Part II

Interacting fields

Chapter 4

Interaction of Photons and Electrons

4.1 The interaction Hamiltonian

4.1.1 Strategy

The aim of this work is to develop a version of quantum electrodynamics in which all shortcomings of the standard approach are avoided. A first modification is the use of a reducible representation of the canonical commutation and anti-commutation relations. The main visible consequence is that many integrations over wave vectors disappear from the defining expressions of key quantities. As a consequence, there is no immediate need for an ultraviolet cutoff. However, this does not mean necessarily that the problem of ultraviolet divergences is solved, only that it is delayed to the moment of evaluation of quantum expectations.

The next intervention concerns the number of degrees of freedom of the electromagnetic field. The free electromagnetic field has two degrees of freedom corresponding with two independent choices of polarization. It is tradition to describe the interacting quantized electromagnetic field by 4 independent vector potentials A_μ, $\mu = 0, \cdots, 3$. This allows for an easy incorporation of Coulomb forces. A further advantage is manifestly Lorentz covariance. However, problems arise because in a quantum field theory it is not obvious how to eliminate spurious degrees of freedom.

The present theory restricts the degrees of freedom to the two transverse polarizations of the free field, treated in the transverse gauge. In this way the difficulties with elimination of degrees of freedom do not occur. The drawback is that the explicit evaluation of Lorentz boosts requires tedious calculations. In addition, it is not immediately clear how Coulomb forces emerge. The latter point is discussed further on in the text.

The interaction between photons and electrons can be implemented in more than one way. The choice made here is to use the Dirac equation only for the description of free electron/positron fields. The interaction between free photons and free electrons/positrons is then determined by the usual interaction Hamiltonian, which involves the contraction of the vector potential and the Dirac current. Because of working in the transverse gauge only the spatial components of the Dirac current appear.

4.1.2 The state space

The reducible representations of the free electromagnetic field and the free electron/positron field each have their own wave vector used to label the irreducible components. They are denoted $\mathbf{k}^{\rm ph}$, respectively $\mathbf{k}$. A field ζ of the interacting system associates with each pair $\mathbf{k}^{\rm ph} \neq 0$, $\mathbf{k}$ of wave vectors a wave function $\zeta_{\mathbf{k}^{\rm ph},\mathbf{k}}$ in the product Hilbert space $\mathcal{H}_{\rm em} \times \mathcal{H}_{16}$, where $\mathcal{H}_{\rm em}$ is the Hilbert space of a two-dimensional harmonic oscillator, and $\mathcal{H}_{16}$ is the 16-dimensional Hilbert space representing the possible states of an electron/positron field. Basis vectors in the product Hilbert space are denoted $|m,n\rangle \times |\Lambda\rangle \equiv |m,n,\Lambda\rangle$, where m,n count photons and Λ is a subset of $\{1,2,3,4\}$. The space of continuous fields of the form

$$(\mathbf{k}^{\rm ph},\mathbf{k}) \in \mathbb{R}^3_{\rm o} \times \mathbb{R}^3 \mapsto \zeta_{\mathbf{k}^{\rm ph},\mathbf{k}} \in \mathcal{H}_{\rm em} \times \mathcal{H}_{16}.$$

is denoted $\Gamma^{\rm ph/el}$.

4.1.3 The interaction Hamiltonian

The Hamiltonian $\hat{H}$ is of the usual form

$$\hat{H} \quad = \quad \hat{H^{\rm ph}} + \hat{H^{\rm el}} + \hat{H^{\rm I}}. \tag{4.1}$$

The kinetic energy of the photon field is given by (2.18)

$$H^{\rm ph}_{\mathbf{k}^{\rm ph}} \quad = \quad \hbar c|\mathbf{k}^{\rm ph}| \left(a^\dagger_{\rm H} a_{\rm H} + a^\dagger_{\rm V} a_{\rm V}\right),$$

that of the electron/positron field by (3.13, 3.15)

$$H^{\rm el}_{\mathbf{k}} = \frac{1}{2}\hbar\omega(\mathbf{k}) \sum_{s=1}^{4} N_s,$$

where N_s is the number operator indicating the presence of a particle of type s (electron or positron, spin up or down). The interaction term involves the Dirac current $\hat{J}^\mu(x)$ and the electromagnetic potential operators $\hat{A}_\mu(x)$. The obvious definition is

$$\hat{H}^{\mathrm{I}}(x^0) \quad = \quad \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x}\, \hat{A}_\mu(x)\hat{J}^\mu(x).$$

For $x^0 = 0$ this defines the interaction Hamiltonian in the Schrödinger picture.

The charge operator $\hat{Q}$ commutes with the Hamiltonian $\hat{H}$ of the interacting system. Indeed, one has

$$\begin{aligned}
[\hat{Q}, \hat{H}]_- &= [\hat{Q}, \hat{H}^{\mathrm{I}}]_- \\
&= \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x}\, \hat{A}_\mu(x)[\hat{Q}, \hat{J}^\mu(x)]_- \\
&= 0.
\end{aligned}$$

The latter commutant vanishes – see (3.43).

4.1.4 Gauge transformations

The charge operator $\hat{Q}$ commutes with $\hat{H^{\mathrm{ph}}}$, $\hat{H^{\mathrm{el}}}$ and $\hat{H^{\mathrm{I}}}$ and hence with the full Hamiltonian $\hat{H}$. The one-parameter group $\Lambda \in \mathbb{R} \mapsto \exp(i\Lambda\hat{Q})$ is a global symmetry of electrodynamics and corresponds with the U(1) gauge group of the Standard Model. This raises the question whether this symmetry group can be extended to include local symmetries. Local means here local in the space of wave vectors.

