approximation theory

mathematical (noise free):

1. build $H_\nu^{(0)}, \quad \nu = 0, 1, 2, \ldots$
2. $H_\nu^{(0)} = U \Sigma V^T$ singular value decomposition
3. $\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\nu \end{pmatrix}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > \sigma_{n+1} = \cdots = \sigma_\nu = 0$
4. find $\Omega_i, \phi_i, \alpha_i, i = 1, \ldots, n$

Approximation theory

numerical (with noise):

1. take ν large enough so that noise is clearly separated from n
2. solve $H_\nu^{(1)} v_i = \lambda_i H_\nu^{(0)} v_i, \quad i = 1, \ldots, \nu, \quad \lambda_i = \Omega_i, \quad i = 1, \ldots, n$
3. find ϕ_i
4. solve $\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_j) = f_j, \quad 0 \leq j \leq 2\nu - 1$

Example: noise

$$\phi_1 = 0, \qquad \alpha_1 = 1,$$

$$\phi_2 = -0.2 + 39.5\mathtt{i}, \qquad \alpha_2 = 2, \qquad x_s = s\frac{2\pi}{100},$$

$$\phi_3 = -0.5 + 40\mathtt{i}, \qquad \alpha_3 = 4, \qquad M = 100$$

$$\phi_4 = -1, \qquad \alpha_4 = 8,$$

$\|\varepsilon(z)\|_\infty = 10^{-2}$, uniform random noise

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Sparse Interpolation
Motivation
Basics: exponential
Basics: polynomial
Approximation theory
Applications
References

Example: noise

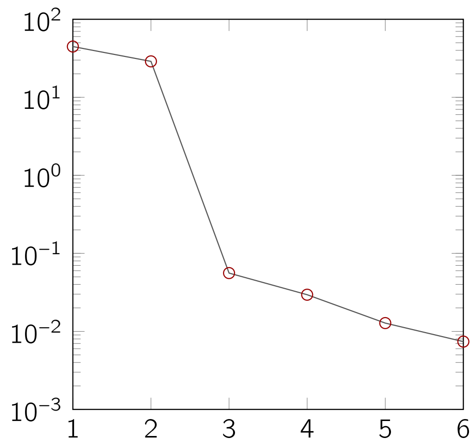

Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 6$

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Example: noise

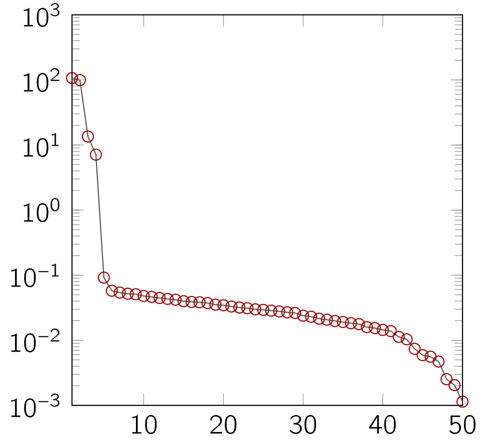


Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 50$

Exercise: approximation

$\phi_1 = 1.5\mathtt{i}$, $\alpha_1 = 10^{-3}$,

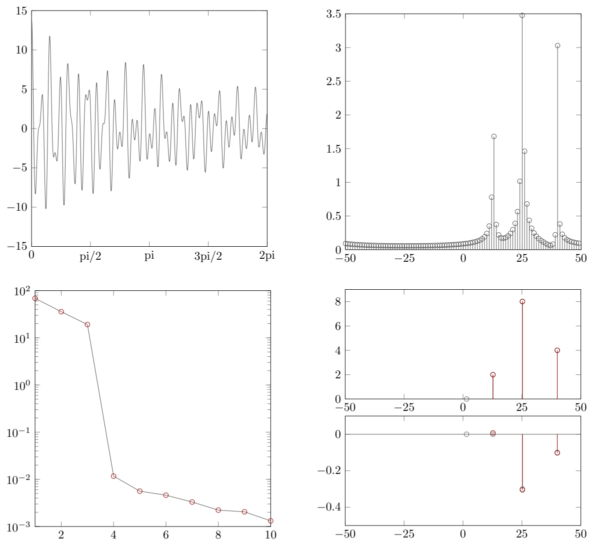
$\phi_2 = 12.7\mathtt{i}$, $\alpha_2 = 2$, $x_s = s\dfrac{2\pi}{100}$,

