Part I

Free Fields

Chapter 1

Fields of Hilbert spaces

1.1 Quantum fields

Throughout this part of the work $\mathcal{H}$ is a given Hilbert space, either finite dimensional or separable, and K is an open subset of $\mathbb{R}^n$. In subsequent parts, K will be either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$. Normalized elements of $\mathcal{H}$ are called *wave functions*, elements of K are called *wave vectors*. Maps of K into $\mathcal{H}$ are called *quantum fields*.

1.1.1 Fields

Let Γ denote the linear space of continuous fields

$$\zeta : \mathbf{k} \in K \mapsto \zeta_{\mathbf{k}} \in \mathcal{H}.$$

In the terminology of [1] Γ is a *continuous field of Hilbert spaces*. A family of sesquilinear forms $(\cdot,\cdot)_{\mathbf{k}}$, $\mathbf{k} \in K$, is defined on Γ by

$$(\phi,\zeta)_{\mathbf{k}} = (\phi_{\mathbf{k}},\zeta_{\mathbf{k}}).$$

The corresponding semi-norms $||\zeta||_{\mathbf{k}} \equiv ||\zeta_{\mathbf{k}}||$ turn Γ into a locally convex Hausdorff space.

A subspace Γ_{norm} of Γ is formed by the $\zeta \in \Gamma$ for which the map $\mathbf{k} \mapsto ||\zeta_{\mathbf{k}}||$ is bounded continuous. A norm is defined on this subspace by

$$||\zeta|| \quad = \quad \sup_{\mathbf{k}\in K} ||\zeta_{\mathbf{k}}||.$$

It turns Γ_{norm} into a Banach space. Fields belonging to this subspace are said to be bounded in norm.

In standard quantum mechanics the normalization of wave functions is important. In the present context this leads to the axiom that *states* of the quantum field theory are represented by elements ζ of Γ which satisfy the normalization condition

$$||\zeta_{\mathbf{k}}|| = 1 \qquad \text{for all } \mathbf{k} \in K.$$

If this condition is satisfied then $\zeta \in \Gamma$ is said to be *properly normalized.*

1.1.2 Transposed fields

The dual Γ^* of Γ consists of all continuous conjugate-linear functions of Γ. Introduce

Definition 1.1.1 *A dual field θ is a map $\mathbf{k} \in K \mapsto \theta_{\mathbf{k}} \in \Gamma^*$ which is point-wise continuous.*

The space of dual fields is denoted $\Gamma^\dagger$. Introduce also the notation

$$(\theta, \zeta)_{\mathbf{k}} \;=\; \overline{\theta_{\mathbf{k}}(\zeta)}, \quad \zeta \in \Gamma, \theta \in \Gamma^\dagger.$$

Because $\zeta \mapsto \theta_{\mathbf{k}}(\zeta)$ is conjugate-linear the form $(\cdot, \cdot)$ is sesquilinear. Following the Physics convention it is linear in the second argument. The requirement of point-wise continuity in the definition means that the map $\mathbf{k} \mapsto (\theta, \zeta)_{\mathbf{k}}$ is continuous for any field ζ in Γ.

Given $\theta \in \Gamma$ let $\theta^{\mathrm{T}}_{\mathbf{k}}$ be defined by

$$\theta^{\mathrm{T}}_{\mathbf{k}} : \zeta \in \Gamma \mapsto \langle \zeta_{\mathbf{k}} | \theta_{\mathbf{k}} \rangle.$$

It belongs to Γ^* and the map $\theta^{\mathrm{T}} : \mathbf{k} \mapsto \theta^{\mathrm{T}}_{\mathbf{k}}$ is a dual field. This shows that Γ is embedded in the set of dual fields by the injection $\theta \mapsto \theta^{\mathrm{T}}$. One has

$$\begin{aligned} (\theta^{\mathrm{T}}, \zeta)_{\mathbf{k}} &= \overline{\theta^{\mathrm{T}}_{\mathbf{k}}(\zeta)} \\ &= \overline{\langle \zeta_{\mathbf{k}} | \theta_{\mathbf{k}} \rangle} \\ &= \langle \theta_{\mathbf{k}} | \zeta_{\mathbf{k}} \rangle \end{aligned}$$

and

$$(\theta^{\mathrm{T}}, \zeta)_{\mathbf{k}} = \overline{(\zeta^{\mathrm{T}}, \theta)_{\mathbf{k}}}.$$

The space of transposed fields θ^{T}, $\theta \in \Gamma$ is denoted Γ^{T} and is a subspace of $\Gamma^\dagger$. The inverse transposition is the map $\theta^{\mathrm{T}} \mapsto \theta$. It is tradition to call this inverse map also a transposition and to convene that $(\theta^{\mathrm{T}})^{\mathrm{T}} = \theta$.

1.2 Linear operators

1.2.1 Diagonal operators

A linear operator $\hat{A}$ in Γ is a *diagonal operator* if there exists a map $\mathbf{k} \in K \mapsto A_{\mathbf{k}}$, where $A_{\mathbf{k}}$ is a linear operator on $\mathcal{H}$, and a subspace $\mathcal{D}$ of Γ, called the domain of $\hat{A}$, such that for all ζ in $\mathcal{D}$

- $\zeta_{\mathbf{k}}$ is in the domain of $A_{\mathbf{k}}$ for all $\mathbf{k}$;
- $\mathbf{k} \mapsto A_{\mathbf{k}}\zeta_{\mathbf{k}}$ is continuous;
- $\hat{A}\zeta$ equals the map $\mathbf{k} \mapsto A_{\mathbf{k}}\zeta_{\mathbf{k}}$.