Given a smooth function $\Lambda(\mathbf{k}^{\mathrm{ph}}, \mathbf{k})$ introduce the diagonal operator $\hat{U}_\Lambda$ defined by

$$[\hat{U}_\Lambda \zeta]_{\mathbf{k}^{\mathrm{ph}},\mathbf{k}} \quad = \quad e^{i\Lambda(\mathbf{k}^{\mathrm{ph}},\mathbf{k})Q}\zeta_{\mathbf{k}^{\mathrm{ph}},\mathbf{k}}.$$

Proposition 4.1.1 *Assume that any of the 4 possibilities* $\mathbf{k}' = \pm\mathbf{k} \pm \mathbf{k}^{ph}$ *implies that* $\Lambda(\mathbf{k}^{ph}, \mathbf{k}') = \Lambda(\mathbf{k}^{ph}, \mathbf{k})$. *Then* $\hat{U}_\Lambda$ *commutes with the Hamiltonian* $\hat{H}$.

Proof

It clearly commutes with $\hat{H}^{\rm ph}$ and $\hat{H}^{\rm el}$. Because $\hat{Q}$ commutes with $\hat{J}^\alpha$ one has

$$[\hat{U}_\Lambda^{-1}\hat{H}^{\rm I}\hat{U}_\Lambda\zeta]_{\mathbf{k}^{\rm ph},\mathbf{k}} = \int_{\mathbb{R}^3} {\rm d}\mathbf{x}\sum_\alpha A_{\alpha,\mathbf{k}^{\rm ph}}(0,\mathbf{x}) \times\ell^3\int {\rm d}\mathbf{k}'\, e^{i[\Lambda(\mathbf{k}^{\rm ph},\mathbf{k}')-\Lambda(\mathbf{k}^{\rm ph},\mathbf{k})]Q} J^\alpha_{\mathbf{k},\mathbf{k}'}(0,\mathbf{x})\zeta_{\mathbf{k}^{\rm ph},\mathbf{k}'}.$$

The $\mathbf{x}$-dependence of $A_{\alpha,\mathbf{k}^{\rm ph}}(0,\mathbf{x})$ involves factors $e^{i\mathbf{k}^{\rm ph}\cdot\mathbf{x}}$. The $\mathbf{x}$-dependence of $J^\alpha_{\mathbf{k},\mathbf{k}'}(0,\mathbf{x})$ involves factors $e^{i(\pm\mathbf{k}\pm\mathbf{k}')}$. Hence the integration over $\mathbf{x}$ produces Dirac delta functions $\delta(\pm\mathbf{k}^{\rm ph}\pm\mathbf{k}\pm\mathbf{k}')$. By assumption these restrictions on the wave vectors imply that $\Lambda(\mathbf{k}^{\rm ph},\mathbf{k}')=\Lambda(\mathbf{k}^{\rm ph},\mathbf{k})$. Therefore the factor $e^{i[\Lambda(\mathbf{k}^{\rm ph},\mathbf{k}')-\Lambda(\mathbf{k}^{\rm ph},\mathbf{k})]Q}$ may be omitted in the above expression. The result is that

$$[\hat{U}_\Lambda^{-1}\hat{H}^{\rm I}\hat{U}_\Lambda\zeta]_{\mathbf{k}^{\rm ph},\mathbf{k}} = [\hat{H}^{\rm I}\zeta]_{\mathbf{k}^{\rm ph},\mathbf{k}}.$$

This implies that $\hat{H}^{\rm I}$ commutes with $\hat{U}_\Lambda$.

□

Let $\mathbf{k}=\mathbf{k}_\parallel+\mathbf{k}_\perp$ be the decomposition of $\mathbf{k}$ into a part parallel to $\mathbf{k}^{\rm ph}$ and an orthogonal part. With this notation, the condition of the proposition is satisfied when $\Lambda(\mathbf{k}^{\rm ph},\mathbf{k})$ is a function of $\mathbf{k}^{\rm ph}$ and $|\mathbf{k}_\perp|$ only. It is easy to construct functions Λ which depend on $\mathbf{k}^{\rm ph}$ and $|\mathbf{k}_\perp|$ in a non-trivial manner. Hence, the Hamiltonian $\hat{H}$ is invariant under non-trivial gauge transformations. On the other hand, it is also easy to construct functions Λ such that $\hat{U}_\Lambda$ does not commute with $\hat{H}$.

4.2 Bound states

4.2.1 The unperturbed vacuum

The ground state of the non-interacting system is given by $\zeta_{\mathbf{k}^{\rm ph},\mathbf{k}}=|0,0,\emptyset\rangle$. The two zeroes indicate the absence of horizontally, respectively vertically polarized photons. The empty set indicates the absence of electrons and positrons. This state is not an eigenstate of the interacting Hamiltonian $\hat{H}$. One has