$\phi_3 = -0.1 + 40\mathtt{i}$, $\alpha_3 = 4$, $M = 100$

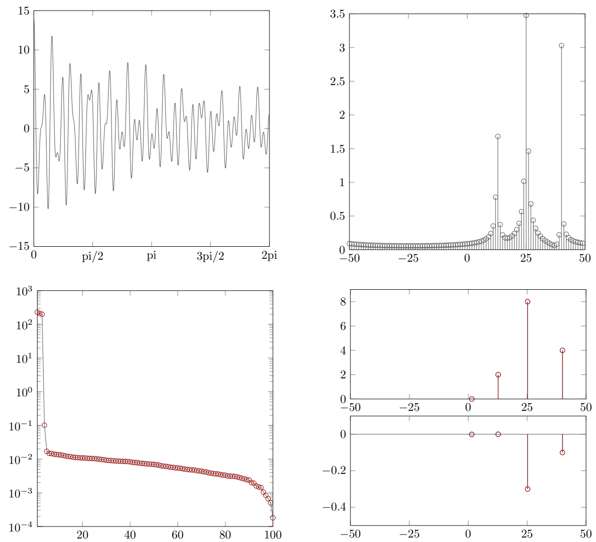
$\phi_4 = -0.3 + 25.2\mathtt{i}$, $\alpha_4 = 8$,

$\|\varepsilon(z)\|_\infty = 2 \times 10^{-3}$, uniform random noise

```
format long;

phi = [1.5*1i, 12.7*1i, -0.1+40*1i, -0.3+25.2*1i];
alpha = [10^(-3), 2, 4, 8];

eps = 2*10^(-3);

M = 100;

plot_signal

pause

plot_fft

pause

% synthesized input data with added noise
N = input('Enter the dimension for SVD: ');
% (10, 6, 3), (100, 100, 4)

randn('seed',0);

omega = 2*pi/M*(0:2*N-1);
f = syn_exp(alpha, phi, omega);

v = randn(size(f))+randn(size(f))*1i;
vv = v/norm(v,Inf);
f = f + eps*vv;

% form Hankel matrices H0 and H1 from y sequence
[H0,H1] = mat_ge(f);

plot_svd

pause

% reconstruct the parameters via generalized eigenvalues
n = input('Size of the model: ');

% compute the generalized eigenvalues and form the Vandermonde system
E = eig(H1(1:n,1:n),H0(1:n,1:n));
V = rot90(vander(E));

% amplitudes
A = V\f(1:n).';

% frequencies and damping factors
alpha_rec = A;
phi_rec = log(E)*M/(2*pi);

pause

% extract the non-zero terms
extract

% plot computed parameters
plot_recontructed_parameters
```



15
10
5
0
−5
−10
−15
0
pi/2
pi
3pi/2
2pi
3.5
3
2.5
2
1.5
1
0.5
0
−50
−25
0
25
50
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
2
4
6
8
10
8
6
4
2
0
−0.2
−0.4



15
10
5
0
−5
−10
−15
0
pi/2
pi
3pi/2
2pi
3.5
3
2.5
2
1.5
1
0.5
0
−50
−25
0
25
50
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
20
40
60
80
100
8
6
4
2
0
−0.2
−0.4

Applications

Applications

Exponential analysis in physical phenomena:

- power system transient detection
- motor fault diagnosis
- drug clearance / glucose tolerance
- magnetic resonance / infrared spectroscopy
- vibration analysis
- seismic data analysis
- music signal processing
- corrosion rate / crack initiation
- odour recognition with electronic nose
- typed keystroke recognition
- liquid (explosive) identification
- ...