The diagonal operators generalize the concept of block-diagonal matrices for which all blocks have the same size. In fact, if K is a finite set and $\mathcal{H}$ is finite-dimensional then any diagonal operator is represented by a block-diagonal matrix.

Any operator A on $\mathcal{H}$ defines a diagonal operator $\hat{A}$ on Γ by

$$[\hat{A}\zeta]_{\mathbf{k}} \quad = \quad A\zeta_{\mathbf{k}} \qquad \text{for all } \mathbf{k} \in K.$$

The domain of this operator is the set

$$\mathcal{D} \quad = \quad \{\zeta \in \Gamma : \zeta_{\mathbf{k}} \text{ is in the domain of } A \text{ for all } \mathbf{k} \in K\}.$$

In particular, the identity operator $\mathbb{I}$ is a diagonal operator which satisfies $\hat{\mathbb{I}}\zeta = \zeta$ for all $\zeta \in \Gamma$.

Proposition 1.2.1 *If A is a bounded operator on $\mathcal{H}$ then*

1) $\hat{A}$ is a continuous operator defined on all of Γ;

2) If $\zeta \in \Gamma$ is bounded in norm then also $\hat{A}\zeta$ is bounded in norm and $||\hat{A}|| = ||A||$.

Proof

1) For any $\zeta \in \Gamma$ is

$$||A\zeta_{\mathbf{k}} - A\zeta_{\mathbf{k}'}|| \quad \leq \quad ||A||\, ||\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'}||.$$

Hence, continuity of $\mathbf{k} \mapsto A\zeta_{\mathbf{k}}$ follows from the continuity of $\mathbf{k} \mapsto \zeta_{\mathbf{k}}$. This shows that any ζ in Γ belongs to the domain of $\hat{A}$. Finally, continuity of $\hat{A}$ follows because it suffices that for each $\mathbf{k}$ the seminorm $||\hat{A}\zeta||_{\mathbf{k}}$ is bounded above by $||A||\, ||\zeta||_{\mathbf{k}}$.

2) If $\zeta \in \Gamma$ is bounded in norm then

$$\begin{aligned} ||\hat{A}\zeta|| &= \sup_{\mathbf{k}} ||[\hat{A}\zeta]_{\mathbf{k}}|| \\ &= \sup_{\mathbf{k}} ||A\zeta_{\mathbf{k}}|| \\ &\leq ||A|| \sup_{\mathbf{k}} ||\zeta_{\mathbf{k}}|| \\ &= ||A||\, ||\zeta||. \end{aligned}$$

Hence $\mathbf{k} \mapsto A\zeta_{\mathbf{k}}$ is bounded in norm and $||\hat{A}|| \leq ||A||$. Equality $||\hat{A}|| = ||A||$ follows from the action of $\hat{A}$ on constant fields.

$\square$

Proposition 1.2.2 *If A is a closed operator on $\mathcal{H}$ then $\hat{A}$ is a closed operator on the Banach space Γ_{norm}.*

Proof

Assume $\zeta^{(n)}$ converge in norm to ζ and $\eta^{(n)} \equiv \hat{A}\zeta^{(n)}$ converge to η. Then for each $\mathbf{k} \in K$ converges $\zeta^{(n)}_{\mathbf{k}}$ to $\zeta_{\mathbf{k}}$ and $\eta^{(n)}_{\mathbf{k}} = A\zeta^{(n)}$ converge to $\eta_{\mathbf{k}}$. Because A is closed with given domain $\mathcal{D}_A \subset \mathcal{H}$ the vector $\zeta_{\mathbf{k}}$ belongs to $\mathcal{D}_A$ and $A\zeta_{\mathbf{k}} = \eta_{\mathbf{k}}$. Because $\eta \in \Gamma_{\mathrm{norm}}$ the map $\mathbf{k} \mapsto \eta_k$ is continuous. Hence, ζ belongs to $\mathcal{D}$ and $\hat{A}\zeta = \eta$.

$\square$

An example of diagonal operators is found in the book of Dixmier [1]. Given two fields ζ, η in Γ introduce the bounded operators $A_{\mathbf{k}}$ defined by

$$A_{\mathbf{k}}\theta_{\mathbf{k}} \quad = \quad \langle \eta_{\mathbf{k}} | \theta_{\mathbf{k}} \rangle \zeta_{\mathbf{k}}.$$

Then the diagonal operator $\hat{A}$ is defined on all of Γ. To prove this use that the map $\mathbf{k} \mapsto \langle \eta_{\mathbf{k}} | \theta_{\mathbf{k}} \rangle$ is continuous.

1.2.2 Integral operators

The diagonal operators generalize a certain type of diagonal block matrices. The analogue of non-diagonal block matrices are then integral operators of the type defined below.

The integral operator $\hat{J}$ with measurable kernel $J(\mathbf{k},\mathbf{k}')$ is defined by

$$[\hat{J}\zeta]_{\mathbf{k}} = \int \mathrm{d}\mathbf{k}'\, J(\mathbf{k},\mathbf{k}')\zeta_{\mathbf{k}'}.$$

The domain of definition of $\hat{J}$ is the subspace of Γ consisting of all ζ for which

- $\zeta_{\mathbf{k}'}$ is in the domain of $J(\mathbf{k},\mathbf{k}')$ for all $\mathbf{k}$ and almost all $\mathbf{k}'$;
- the map $\mathbf{k}' \mapsto J(\mathbf{k},\mathbf{k}')\zeta_{\mathbf{k}'}$ is integrable for all $\mathbf{k}$;
- the map $\mathbf{k} \mapsto \int \mathrm{d}\mathbf{k}'\, J(\mathbf{k},\mathbf{k}')\zeta_{\mathbf{k}'}$ is continuous;

Formally, a diagonal operator $\hat{A}$ is an integral operator $\hat{J}$ with kernel $J(\mathbf{k},\mathbf{k}') = \delta(\mathbf{k}-\mathbf{k}')A_{\mathbf{k}}$. However, this kernel does not satisfy the condition of integrability.