$$\begin{aligned}[\hat{H}\zeta]_{\mathbf{k}^{\rm ph},\mathbf{k}} &= [\hat{H}^{\rm I}\zeta]_{\mathbf{k}^{\rm ph},\mathbf{k}}\\ &= \int_{\mathbb{R}^3}{\rm d}\mathbf{x}\,[\hat{A}_\mu(x)\hat{J}^\mu(x)\zeta]_{\mathbf{k}^{\rm ph},\mathbf{k}}\Big|_{x^0=0}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x}\, A_{\mu,\mathbf{k}^{\mathrm{ph}}}(x)|0,0\rangle \int \mathrm{d}\mathbf{k}'\, J^{\mu}_{\mathbf{k},\mathbf{k}'}(x)|\emptyset\rangle\Big|_{x^0=0}\\
&= \frac{\lambda}{2N_0(\mathbf{k}^{\mathrm{ph}})}\frac{qc}{2(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}\mathbf{x}\, e^{ik^{\mathrm{ph}}_{\mu}x^{\mu}} \int \mathrm{d}\mathbf{k}'\, e^{i(k_\nu+k'_\nu)x^\nu}\Big|_{x^0=0}\\
&\quad\times \sum_\alpha \left[\varepsilon^{(H)}_\alpha(\mathbf{k}^{\mathrm{ph}})|1,0\rangle + \varepsilon^{(V)}_\alpha(\mathbf{k}^{\mathrm{ph}})|0,1\rangle\right]\\
&\quad\times \sum_{s=1,2}\sum_{t=3,4}\left[\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\alpha v^{(t)}(\mathbf{k}')\rangle + \langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\alpha v^{(t)}(\mathbf{k})\rangle\right]\\
&\quad\times \sigma^{(-)}_s\sigma^{(-)}_t|\emptyset\rangle\\
&= -\frac{\lambda qc}{2N_0(\mathbf{k}^{\mathrm{ph}})}\sum_\alpha \left[\varepsilon^{(H)}_\alpha(\mathbf{k}^{\mathrm{ph}})|1,0\rangle + \varepsilon^{(V)}_\alpha(\mathbf{k}^{\mathrm{ph}})|0,1\rangle\right]\\
&\quad\times \sum_{s=1,2}\sum_{t=3,4}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\alpha v^{(t)}(-\mathbf{k}-\mathbf{k}^{\mathrm{ph}})\rangle\,|\{s,t\}\rangle.
\end{aligned}
$$

To obtain the last line the identities (D.5 — D.7), found in the Appendix D, are used. The action of the Hamiltonian $\hat{H}$ on the free vacuum creates an electron/positron pair together with a single photon.

The quantum expectation $\langle\zeta|\hat{H}\zeta\rangle$ of the energy vanishes. An important question is whether there exist fields ζ in $\Gamma^{\mathrm{ph/el}}$ for which $\langle\zeta|\hat{H}\zeta\rangle$ is negative, or even diverges to minus infinity. This question is related to the problem of stability of matter and has been studied extensively. See for instance the work of Lieb and coworkers [12]. A systematic study in the present context is postponed. A partial answer is given in the following sections.

4.2.2 Trial wave function

Let us try to find a wave function $\zeta_{\mathbf{k}^{\mathrm{ph}},\mathbf{k}}$ for which the expectation value $\langle\zeta|\hat{H}\zeta\rangle$ is strictly negative.

For all ζ of the form $\zeta_{\mathbf{k}^{\mathrm{ph}},\mathbf{k}} = \sum_{m,n} a_{m,n}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|m,n,\emptyset\rangle$ is $\langle\zeta|\hat{H}\zeta\rangle \geq 0$. As a next step consider wave functions of the form

$$
\begin{aligned}
\zeta_{\mathbf{k}^{\mathrm{ph}},\mathbf{k}} &= \sqrt{\rho^{\mathrm{vac}}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})}|0,0,\emptyset\rangle\\
&\quad + a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|0,0,\{1\}\rangle + a_{1,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|1,0,\{1\}\rangle + a_{0,1}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|0,1,\{1\}\rangle.
\end{aligned}
$$

They describe a single electron with spin up, eventually accompanied by an electromagnetic wave which is a superposition of a horizontally and a vertically polarized photon. The superposition with a wave function with vanishing electron/positron field is needed in order to satisfy the conflicting requirements of proper normalization and of a finite quantum expectation of the energy. Proper normalization requires that for all $\mathbf{k}^{\rm ph}, \mathbf{k}$

$$1 = \rho^{\rm vac}(\mathbf{k}^{\rm ph}, \mathbf{k}) + |a_{0,0}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2 + |a_{1,0}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2 + |a_{0,1}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2.$$

Finiteness of the total energy requires that the density of the vacuum $\rho^{\rm vac}(\mathbf{k}^{\rm ph}, \mathbf{k})$ tends to 1 fast enough when $|\mathbf{k}^{\rm ph}|$ and $|\mathbf{k}|$ become large.

The kinetic energy of the fields equals

$$\begin{aligned} \mathcal{E}^{\rm ph} &= \ell^3 \int \mathrm{d}\mathbf{k}^{\rm ph}\, \hbar c |\mathbf{k}^{\rm ph}| \rho^{\rm ph}(\mathbf{k}^{\rm ph}), \\ \mathcal{E}^{\rm el} &= \ell^3 \int \mathrm{d}\mathbf{k}\, \hbar\omega(\mathbf{k}) \rho^{\rm el}(\mathbf{k}), \end{aligned}$$

with

$$\begin{aligned} \rho^{\rm ph}(\mathbf{k}^{\rm ph}) &= \ell^3 \int \mathrm{d}\mathbf{k}\, \left[|a_{1,0}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2 + |a_{0,1}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2)\right], \\ \rho^{\rm el}(\mathbf{k}) &= \ell^3 \int \mathrm{d}\mathbf{k}^{\rm ph}\, \left[|a_{0,0}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2 + |a_{1,0}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2 + |a_{0,1}(\mathbf{k}^{\rm ph}, \mathbf{k})|^2)\right] \\ &= \ell^3 \int \mathrm{d}\mathbf{k}^{\rm ph}\, \left[1 - \rho^{\rm vac}(\mathbf{k}^{\rm ph}, \mathbf{k})\right]. \end{aligned}$$