Sparse Interpolation

Transients

Figure: Transient detection and characterization

Sparse Interpolation

Transients

short lived high frequency signal:

- speech processing
- turbulent flow
- power lines
- ...

Transients

- model with $\phi_i = 120\pi\mathtt{i}$,

$$\sum_{i=1}^{n} \alpha_i \cos(120\pi x + \gamma_i)\mathbf{1}_{[A_i, Z_i[}$$

- $n = 3, \alpha_i = 1, \gamma_{1,3} = -\pi/2, \gamma_2 = 3\pi/4$
- $[A_1, Z_1[= [0, 0.0308[$
  $[A_2, Z_2[= [0.0308, 0.0625[$
  $[A_3, Z_3[= [0.0625, 0.1058[$
- $M = 1200$
- uniformly distributed noise in $[-0.05, 0.05]$

Transients

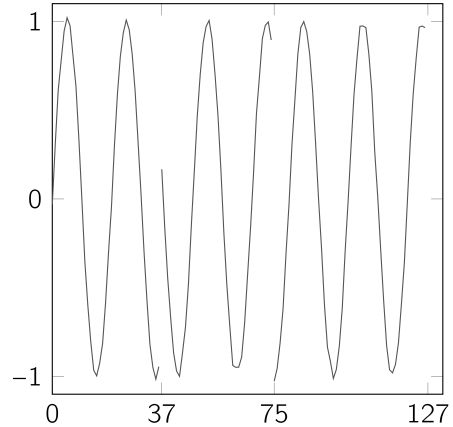


Figure: Given transient signal

Transients

- at each instance: 2 exponential terms
- characteristics of terms change
- inspect rank of

$$H_4^{(1)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ f_2 & f_3 & f_4 & f_5 \\ f_3 & f_4 & f_5 & f_6 \\ f_4 & f_5 & f_6 & f_7 \end{pmatrix}$$

$$[A_1, Z_1[= [0/M, 37/M[,$$
$$[A_2, Z_2[= [37/M, 75/M[,$$
$$[A_3, Z_3[= [75/M, 127/M[$$

Transients

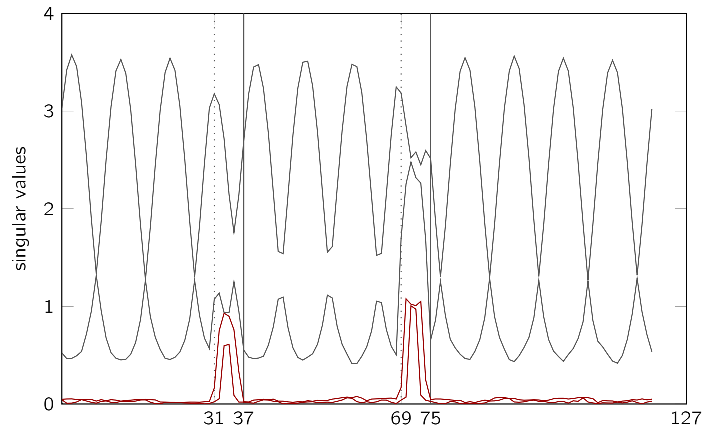


Figure: Numerical rank of $H_4^{(r)}$ evolving over time x

Audio signals

Figure: Reconstructing undersampled audio signals

Audio signals

song containing 29 notes of 0.25 seconds each:

- $M = 44100$ (Hz)
- 11025 samples per note, 319725 in total
- $16.35 \leq \phi_i \leq 4978.03, \; i = 1, \ldots, 100$
- complex exponential model

Audio signals

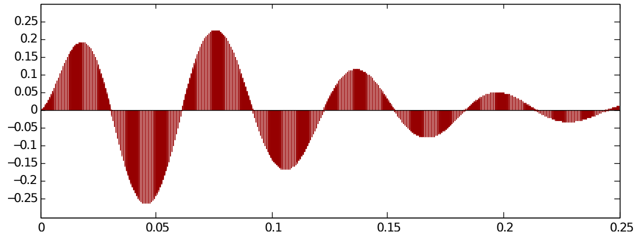


Figure: Sampled signal produced by 1 note

Audio signals

compressive sensing (optimisation, probabilistic)