Given kernels $J(\mathbf{k},\mathbf{k}')$ and $L(\mathbf{k},\mathbf{k}')$ the product of the operators $\hat{J}$ and $\hat{L}$ involves a convolution of their kernels and can be written as $\hat{J}\hat{L} = \widehat{(J*L)}$. This follows from

$$\begin{aligned}
[\hat{J}\hat{L}\zeta]_{\mathbf{k}} &= \int \mathrm{d}\mathbf{k}'\, J(\mathbf{k},\mathbf{k}')[\hat{L}\zeta]_{\mathbf{k}'} \\
&= \int \mathrm{d}\mathbf{k}'\, J(\mathbf{k},\mathbf{k}') \int \mathrm{d}\mathbf{k}''\, L(\mathbf{k}',\mathbf{k}'')\zeta_{\mathbf{k}''} \\
&= \int \mathrm{d}\mathbf{k}'' \left(\int \mathrm{d}\mathbf{k}'\, J(\mathbf{k},\mathbf{k}')L(\mathbf{k}',\mathbf{k}'') \right) \zeta_{\mathbf{k}''} \\
&= [\widehat{(J*L)}\zeta]_{\mathbf{k}}
\end{aligned}$$

with the convolution of kernels J and L defined by

$$(J*L)(\mathbf{k},\mathbf{k}'') = \int \mathrm{d}\mathbf{k}'\, J(\mathbf{k},\mathbf{k}')L(\mathbf{k}',\mathbf{k}'').$$

1.2.3 Adjoint operators

The adjoint $\hat{A}^\dagger$ of an operator $\hat{A}$ on Γ is an operator on $\Gamma^\dagger$ satisfying

$$(\hat{A}^\dagger\theta,\zeta)_{\mathbf{k}} = (\theta,\hat{A}\zeta)_{\mathbf{k}} \quad \text{for all } \mathbf{k}\in K, \theta\in\Gamma^\dagger, \zeta\in\Gamma.$$

The operator $\hat{A}$ on Γ is said to be symmetric if $\hat{A}^\dagger\zeta^{\mathrm{T}} = (\hat{A}\zeta)^{\mathrm{T}}$ for all $\zeta\in\Gamma$.

Proposition 1.2.3 *Consider an operator $\hat{A}$ on Γ, which is everywhere defined and continuous. Then there exists a unique adjoint $\hat{A}^\dagger$ with domain all of $\Gamma^\dagger$.*

Proof

Fix θ in $\Gamma^\dagger$. Let $\eta^\dagger_{\mathbf{k}}(\zeta) = \theta^\dagger_{\mathbf{k}}(\hat{A}\zeta)$. Then the map $\zeta \mapsto \eta^\dagger_{\mathbf{k}}(\zeta)$ is continuous because $\zeta \mapsto \hat{A}\zeta$ is continuous by assumption and $\theta^\dagger_{\mathbf{k}}$ belongs to Γ^*. In addition is $\mathbf{k} \mapsto \eta^\dagger_{\mathbf{k}}(\zeta)$ continuous for any $\zeta \in \Gamma$ because $\mathbf{k} \mapsto \theta^\dagger_{\mathbf{k}}$ is pointwise continuous. Hence, $\mathbf{k} \mapsto \eta^\dagger_{\mathbf{k}}$ belongs to $\Gamma^\dagger$.

Define the linear operator $\hat{A}^\dagger$ by $\hat{A}^\dagger\theta = \eta$. One verifies that

$$(\hat{A}^\dagger\theta, \zeta)_{\mathbf{k}} = (\eta, \zeta)_{\mathbf{k}} = \overline{\eta_{\mathbf{k}}(\zeta)} = \overline{\zeta_{\mathbf{k}}(\hat{A}\zeta)} = (\zeta, \hat{A}\zeta)_{\mathbf{k}}.$$

This shows that $\hat{A}^\dagger$ is an adjoint of $\hat{A}$.

Assume now that $(\zeta, \theta)_{\mathbf{k}} = (\zeta, \eta)_{\mathbf{k}}$ for all ζ and $\mathbf{k}$. This means $\theta_{\mathbf{k}}(\zeta) = \eta_{\mathbf{k}}(\zeta)$ so that the functions $\theta_{\mathbf{k}}$ and $\eta_{\mathbf{k}}$ coincide for all $\mathbf{k}$. This implies $\theta = \eta$ and hence uniqueness of the adjoint $\hat{A}^\dagger$.

□

Consider a diagonal operator $\hat{A}$ defined by bounded operators $A_{\mathbf{k}}$. Then $\hat{A}$ is continuous and everywhere defined. Hence the proposition applies and the adjoint $\hat{A}^\dagger$ is well-defined. In addition one has for all $\theta \in \Gamma$ that $\hat{A}^\dagger\theta^{\mathrm{T}} = \eta^{\mathrm{T}}$ with the field η defined by $\eta_{\mathbf{k}} = A^\dagger_{\mathbf{k}}\theta_{\mathbf{k}}$. This implies that $\hat{A}^\dagger$ maps the subspace Γ^{T} of $\Gamma^\dagger$ into itself.