Before evaluating the interaction energy first consider

$$\begin{aligned} & [\hat{H}^{\rm I}\zeta]_{\mathbf{k}^{\rm ph},\mathbf{k}} \\ = & \int \mathrm{d}\mathbf{x} \sum_\alpha A_{\alpha,\mathbf{k}^{\rm ph}}(x)|0,0\rangle \\ & \times \int \mathrm{d}\mathbf{k}'\, J_{\alpha,\mathbf{k},\mathbf{k}'}(x) a_{0,0}(\mathbf{k}^{\rm ph}, \mathbf{k}')|\{1\}\rangle\Big|_{x^0=0} \\ & + \int \mathrm{d}\mathbf{x} \sum_\alpha A_{\alpha,\mathbf{k}^{\rm ph}}(x)|1,0\rangle \\ & \times \int \mathrm{d}\mathbf{k}'\, J_{\alpha,\mathbf{k},\mathbf{k}'}(x) a_{1,0}(\mathbf{k}^{\rm ph}, \mathbf{k}')|\{1\}\rangle\Big|_{x^0=0} \\ & + \int \mathrm{d}\mathbf{x} \sum_\alpha A_{\alpha,\mathbf{k}^{\rm ph}}(x)|0,1\rangle \\ & \times \int \mathrm{d}\mathbf{k}'\, J_{\alpha,\mathbf{k},\mathbf{k}'}(x) a_{0,1}(\mathbf{k}^{\rm ph}, \mathbf{k}')|\{1\}\rangle\Big|_{x^0=0} \\ = & \frac{\lambda q c}{4N_0(\mathbf{k}^{\rm ph})(2\pi)^3} \int \mathrm{d}\mathbf{x} \sum_\alpha \left[\varepsilon_\alpha^{(H)}(\mathbf{k}^{\rm ph}) e^{-i\mathbf{k}^{\rm ph}\cdot\mathbf{x}} a_{\rm H}^\dagger + \varepsilon_\alpha^{(V)}(\mathbf{k}^{\rm ph}) e^{-i\mathbf{k}^{\rm ph}\cdot\mathbf{x}} a_{\rm V}^\dagger\right] |0,0\rangle \end{aligned}$$

$$
\begin{aligned}
&\times \int \mathrm{d}\mathbf{k}' \left[e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle u^{(1)}(\mathbf{k}) | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}') \rangle + e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle u^{(1)}(\mathbf{k}') | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}) \rangle \right] \\
&\quad \times a_{0,0}(\mathbf{k}^{\mathrm{ph}}, \mathbf{k}') | \{1\} \rangle \\
&+ \frac{\lambda q c}{4 N_0(\mathbf{k}^{\mathrm{ph}})(2\pi)^3} \int \mathrm{d}\mathbf{x} \sum_\alpha \varepsilon_\alpha^{(H)}(\mathbf{k}^{\mathrm{ph}}) e^{i\mathbf{k}^{\mathrm{ph}}\cdot\mathbf{x}} a_{\mathrm{H}} |1,0\rangle \\
&\times \int \mathrm{d}\mathbf{k}' \left[e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle u^{(1)}(\mathbf{k}) | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}') \rangle + e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle u^{(1)}(\mathbf{k}') | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}) \rangle \right] \\
&\quad \times a_{1,0}(\mathbf{k}^{\mathrm{ph}}, \mathbf{k}') | \{1\} \rangle \\
&+ \frac{\lambda q c}{4 N_0(\mathbf{k}^{\mathrm{ph}})(2\pi)^3} \int \mathrm{d}\mathbf{x} \sum_\alpha \varepsilon_\alpha^{(V)}(\mathbf{k}^{\mathrm{ph}}) e^{i\mathbf{k}^{\mathrm{ph}}\cdot\mathbf{x}} a_{\mathrm{V}} |0,1\rangle \\
&\times \int \mathrm{d}\mathbf{k}' \left[e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle u^{(1)}(\mathbf{k}) | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}') \rangle + e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle u^{(1)}(\mathbf{k}') | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}) \rangle \right] \\
&\quad \times a_{0,1}(\mathbf{k}^{\mathrm{ph}}, \mathbf{k}') | \{1\} \rangle \\
&+ \cdots .
\end{aligned}
$$

The omitted terms are orthogonal to $|0,0,\emptyset\rangle$ and $|m,n,\{1\}\rangle$. Integration over $\mathbf{x}$ gives