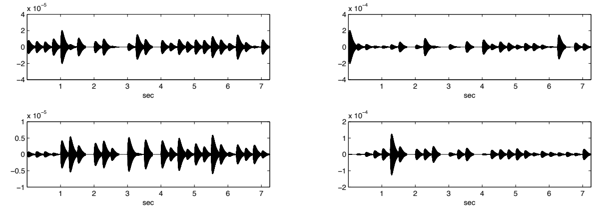


Figure: 4 runs with 1229 samples

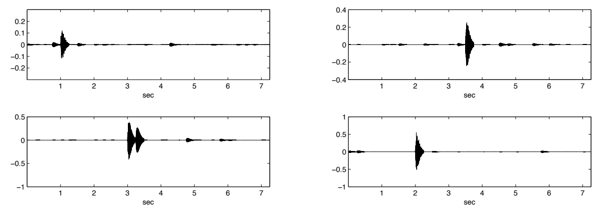


Figure: 4 runs with 456 samples

Audio signals

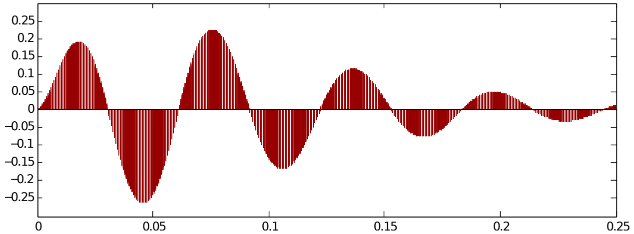


Figure: Full set of samples per note

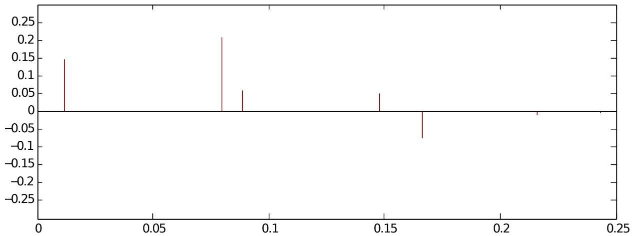


Figure: Sparse interpolation with 7 samples per note

MCSA

Figure: Preventive diagnosis of a broken rotor bar

MCSA

3-phase induction motors:

- consume 40 – 50% of all electricity in industrialized countries
- rotor made up metal bars
- stator current signal analysed
- broken bar(s) characterized by sideband frequencies
- difficult to diagnoze under low or no load