On the other hand, if $\hat{J}$ is an integral operator with kernel $J_{\mathbf{k},\mathbf{k}'}$, then one cannot expect that $\hat{J}^\dagger$ maps the subspace Γ^{T} of $\Gamma^\dagger$ into itself. Indeed, one calculates

$$\begin{aligned}
\left(\hat{J}^\dagger\theta^{\mathrm{T}}, \zeta\right)_{\mathbf{k}} &= \left(\theta^{\mathrm{T}}, \hat{J}\zeta\right)_{\mathbf{k}} \\
&= \left\langle \theta_{\mathbf{k}} | [\hat{J}\zeta]_{\mathbf{k}} \right\rangle \\
&= \int \mathrm{d}\mathbf{k}' \, \langle \theta_{\mathbf{k}} | J_{\mathbf{k},\mathbf{k}'}\zeta_{\mathbf{k}'} \rangle \\
&= \int \mathrm{d}\mathbf{k}' \, \left\langle J^\dagger_{\mathbf{k},\mathbf{k}'}\theta_{\mathbf{k}} | \zeta_{\mathbf{k}'} \right\rangle \\
&= \int \mathrm{d}\mathbf{k}' \, (\eta^{\mathrm{T}}(\mathbf{k}), \zeta_{\mathbf{k}'})_{\mathbf{k}'}
\end{aligned}$$

with $\eta_{\mathbf{k}'}(\mathbf{k}) = J^\dagger_{\mathbf{k},\mathbf{k}'}\theta_{\mathbf{k}}$. This result is not of the form $(\eta^{\mathrm{T}}, \zeta)_{\mathbf{k}}$.

1.2.4 Isometries

Consider an operator $\hat{U}$ on Γ which conserves field normalization. Continuity of the map $\hat{U}$ follows from

$$||[\hat{U}\zeta]_{\mathbf{k}}|| = ||\zeta_{\mathbf{k}}|| \quad \text{for all } \mathbf{k} \in K.$$

Hence, by Proposition 1.2.3, $\hat{U}^\dagger$ is defined on all of $\Gamma^\dagger$. In addition, if ζ and θ belong to Γ then one has

$$\left(\hat{U}^\dagger(\hat{U}\theta)^{\mathrm{T}}, \zeta\right)_{\mathbf{k}} = \left((\hat{U}\theta)^{\mathrm{T}}, \hat{U}\zeta\right)_{\mathbf{k}} = \langle[\hat{U}\theta]_{\mathbf{k}}|[\hat{U}\zeta]_{\mathbf{k}}\rangle = \langle\theta_{\mathbf{k}}|\zeta_{\mathbf{k}}\rangle = (\theta^{\mathrm{T}}, \zeta)_{\mathbf{k}}.$$

This implies $\hat{U}^\dagger(\hat{U}\theta)^{\mathrm{T}} = \theta^{\mathrm{T}}$ for all $\theta \in \Gamma$.

Proposition 1.2.4 *Any strongly continuous map $\mathbf{k} \mapsto U_{\mathbf{k}}$ into the isometries of $\mathcal{H}$ defines a diagonal operator $\hat{U}$ which conserves field normalization.*

Proof

One has

$$\begin{aligned} ||U_{\mathbf{k}}\zeta_{\mathbf{k}} - U_{\mathbf{k}'}\zeta_{\mathbf{k}'}|| &\le ||(U_{\mathbf{k}} - U_{\mathbf{k}'})\zeta_{\mathbf{k}}|| + ||U_{\mathbf{k}'}(\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'})|| \\ &= ||(U_{\mathbf{k}} - U_{\mathbf{k}'})\zeta_{\mathbf{k}}|| + ||\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'}||. \end{aligned}$$

Hence continuity of the map $\mathbf{k} \mapsto ||U_{\mathbf{k}}\zeta_{\mathbf{k}}||$ follows from the strong continuity of $\mathbf{k} \mapsto U_{\mathbf{k}}$ and continuity of $\mathbf{k} \mapsto \zeta_{\mathbf{k}}$. This shows that $\hat{U}\zeta$ belongs to Gamma for all ζ and therefore that $\hat{U}$ is defined on all of Γ. That it conserves field normalization follows immediately.

□

1.3 Quantum expectations

The quantum expectation of an operator $\hat{A}$ on Γ, given a properly normalized field ζ belonging to its domain, equals

$$\langle\hat{A}\rangle \quad = \quad \ell^3 \int \mathrm{d}\mathbf{k} \, \left(\zeta^{\mathrm{T}}, \hat{A}\zeta\right)_{\mathbf{k}},$$

whenever this integral converges. The constant length ℓ has been added to make the field ζ dimensionless. The quantity $\left(\zeta^{\mathrm{T}}, \hat{A}\zeta\right)_{\mathbf{k}}$ is interpreted as being the quantum expectation of $\hat{A}$ conditioned on the knowledge of the value of the wave vector $\mathbf{k}$. This conditioning is meaningful because the reduction of the representation over the wave vector is a classical, i.e. non-quantum aspect of the theory. A weighing of the integration over $\mathbf{k}$ may be added if there is physical evidence for it.