$$
\begin{aligned}
& [\hat{H}^{\mathrm{I}}\zeta]_{\mathbf{k}^{\mathrm{ph}},\mathbf{k}} \\
= \;& \frac{\lambda q c}{4 N_0(\mathbf{k}^{\mathrm{ph}})} \sum_\alpha \left[\varepsilon_\alpha^{(H)}(\mathbf{k}^{\mathrm{ph}}) |1,0\rangle + \varepsilon_\alpha^{(V)}(\mathbf{k}^{\mathrm{ph}}) |0,1\rangle \right] \\
& \times \int \mathrm{d}\mathbf{k}' \Big[\delta(-\mathbf{k}^{\mathrm{ph}} - \mathbf{k} + \mathbf{k}') \langle u^{(1)}(\mathbf{k}) | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}') \rangle \\
& \quad + \delta(-\mathbf{k}^{\mathrm{ph}} + \mathbf{k} - \mathbf{k}') \langle u^{(1)}(\mathbf{k}') | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}) \rangle \Big] a_{0,0}(\mathbf{k}^{\mathrm{ph}}, \mathbf{k}') | \{1\} \rangle \\
& + \frac{\lambda q c}{4 N_0(\mathbf{k}^{\mathrm{ph}})} \sum_\alpha \varepsilon_\alpha^{(H)}(\mathbf{k}^{\mathrm{ph}}) |0,0\rangle \\
& \times \int \mathrm{d}\mathbf{k}' \Big[\delta(\mathbf{k}^{\mathrm{ph}} - \mathbf{k} + \mathbf{k}') \langle u^{(1)}(\mathbf{k}) | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}') \rangle \\
& \quad + \delta(\mathbf{k}^{\mathrm{ph}} + \mathbf{k} - \mathbf{k}') \langle u^{(1)}(\mathbf{k}') | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}) \rangle \Big] a_{1,0}(\mathbf{k}^{\mathrm{ph}}, \mathbf{k}') | \{1\} \rangle \\
& + \frac{\lambda q c}{4 N_0(\mathbf{k}^{\mathrm{ph}})} \sum_\alpha \varepsilon_\alpha^{(V)}(\mathbf{k}^{\mathrm{ph}}) |0,0\rangle \\
& \times \int \mathrm{d}\mathbf{k}' \Big[\delta(\mathbf{k}^{\mathrm{ph}} - \mathbf{k} + \mathbf{k}') \langle u^{(1)}(\mathbf{k}) | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}') \rangle \\
& \quad + \delta(\mathbf{k}^{\mathrm{ph}} + \mathbf{k} - \mathbf{k}') \langle u^{(1)}(\mathbf{k}') | \gamma^0 \gamma^\alpha u^{(1)}(\mathbf{k}) \rangle \Big] a_{0,1}(\mathbf{k}^{\mathrm{ph}}, \mathbf{k}') | \{1\} \rangle \\
& + \cdots \\
= \;& \frac{\lambda q c}{4 N_0(\mathbf{k}^{\mathrm{ph}})} \sum_\alpha \left[\varepsilon_\alpha^{(H)}(\mathbf{k}^{\mathrm{ph}}) |1,0,\{1\}\rangle + \varepsilon_\alpha^{(V)}(\mathbf{k}^{\mathrm{ph}}) |0,1,\{1\}\rangle \right]
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big[\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\text{ph}})\rangle a_{0,0}(\mathbf{k}^{\text{ph}},\mathbf{k}+\mathbf{k}^{\text{ph}}) \\
&\quad+\langle u^{(1)}(\mathbf{k}-\mathbf{k}^{\text{ph}})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k})\rangle a_{0,0}(\mathbf{k}^{\text{ph}},\mathbf{k}-\mathbf{k}^{\text{ph}})\Big] \\
&+\frac{\lambda qc}{4N_0(\mathbf{k}^{\text{ph}})}\sum_\alpha \varepsilon_\alpha^{(H)}(\mathbf{k}^{\text{ph}})|0,0,\{1\}\rangle \\
&\times\Big[\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}-\mathbf{k}^{\text{ph}})\rangle a_{1,0}(\mathbf{k}^{\text{ph}},\mathbf{k}-\mathbf{k}^{\text{ph}}) \\
&\quad+\langle u^{(1)}(\mathbf{k}+\mathbf{k}^{\text{ph}})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k})\rangle a_{1,0}(\mathbf{k}^{\text{ph}},\mathbf{k}+\mathbf{k}^{\text{ph}})\Big] \\
&+\frac{\lambda qc}{4N_0(\mathbf{k}^{\text{ph}})}\sum_\alpha \varepsilon_\alpha^{(V)}(\mathbf{k}^{\text{ph}})|0,0,\{1\}\rangle \\
&\times\Big[\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}-\mathbf{k}^{\text{ph}})\rangle a_{0,1}(\mathbf{k}^{\text{ph}},\mathbf{k}-\mathbf{k}^{\text{ph}}) \\
&\quad+\langle u^{(1)}(\mathbf{k}+\mathbf{k}^{\text{ph}})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k})\rangle a_{0,1}(\mathbf{k}^{\text{ph}},\mathbf{k}+\mathbf{k}^{\text{ph}})\Big] \\
&+\cdots.
\end{aligned}
$$

The quantum expectation of the interaction energy becomes

$$
\begin{aligned}
\mathcal{E}^{\text{int}} &= -\ell^3\int \mathrm{d}\mathbf{k}^{\text{ph}}\,\frac{\lambda qc}{4N_0(\mathbf{k}^{\text{ph}})}\sum_\alpha \\
&\quad\times\left[\varepsilon_\alpha^{(H)}(\mathbf{k}^{\text{ph}})w_\alpha^{(H)}(\mathbf{k}^{\text{ph}})+\varepsilon_\alpha^{(V)}(\mathbf{k}^{\text{ph}})w_\alpha^{(V)}(\mathbf{k}^{\text{ph}})\right]
\end{aligned}
$$

with

$$
\begin{aligned}
w_\alpha^{(H)}(\mathbf{k}^{\text{ph}}) &= -2\,\mathrm{Re}\,\ell^3\int \mathrm{d}\mathbf{k}\,\overline{a_{1,0}(\mathbf{k}^{\text{ph}},\mathbf{k})}a_{0,0}(\mathbf{k}^{\text{ph}},\mathbf{k}+\mathbf{k}^{\text{ph}}) \\
&\quad\times\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\text{ph}})\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
w_\alpha^{(V)}(\mathbf{k}^{\text{ph}}) &= -2\,\mathrm{Re}\,\ell^3\int \mathrm{d}\mathbf{k}\,\overline{a_{0,1}(\mathbf{k}^{\text{ph}},\mathbf{k})}a_{0,0}(\mathbf{k}^{\text{ph}},\mathbf{k}+\mathbf{k}^{\text{ph}}) \\
&\quad\times\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\text{ph}})\rangle.
\end{aligned}
$$

The terms which contribute describe the interaction of the photon with the spin of the electron.