Figure: Stator and rotor

MCSA

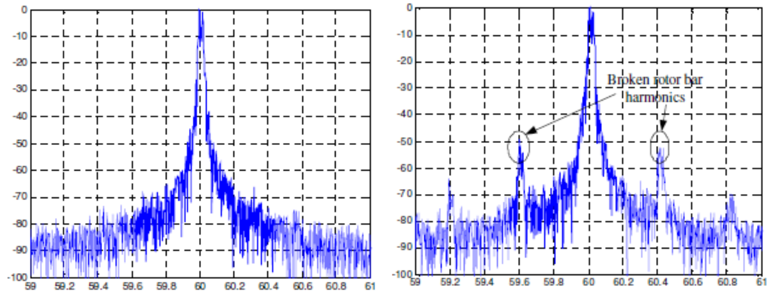


Figure: Stator current FFT spectra: healthy and with 1 broken bar

MCSA

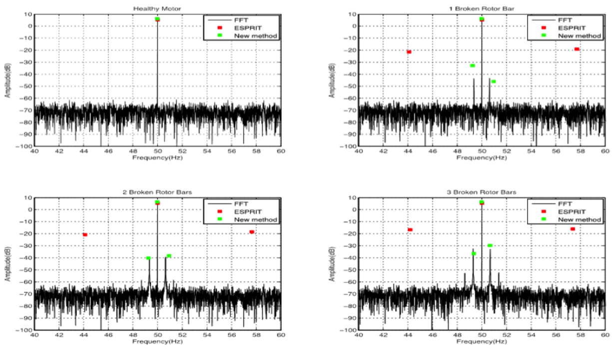


Figure: 10% load, 16dB noise, $\nu = 400$

Bio-electrical

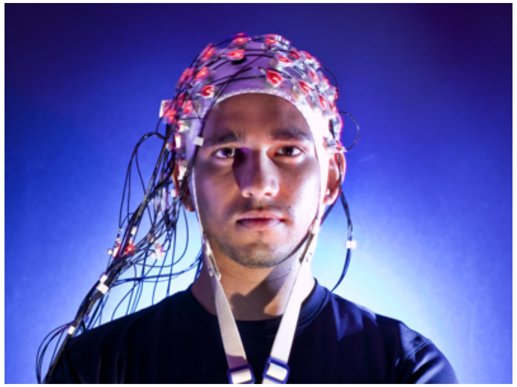


Figure: Sparse EEG approximation

Bio-electrical

Bio-electrical signals:

- electrical activity of cells and tissues
- clinical studies of health status
- ECG, EEG, EMG, EOG, ...
- sparse model $(n = 8)$ is approximate

Sparse
Interpolation

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Bio-electrical

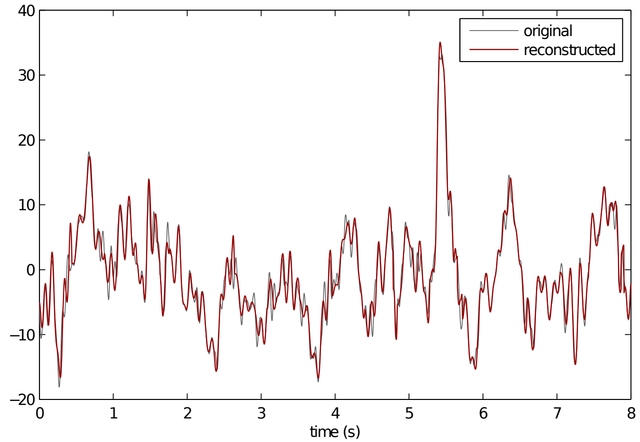

Figure: Reconstruction of 8 second $[1 - 20]$ Hz bandpass filtered EEG

Bio-electrical

Figure: Sparse EOG approximation

Bio-electrical

Polysomnogram:

- 12 channels
- 22 wire attachments to patient
- heart rate, leg measurement, airflow (chest, abdomen), chin muscle, EEG, EOG, ...

Sparse Interpolation

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Basics: exponential

Basics: polynomial

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Bio-electrical

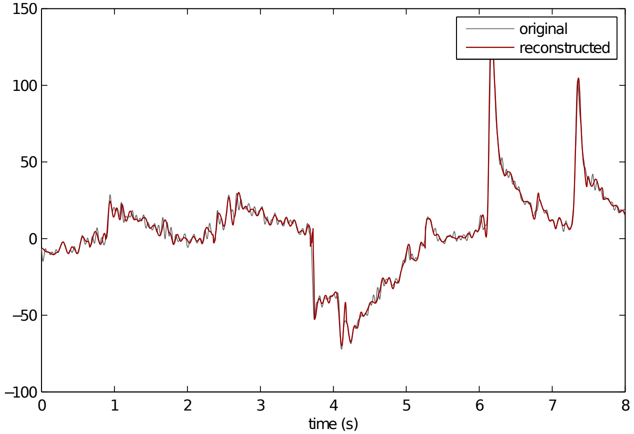

Figure: Reconstruction of 8 second EOG (CC = 99.2%)