Chapter 2

Boson Fields

2.1 The scalar boson field

The standard representation of quantum mechanics is said to be irreducible because the only operators commuting with momentum and position operators P and Q are the multiples of the identity operator. The reducible representation, used in the present work, is built by integrating irreducible representations. Following the original work of Marek Czachor and coworkers (see [2, 3, 4, 5] and papers cited in these works) integration over the wave vector $\mathbf{k}$ is used to decompose the reducible representation into irreducible ones. This means that for a given wave vector $\mathbf{k}$ the standard representation of quantum mechanics in a Hilbert space $\mathcal{H}$ is used. The dependence of the wave vector involves a field of Hilbert spaces Γ. It is the linear space which consists of all continuous fields $\zeta : \mathbf{k} \in \mathbb{R}^3_{\mathrm{o}} \mapsto \zeta_{\mathbf{k}} \in \mathcal{H}$. Note the exclusion of $\mathbf{k} = 0$. It is assumed that the wave vector of a massless boson field cannot vanish.

2.1.1 The irreducible components

A scalar boson at a given wave vector $\mathbf{k}$ in $\mathbb{R}^3_{\mathrm{o}}$ is described by a quantum harmonic oscillator. For further use and in order to establish notations some standard knowledge about the quantum harmonic oscillator is repeated here.

The Hilbert space $\mathcal{H}$ equals the space $\mathcal{L}^2(\mathbb{R}, \mathbb{C})$ of quadratically integrable complex functions over the real line. The momentum operator P and the position operator Q are self-adjoint operators defined in the usual manner. The annihilation operator a, and its adjoint $a^\dagger$, are defined by

$$a \quad = \quad \frac{1}{r\sqrt{2}} Q + i \frac{r}{\hbar\sqrt{2}} P.$$

The positive constant length r is introduced to make the operators a, and $a^\dagger$ dimensionless. The Hamiltonian H of the harmonic oscillator can then be written as

$$H = \hbar\omega a^\dagger a,$$

with $\omega > 0$ the frequency of the oscillator. Note that the so-called ground state energy is omitted. In what follows the frequency ω will depend on a 3-dimensional wave vector $\mathbf{k}$, with a so-called *linear dispersion relation*

$$\omega(\mathbf{k}) = c|\mathbf{k}|.$$

Here, c is the speed of light.

The ground state of the harmonic oscillator is described by the wave function

$$|0\rangle(y) = \frac{1}{\pi^{1/4}r^{1/2}}e^{-y^2/2r^2}, \quad y \in \mathbb{R}.$$

It is normalized to one

$$||\,|0\rangle||^2 = \int |\,|0\rangle(y)|^2\,\mathrm{d}y = 1.$$

It satisfies $a|0\rangle = 0$.

The eigenstates of the Hamiltonian H are denoted $|n\rangle$, $n = 0, 1, 2\cdots$. They can be constructed starting from the ground state $|0\rangle$ by the action of the creation operator $a^\dagger$. Indeed, one has

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \text{and} \quad a|n+1\rangle = \sqrt{n+1}|n\rangle.$$

This implies $a^\dagger a|n\rangle = n|n\rangle$ so that

$$H|n\rangle = n\hbar\omega|n\rangle.$$

A formally definition of the annihilation operator a is needed in what follows. It is the smallest closed linear operator satisfying $a|0\rangle = 0$ and $a|n\rangle = \sqrt{n}|n-1\rangle$, $n = 1, 2\cdots$.

2.1.2 Coherent states

Fix a complex number z. The following wave function determines a *coherent state*

$$|z\rangle^{\mathrm{c}} = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}z^n|n\rangle$$

The sum is convergent and the wave function is normalized to one:

$$||\,|z\rangle^{\mathrm{c}}|| = \sqrt{{}^{\mathrm{c}}\langle z|z\rangle^{\mathrm{c}}} = 1.$$

Proposition 2.1.1 *All coherent states belong to the domain of the annihilation operator a.*

Proof

The wave functions ζ_m defined by

$$\zeta_m = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{m}\frac{1}{\sqrt{n!}}z^n|n\rangle$$

belong to the domain of a and converge to $|z\rangle^{\mathrm{c}}$. Their image $a\zeta_m$ satisfies

$$\begin{aligned} a\zeta_m &= e^{-\frac{1}{2}|z|^2}\sum_{n=1}^{m}\frac{1}{\sqrt{n!}}z^n\sqrt{n}|n-1\rangle \\ &= ze^{-\frac{1}{2}|z|^2}\sum_{n=0}^{m-1}\frac{1}{\sqrt{n!}}z^n|n\rangle \\ &= z\zeta_{m-1}. \end{aligned}$$

In particular, the latter converges to $z|z\rangle^{\mathrm{c}}$. Therefore one concludes that $|z\rangle^{\mathrm{c}}$ belongs to the domain of the closed operator a.

□

From the proof follows the well-known result that the coherent states are eigenstates of the annihilation operator

$$a|z\rangle^{\mathrm{c}} = z|z\rangle^{\mathrm{c}}.$$

Note that $|0\rangle^{\mathrm{c}} = |0\rangle$ is the ground state of the harmonic oscillator.

Proposition 2.1.2 *The maps $w \to |w\rangle^{\mathrm{c}}$ and $|w\rangle^{\mathrm{c}} \to w$ are one-to-one and continuous.*

Proof

The norm continuity of the map $z \mapsto |z\rangle^{\rm c}$ follows immediately from

$$\begin{aligned} ||\,|z\rangle^{\rm c} - |w\rangle^{\rm c}||^2 &= 2\,(1 - \,\mathrm{Re}\,\langle z|w)\rangle^{\rm c})^2 \\ &= 2\left[1 - e^{-\frac{1}{2}|z-w|^2}\cos\left(z\wedge w\right)\right], \end{aligned}$$

with

$$w \wedge z \quad = \quad (\,\mathrm{Re}\, w)(\,\mathrm{Im}\, z) - (\,\mathrm{Im}\, w)(\,\mathrm{Re}\, z).$$

Conversely, if $||\,|z\rangle^{\rm c} - |w\rangle^{\rm c}||$ tends to zero then both $e^{-\frac{1}{2}|z-w|^2}$ and $\cos\left(z\wedge w\right)$ converge to 1. This implies that $|z-w|$ tends to zero.