4.2.3 Variational approach

Consider now the problem of minimizing the total energy given a fixed value for the density of the vacuum $\rho^{\text{vac}}(\mathbf{k}^{\text{ph}},\mathbf{k})$. Variation of $\overline{a_{1,0}(\mathbf{k}^{\text{ph}},\mathbf{k})}$ gives

$$
a_{1,0}(\mathbf{k}^{\text{ph}},\mathbf{k}) = -U^{(H)}(\mathbf{k}^{\text{ph}},\mathbf{k})a_{0,0}(\mathbf{k}^{\text{ph}},\mathbf{k}+\mathbf{k}^{\text{ph}})
$$

with

$$U^{(H/V)}(\mathbf{k}^{\mathrm{ph}},\mathbf{k}) \;=\; \frac{\lambda q}{4N_0(\mathbf{k}^{\mathrm{ph}})\hbar|\mathbf{k}^{\mathrm{ph}}|}\sum_\alpha \varepsilon_\alpha^{(H/V)}(\mathbf{k}^{\mathrm{ph}})\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})\rangle.$$

Similarly,

$$a_{0,1}(\mathbf{k}^{\mathrm{ph}},\mathbf{k}) \;=\; -U^{(V)}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k}+\mathbf{k}^{\mathrm{ph}}).$$

The normalization condition becomes

$$1=\rho^{\mathrm{vac}}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})+|a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|^2+U_\perp^2(\mathbf{k}^{\mathrm{ph}},\mathbf{k})\,|a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k}+\mathbf{k}^{\mathrm{ph}})|^2, \quad (4.2)$$

with

$$U_\perp^2(\mathbf{k}^{\mathrm{ph}},\mathbf{k}) \;=\; |U^{(H)}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|^2+|U^{(V)}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|^2.$$

The functions $w_\alpha^{(H)}(\mathbf{k}^{\mathrm{ph}})$ and $w_\alpha^{(V)}(\mathbf{k}^{\mathrm{ph}})$ are of the form

$$\begin{aligned} w_\alpha^{(H/V)}(\mathbf{k}^{\mathrm{ph}}) \;&=\; 2\,\mathrm{Re}\,\ell^3\int \mathrm{d}\mathbf{k}\,\overline{U^{(H/V)}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})}|a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k}+\mathbf{k}^{\mathrm{ph}})|^2 \\ &\quad\times\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})\rangle. \end{aligned}$$

The interaction energy becomes

$$\begin{aligned} \mathcal{E}^{\mathrm{int}} \;&=\; -2\ell^6\int \mathrm{d}\mathbf{k}^{\mathrm{ph}}\int \mathrm{d}\mathbf{k}\,\frac{\lambda qc}{4N_0(\mathbf{k}^{\mathrm{ph}})}|a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k}+\mathbf{k}^{\mathrm{ph}})|^2 \\ &\quad\times\sum_\alpha \mathrm{Re}\left[\varepsilon_\alpha^{(H)}(\mathbf{k}^{\mathrm{ph}})\overline{U^{(H)}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})}+\varepsilon_\alpha^{(V)}(\mathbf{k}^{\mathrm{ph}})\overline{U^{(V)}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})}\right] \\ &\quad\times\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})\rangle \\ &=\; -2\ell^6\int \mathrm{d}\mathbf{k}^{\mathrm{ph}}\,\hbar c|\mathbf{k}^{\mathrm{ph}}|\int \mathrm{d}\mathbf{k}\,|a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k}+\mathbf{k}^{\mathrm{ph}})|^2U_\perp^2(\mathbf{k}^{\mathrm{ph}},\mathbf{k}) \\ &=\; -2\mathcal{E}^{\mathrm{ph}}. \end{aligned}$$

The interaction energy is minus twice the kinetic energy of the photon field. This result is typical for a quadratic minimization problem. During the minimization the energy of the electron field is kept constant. Hence one can conclude that for an electron field with a given kinetic energy and no photons present there always exists an interacting system where the energy spectrum of the electron field is unmodified but the total energy is lowered by adding the photon field.

For further use, note that the kinetic energy of the photon field can be written as

$$\mathcal{E}^{\mathrm{ph}} \;=\; \ell^6\int \mathrm{d}\mathbf{k}^{\mathrm{ph}}\,\hbar c|\mathbf{k}^{\mathrm{ph}}|\int \mathrm{d}\mathbf{k}\,|a_{0,0}(\mathbf{k}^{\mathrm{ph}},\mathbf{k})|^2U_\perp^2(\mathbf{k}^{\mathrm{ph}},\mathbf{k}-\mathbf{k}^{\mathrm{ph}}).$$

(4.3)