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Sparse Interpolation

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Approximation theory

Applications

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Spectroscopy

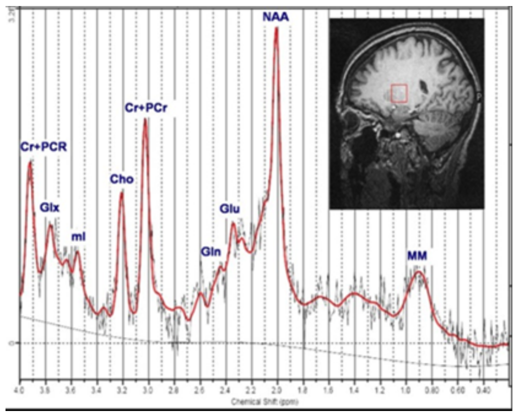

Figure: Spectral analysis of FID

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Spectroscopy

Magnetic resonance spectroscopy:

- physical and chemical properties of molecules
- a.o. concentration of metabolites in the brain
- frequencies clustered → high frequence resolution
- free induction decay → time constraint
- Fourier methods need additional tools

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Spectroscopy

$$\phi(x) = 5 \times 10^{-2} + 2e^{(-0.97+i79.94\pi)x} + 4e^{(-1+i80\pi)x} + 8e^{-1.1x} + \varepsilon(x)$$

$$\|\varepsilon(x)\|_\infty = 10^{-3}, \qquad \text{circular Gaussian noise}$$

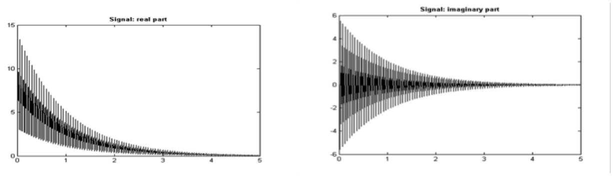

Figure: The real (left) and imaginary (right) part of $\phi(x)$

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Spectroscopy

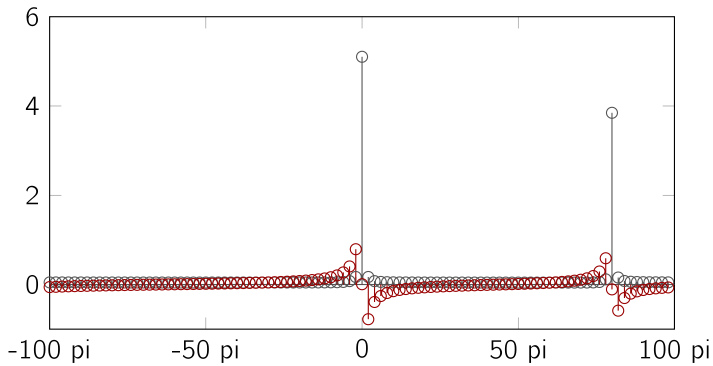

Figure: Real (black) and imaginary (red) parts of FFT

Spectroscopy

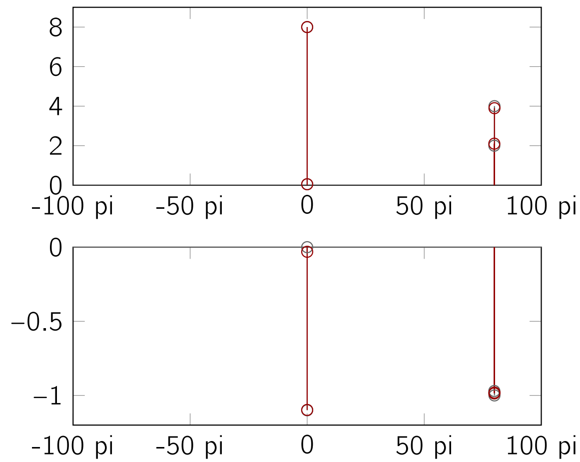

Figure: Amplitudes (top) and damping factors (bottom) of $\phi(x)$

Sparse Interpolation

Motivation

Basics: exponential

Basics: polynomial

Approximation theory

Applications

References

References

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Sparse Interpolation

Motivation

Basics: exponential

Basics: polynomial

Approximation theory

Applications

References

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