$\square$

Proposition 2.1.3 *All coherent states belong to the domain of the creation operator $a^\dagger$.*

Proof

The domain of $a^\dagger$ consists of all wave functions ϕ for which the map

$$\zeta \in \mathrm{Dom}(a) \mapsto \langle \phi | a\zeta\rangle$$

is continuous. Let $\phi = |z\rangle^{\rm c}$ and $\zeta = |w\rangle^{\rm c}$. It suffices to show that the map

$$|w\rangle^{\rm c} \mapsto w\langle z|w\rangle^{\rm c}$$

is continuous. From the lemma follows that $|w\rangle^{\rm c} \mapsto w$ is continuous. In addition the map $w \mapsto w\langle z|w\rangle^{\rm c}$ is continuous as well. This follows from

$$w\langle z|w\rangle^{\rm c} \quad = \quad w e^{i(w\wedge z)} e^{-\frac{1}{2}|w-z|^2}.$$

$\square$

2.1.3 Coherent fields

Let be given a continuous complex function $F(\mathbf{k})$. Use it to define the wave function $|F\rangle^{\rm c}$ of a *coherent field* by $[|F\rangle^{\rm c}]_{\mathbf{k}} = |F(\mathbf{k})\rangle^{\rm c}$. This coherent field is a properly normalized element of Γ. This follows from Proposition 2.1.2. Clearly is

$$a[|F\rangle^{\rm c}]_{\mathbf{k}} = F(\mathbf{k})[|F\rangle^{\rm c}]_{\mathbf{k}} \quad \text{for all } \mathbf{k} \in \mathbb{R}^3_{\rm o}.$$

Coherent fields play an important role further on in the development of the free field theory.

With some abuse of notation the constant field $\mathbf{k} \mapsto |0\rangle^c$ will be denoted $|0\rangle^c$ as well as $|0\rangle$. It is the vacuum state of the free field theory.

The extension $\hat{a}$ of the annihilation operator a to a diagonal operator on Γ satisfies the following property.

Proposition 2.1.4 *Let be given a continuous complex function $F(\mathbf{k})$. The coherent field $|F\rangle^c$ belongs to the domain of the diagonal operator $\hat{a}$ and satisfies $[\hat{a}|F\rangle^c]_{\mathbf{k}} = F(\mathbf{k})|F(\mathbf{k})\rangle^c$ for all $\mathbf{k}$ in $\mathbb{R}^3_o$. If $F(\mathbf{k})$ is bounded then $\hat{a}|F\rangle^c$ is bounded in norm.*

Proof

The conditions for $|F\rangle^c$ to belong to the domain of $\hat{a}$ are that $|F(\mathbf{k})\rangle^c$ is in the domain of a for any $\mathbf{k}$ and that the map $\mathbf{k} \mapsto F(\mathbf{k})|F(\mathbf{k})\rangle^c$ is continuous. Proposition 2.1.1 shows that $|F(\mathbf{k})\rangle^c$ is in the domain of a. Continuity follows from Proposition 2.1.2 and continuity of the function F. If $F(\mathbf{k})$ is bounded then

$$||[\hat{a}|F\rangle^c]_{\mathbf{k}}|| = |F(\mathbf{k})|\,|||F\rangle^c|| = |F(\mathbf{k})|$$

is bounded as well.

□

A similar result holds for the creation operator $\hat{a}^\dagger$.

Proposition 2.1.5 *Let be given a continuous complex function $F(\mathbf{k})$. The coherent field $|F\rangle^c$ belongs to the domain of the diagonal operator $\hat{a}^\dagger$. If $F(\mathbf{k})$ is bounded then $\hat{a}^\dagger|F\rangle^c$ is bounded in norm.*

Proof

Use now Proposition 2.1.3 to show that $|F\rangle^c$ belongs to the domain of $\hat{a}^\dagger$. Denote $\zeta = |F\rangle^c$. Continuity of the map $\mathbf{k} \mapsto a^\dagger\zeta_{\mathbf{k}}$ follows from

$$\begin{aligned}
||a^\dagger\zeta_{\mathbf{k}} - a^\dagger\zeta_{\mathbf{k}'}||^2 &= \langle\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'}|aa^\dagger(\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'})\rangle \\
&= \langle\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'}|(1 + a^\dagger a(\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'})\rangle \\
&= ||\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'}||^2 + ||a(\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'})||^2 \\
&= ||\zeta_{\mathbf{k}} - \zeta_{\mathbf{k}'}||^2 + ||F(\mathbf{k})\zeta_{\mathbf{k}} - F(\mathbf{k}')\zeta_{\mathbf{k}'}||^2
\end{aligned}$$

and the continuity of both $F(\mathbf{k})$ and $\mathbf{k} \mapsto \zeta_{\mathbf{k}}$.

In addition one has

$$\begin{aligned}
||[\hat{a}^\dagger\zeta]_{\mathbf{k}}||^2 &= \langle\zeta_{\mathbf{k}}|aa^\dagger\zeta_{\mathbf{k}}\rangle \\
&= \langle\zeta_{\mathbf{k}}|(1+a^\dagger a]\zeta_{\mathbf{k}}\rangle \\
&= [1+|F(k)|^2]\,|||\zeta_{\mathbf{k}}||^2 \\
&= 1+|F(k)|^2.
\end{aligned}$$

This shows that $\hat{a}^\dagger|F\rangle^{\mathrm{c}}$ is bounded in norm when $F(\mathbf{k})$ is bounded.