The density of the electron field equals

$$\begin{aligned}
\rho^{\rm el}(\mathbf{k}) &= \ell^3 \int \mathrm{d}\mathbf{k}^{\rm ph}\ \left[|a_{0,0}(\mathbf{k}^{\rm ph},\mathbf{k})|^2 + |a_{1,0}(\mathbf{k}^{\rm ph},\mathbf{k})|^2 + |a_{0,1}(\mathbf{k}^{\rm ph},\mathbf{k})|^2)\right]\\
&= \ell^3 \int \mathrm{d}\mathbf{k}^{\rm ph}\ \left[|a_{0,0}(\mathbf{k}^{\rm ph},\mathbf{k})|^2 1 + U_\perp^2(\mathbf{k}^{\rm ph},\mathbf{k})|a_{0,0}(\mathbf{k}^{\rm ph},\mathbf{k}+\mathbf{k}^{\rm ph})|^2\right].
\end{aligned} \tag{4.4}$$

4.2.4 Long-wavelength analysis

In the previous section it is shown that for a wave function of the form

$$\begin{aligned}
\zeta_{\mathbf{k}^{\rm ph},\mathbf{k}} &= \sqrt{\rho^{\rm vac}(\mathbf{k}^{\rm ph},\mathbf{k})}|0,0,\emptyset\rangle + a_{0,0}(\mathbf{k}^{\rm ph},\mathbf{k})|0,0,\{1\}\rangle\\
&\quad -U^{(H)}(\mathbf{k}^{\rm ph},\mathbf{k})a_{0,0}(\mathbf{k}^{\rm ph},\mathbf{k}+\mathbf{k}^{\rm ph})|1,0,\{1\}\rangle\\
&\quad -U^{(V)}(\mathbf{k}^{\rm ph},\mathbf{k})a_{0,0}(\mathbf{k}^{\rm ph},\mathbf{k}+\mathbf{k}^{\rm ph})|0,1,\{1\}\rangle
\end{aligned}$$

the interaction energy is minus twice the kinetic energy of the photon field. Let us now check by explicit calculation that coefficients $a_{0,0}(\mathbf{k}^{\rm ph},\mathbf{k})$ exist such that the wave function is physically acceptable.

It is shown in the Appendix H that

$$\begin{aligned}
U^{(H/V)}(\mathbf{k}^{\rm ph},\mathbf{k}) &= -\frac{\lambda q c}{4N_0(\mathbf{k}^{\rm ph})\hbar|\mathbf{k}^{\rm ph}|}\sum_\alpha \varepsilon_\alpha^{(H/V)}(\mathbf{k}^{\rm ph})k_\alpha\\
&\quad\times\frac{\omega(\mathbf{k})+\omega(\mathbf{k}+\mathbf{k}^{\rm ph})+2c\kappa}{\sqrt{\omega(\mathbf{k}+\mathbf{k}^{\rm ph})[\omega(\mathbf{k}+\mathbf{k}^{\rm ph})+c\kappa]}\sqrt{\omega(\mathbf{k})[\omega(\mathbf{k})+c\kappa]}}.
\end{aligned} \tag{4.5}$$

This gives

$$\begin{aligned}
U_\perp^2(\mathbf{k}^{\rm ph},\mathbf{k}) &= \left(\frac{\lambda q c}{4N_0(\mathbf{k}^{\rm ph})\hbar|\mathbf{k}^{\rm ph}|}\right)^2\left(|\mathbf{k}|^2 - \frac{(\mathbf{k}\cdot\mathbf{k}^{\rm ph})^2}{|\mathbf{k}^{\rm ph}|^2}\right)\\
&\quad\times\frac{[\omega(\mathbf{k})+\omega(\mathbf{k}+\mathbf{k}^{\rm ph})+2c\kappa]^2}{\omega(\mathbf{k}+\mathbf{k}^{\rm ph})[\omega(\mathbf{k}+\mathbf{k}^{\rm ph})+c\kappa]\,\omega(\mathbf{k})[\omega(\mathbf{k})+c\kappa]}\\
&\geq \left(\frac{\lambda q c}{4N_0(\mathbf{k}^{\rm ph})\hbar|\mathbf{k}^{\rm ph}|}\right)^2\left(|\mathbf{k}|^2 - \frac{(\mathbf{k}\cdot\mathbf{k}^{\rm ph})^2}{|\mathbf{k}^{\rm ph}|^2}\right)\frac{4}{\omega(\mathbf{k})\omega(\mathbf{k}+\mathbf{k}^{\rm ph})}.
\end{aligned}$$

A long wavelength approximation is

$$U_\perp^2(\mathbf{k}^{\rm ph},\mathbf{k}) = \left(\frac{\lambda q c}{4N_0(\mathbf{k}^{\rm ph})\hbar|\mathbf{k}^{\rm ph}|}\right)^2\left(|\mathbf{k}|^2 - \frac{(\mathbf{k}\cdot\mathbf{k}^{\rm ph})^2}{|\mathbf{k}^{\rm ph}|^2}\right)$$

$$\times \frac{4}{\omega^2(\mathbf{k})} \left[1 - \frac{\mathbf{k}\cdot\mathbf{k}^{\rm ph}}{\kappa^2 + |\mathbf{k}|^2} + \mathrm{O}(|\mathbf{k}^{\rm ph}|^2) \right].$$

If $\mathbf{k}^{\rm ph}$ and $\mathbf{k}$ are not parallel then this expression diverges as $|\mathbf{k}^{\rm ph}|^{-3}$. From the normalization condition (4.2) then follows that $a_{0,0}(\mathbf{k}^{\rm ph}, \mathbf{k})$ should vanish in the long-wavelength limit as $|\mathbf{k}^{\rm ph}|^3$ or it should vanish in all regions away from the longitudinal direction. This observation leaves room for two types of solutions.

4.3 Emergent Coulomb forces

In standard QED the electromagnetic field is described by 4 independent operators $\hat{A}_\mu(x)$. In the present approach one component, namely $\hat{A}_0(x)$ is identically zero and the 3 remaining components $\hat{A}_\alpha(x)$, $\alpha = 1, 2, 3$, satisfy the orthogonality relation

$$\sum_\alpha k^{\rm ph}_\alpha A_{\alpha,\mathbf{k}^{\rm ph}}(x) = 0.$$

Hence, the electromagnetic quantum field, as treated in the present work, has only two degrees of freedom. In particular, the electric field operators $\hat{E}_\alpha(x)$ satisfy Gauss' law in absence of charges. See (2.21). This can be justified with the argument that the full law, including a source term in the r.h.s., will emerge after the interaction with the electron/positron field is taken into account. This argument is supported by the existence [13] of a transformation of the field operators $\hat{E}_\alpha(x)$ which maps the homogeneous law of Gauss onto the full version of the law.

Introduce new electric field operators

$$\begin{aligned} \hat{E}''_\alpha(x) &= \hat{E}'_\alpha(x) \\ &+ \frac{\mu_0 c}{4\pi} \frac{\partial}{\partial x^\alpha} \int \mathrm{d}\mathbf{y}\, \frac{1}{|\mathbf{x}-\mathbf{y}|} \\ &\times \hat{U}(-x^0)\hat{j}^0(\mathbf{y}, 0)\hat{U}(x^0). \end{aligned} \tag{4.6}$$

Here, $\hat{U}(x^0) = \exp(-ix^0\hat{H}/\hbar c)$ is the time evolution of the interacting system. The new operators are marked with a double prime to distinguish them from the operators of the non-interacting system and those of the interacting system. The latter are denoted with a single prime. One verifies immediately that Gauss' law is satisfied

$$\sum_\alpha \frac{\partial}{\partial x^\alpha} \hat{E}''_\alpha(x) = -\mu_0 c\, \hat{j}^{0\prime}(x).$$

The second term of (4.6) is the Coulomb contribution to the electric field. The curl of this term vanishes. Hence it is obvious to take

$$\hat{B}''_\alpha(x) \equiv \hat{B}'_\alpha(x).$$

This implies the second of the four equations of Maxwell, stating that the divergence of $\hat{B}''_\alpha(x)$ vanishes. In addition, the fourth equation, absence of magnetic charges, follows immediately because $\hat{E}''(x)$ and $\hat{E}'(x)$ have the same curl. Remains Faraday's law to be written as

$$(\nabla \times \hat{B}''(x))_\alpha - \frac{1}{c}\frac{\partial}{\partial x^0}\hat{E}''_\alpha(x) = -\mu_0\,\hat{j}''_\alpha(x)$$

with

$$\hat{j}''_\alpha(x) = -\frac{1}{\mu_0 c}\frac{\partial}{\partial x^0}\left(\hat{E}''_\alpha(x) - \hat{E}'_\alpha(x)\right).$$

Finally, take $\hat{j}''_0(x) = \hat{j}'_0(x)$. A short calculation shows that the newly defined current operators $\hat{j}''_\mu(x)$ satisfy the continuity equation.

One concludes that a formalism of QED is possible which does not postulate the existence of longitudinal or scalar photons. Two pictures coexist: the original Heisenberg picture and what is called here the *emergent* picture. In both pictures the time evolution of all operators is the same, but the definition of the electromagnetic field operators differs. In the original description only transversely polarized photons exist. On the other hand, the field operators of the emergent picture satisfy the full Maxwell equations, including Coulomb forces.

Appendix H

Long wavelength analysis

One has

$$\begin{aligned} & \langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})\rangle \\ = \ & -c\frac{k_\alpha[\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})+c\kappa]+(k_\alpha+k^{\mathrm{ph}}_\alpha)[\omega(\mathbf{k})+c\kappa]}{\sqrt{\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})[\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})+c\kappa]}\,\sqrt{\omega(\mathbf{k})[\omega(\mathbf{k})+c\kappa]}}. \end{aligned}$$

This implies

$$\sum_\alpha \varepsilon^{(H/V)}_\alpha(\mathbf{k}^{\mathrm{ph}})\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})\rangle = -\sum_\alpha \varepsilon^{(H/V)}_\alpha(\mathbf{k}^{\mathrm{ph}})\left[Ak_\alpha+Bk^{\mathrm{ph}}_\alpha\right],$$

with

$$\begin{aligned} A &= c\frac{\omega(\mathbf{k})+\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})+2c\kappa}{\sqrt{\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})[\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})+c\kappa]}\,\sqrt{\omega(\mathbf{k})[\omega(\mathbf{k})+c\kappa]}} \text{ and} \\ B &= c\frac{\omega(\mathbf{k})+c\kappa}{\sqrt{\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})[\omega(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})+c\kappa]}\,\sqrt{\omega(\mathbf{k})[\omega(\mathbf{k})+c\kappa]}}. \end{aligned}$$

Because $\sum_\alpha \varepsilon^{(H/V)}_\alpha(\mathbf{k}^{\mathrm{ph}})k^{\mathrm{ph}}_\alpha = 0$ there follows

$$\sum_\alpha \varepsilon^{(H/V)}_\alpha(\mathbf{k}^{\mathrm{ph}})\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}+\mathbf{k}^{\mathrm{ph}})\rangle = -A\sum_\alpha \varepsilon^{(H/V)}_\alpha(\mathbf{k}^{\mathrm{ph}})k_\alpha.$$

This implies (4.5).