□

2.1.4 The free field Hamiltonian

The free-field Hamiltonian $\hat{H}$ is an unbounded symmetric operator on Γ. It is the diagonal operator defined by

$$[\hat{H}\zeta]_{\mathbf{k}} = H_{\mathbf{k}}\zeta_{\mathbf{k}} \quad \text{with} \quad H_k = \hbar c|\mathbf{k}|a^\dagger a, \tag{2.1}$$

where a is the annihilation operator introduced before. Its domain of definition is the subspace of Γ consisting of all ζ in Γ such that

- $\zeta_{\mathbf{k}}$ is in the domain of the self-adjoint operator $a^\dagger a$ for all $\mathbf{k}$;
- $\mathbf{k} \in \mathbb{R}^3_{\mathrm{o}} \mapsto |\mathbf{k}|a^\dagger a\zeta_{\mathbf{k}}$ is continuous.

The following result shows that physically acceptable free fields necessarily are superpositions with the vacuum field.

Proposition 2.1.6 *Let be given a field ζ in the domain of $\hat{H}$ and assume that $\hat{H}\zeta$ is bounded in norm. Then there exists a complex function $c_0(\mathbf{k})$ such that $\zeta_{\mathbf{k}} - c_0(\mathbf{k})|0\rangle$ converges in norm to zero as $|\mathbf{k}|$ diverges.*

Proof

Let C denote the uniform bound of $|\mathbf{k}|^2\,||a^\dagger a\zeta_{\mathbf{k}}||^2$. Decompose $\zeta_{\mathbf{k}}$ in the basis of eigenvectors of the harmonic oscillator

$$\zeta_{\mathbf{k}} = \sum_{n=0}^{\infty} c_n(\mathbf{k})|n\rangle.$$

Then one has

$$C \geq |\mathbf{k}|^2\,||a^\dagger a\zeta_{\mathbf{k}}||^2 = |\mathbf{k}|^2 \sum_n n^2|c_n(\mathbf{k})|^2.$$

This implies

$$|\mathbf{k}|^2 n^2\,|c_n(\mathbf{k})|^2 \leq C \quad \text{for all } \mathbf{k}, n.$$

One obtains

$$\begin{aligned} ||\sum_{n=1}^{\infty} c_n(\mathbf{k})|n\rangle||^2 &= \sum_{n=1}^{\infty} |c_n(\mathbf{k})|^2 \\ &\leq \frac{C}{|\mathbf{k}|^2}\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty. \end{aligned}$$

This shows that $\sum_{n=1}^{\infty} c_n(\mathbf{k})|n\rangle$ converges in norm to zero as $|\mathbf{k}|$ tends to infinity.

□

2.1.5 The classical wave equation

A large class of solutions of the free wave equation $\Box_x\phi = 0$ consists of functions $\phi(x)$ of the form

$$\phi(x) = 2\,\mathrm{Re}\int \mathrm{d}\mathbf{k}\,\frac{\ell^{3/2}}{N_0(\mathbf{k})} f(\mathbf{k}) e^{-ik_\mu x^\mu}, \tag{2.2}$$

where f is a continuous function of $\mathbb{R}^3_{\mathrm{o}}$.

The so-called normalization factor $N_0(\mathbf{k})$ is the usual one

$$N_0(\mathbf{k}) = \sqrt{(2\pi)^3 2|\mathbf{k}|\ell}, \tag{2.3}$$

except that the constant ℓ is inserted also here to make it dimensionless. The insertion of this normalization factor leads further on to a satisfactory physical interpretation of the profile function $f(\mathbf{k})$.

The total energy of the classical field ϕ is given by

$$\mathcal{E}^{\mathrm{cl}} = \frac{\hbar c}{2\ell^2}\int_{\mathbb{R}^3} \mathrm{d}\mathbf{x}\,\left[\left(\frac{\partial\phi}{\partial x^0}\right)^2 + \sum_\alpha\left(\frac{\partial\phi}{\partial x^\alpha}\right)^2\right]. \tag{2.4}$$

From (2.2) one obtains

$$\mathcal{E}^{\mathrm{cl}} = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\,\hbar c|\mathbf{k}||f(\mathbf{k})|^2. \tag{2.5}$$

The interpretation in the context of quantum mechanics is standard. The factor $|f(\mathbf{k})|^2$ is the density of particles with wave vector $\mathbf{k}$ and corresponding energy $\hbar c|\mathbf{k}|$.

The particle density $|f(\mathbf{k})|^2$ has the dimension of the inverse of a volume in $\mathbf{k}$-space. Introduce therefore the dimensionless function

$$F(\mathbf{k}) \quad = \quad l^{-3/2} f(\mathbf{k})$$

and use it to construct the coherent field $|F\rangle^{\mathrm{c}}$ — see Section 2.1.3. Consider now the Hamiltonian $\hat{H}$ of the free boson field as given by (2.1). Its irreducible components satisfy

$$\begin{aligned} {}^{\mathrm{c}}\langle F(\mathbf{k})|H_{\mathbf{k}}|F(\mathbf{k})\rangle^{\mathrm{c}} &= \hbar c|\mathbf{k}|^{\mathrm{c}}\langle F(\mathbf{k})|a^\dagger a|F(\mathbf{k})\rangle^{\mathrm{c}} \\ &= \hbar c|\mathbf{k}|\,|F(\mathbf{k})|^2 \\ &= \ell^{-3}\hbar c|\mathbf{k}||f(\mathbf{k})|^2. \end{aligned}$$

This result allows us to write the classical energy (2.5) in terms of the free-field Hamiltonian $\hat{H}$ and the coherent field $|F\rangle^{\mathrm{c}}$

$$\mathcal{E}^{\mathrm{cl}} \quad = \quad \ell^3 \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, \langle F(\mathbf{k})|H_{\mathbf{k}}|F(\mathbf{k})\rangle^{\mathrm{c}} \leq +\infty. \tag{2.6}$$

2.1.6 Correspondence principle

Introduce field operators $\hat{\phi}(x)$, with x in Minkowski space $\mathbb{R}^4$, defined by

$$[\hat{\phi}(x)\zeta]_{\mathbf{k}} = \phi_{\mathbf{k}}(x)\zeta_{\mathbf{k}}$$

with

$$\phi_{\mathbf{k}}(x) \quad = \quad \frac{1}{N_0(\mathbf{k})}\left(e^{-ik_\mu x^\mu} a + e^{ik_\mu x^\mu} a^\dagger\right). \tag{2.7}$$

The eigenstates $|n\rangle$, $n = 0, 1, \cdots$ of the harmonic oscillator belong to the domain of the r.h.s. of (2.7), as well as all coherent states $|z\rangle$, $z \in \mathbb{C}$. It is obvious to define $\phi_{\mathbf{k}}$ as the self-adjoint extension of the r.h.s. of (2.7). The map $\mathbf{k} \mapsto \phi_{\mathbf{k}}(x)$ defines a diagonal operator $\hat{\phi}(x)$ of Γ. It is called the *free field operator.*

The free field operators satisfy the commutation relations

$$[\hat{\phi}(x), \hat{\phi}(y)]_- \quad = \quad \left(\mathbf{k} \mapsto \frac{1}{(2\pi)^3 \ell |\mathbf{k}|} i \sin(k_\mu(y^\mu - x^\mu)\right).$$

The r.h.s. of this expression is a bounded diagonal operator which commutes with all other diagonal operators of Γ.

Derivatives to all orders of $\hat{\phi}(x)$ with respect to x^μ are again diagonal operators. In particular the free field operators satisfy the operator-valued wave equation

$$\Box_x \hat{\phi}(x) = 0.$$

Proposition 2.1.7 *Given any continuous complex function f of $\mathbb{R}^3_o$ and the corresponding function $F(\mathbf{k}) = \ell^{-3/2} f(\mathbf{k})$, the coherent field $|F\rangle^c$ belongs to the domain of the free field operator $\hat{\phi}(x)$ for any x in Minkowski space $\mathbb{R}^4$.*

Proof

The proofs of Propositions 2.1.4 and 2.1.5 can be adapted to show that $|F(\mathbf{k})\rangle^c$ is in the domain of the operators $\phi_\mathbf{k}(x)$ and that the map $\mathbf{k} \in \mathbb{R}^3_o \mapsto \phi_\mathbf{k}(x)|F(\mathbf{k})\rangle^c$ is continuous. This suffices to conclude that $|F\rangle^c$ belongs to the domain of $\hat{\phi}(x)$.

$\Box$

The calculations of the previous section can be summarized as follows.

Theorem 2.1.8 *Let be given a solution $\phi(x)$ of the wave equation. Assume it is of the form (2.2) with a complex function $f(\mathbf{k})$ continuously defined on $\mathbb{R}^3_o$. Then there exists a coherent field ζ in Γ such that the classical field $\phi(x)$ and its classical energy $\mathcal{E}^{cl}$ are given by*

$$\begin{aligned} \phi(x) &= \ell^3 \int \mathrm{d}\mathbf{k}\, (\zeta, \phi_\mathbf{k}(x)\zeta)_\mathbf{k}, \qquad (2.8) \\ \mathcal{E}^{cl} &= \ell^3 \int \mathrm{d}\mathbf{k}\, (\zeta, \hat{H}\zeta)_\mathbf{k} \le +\infty. \end{aligned}$$

Proof

Take $\zeta_\mathbf{k} = |F(\mathbf{k})\rangle^c$ with $F(\mathbf{k}) = \ell^{-3/2} f(\mathbf{k})$. From the definitions follows

$$\begin{aligned} \ell^3 \int \mathrm{d}\mathbf{k}\, \langle \zeta_\mathbf{k} | [\hat{\phi}(x)\zeta]_\mathbf{k}\rangle &= \ell^3 \int \mathrm{d}\mathbf{k}\, \frac{1}{N_0(\mathbf{k})} 2\,\mathrm{Re}\, \langle \zeta_\mathbf{k} | e^{-ik_\mu x^\mu} a \zeta_\mathbf{k}\rangle \\ &= \ell^3 \int \mathrm{d}\mathbf{k}\, \frac{1}{N_0(\mathbf{k})} 2\,\mathrm{Re}\, e^{-ik_\mu x^\mu} F(\mathbf{k})\rangle \\ &= \ell^{3/2} \int \mathrm{d}\mathbf{k}\, \frac{1}{N_0(\mathbf{k})} 2\,\mathrm{Re}\, e^{-ik_\mu x^\mu} f(\mathbf{k})\rangle \end{aligned}$$

$$= \phi(x).$$

This proves the former of the two claims. Next calculate

$$\begin{aligned}
\ell^3 \int \mathrm{d}\mathbf{k}\, \langle \zeta_{\mathbf{k}} | [\hat{H}\zeta]_{\mathbf{k}} \rangle &= \hbar c \ell^3 \int \mathrm{d}\mathbf{k}\, |\mathbf{k}| \langle \zeta_{\mathbf{k}} | a^\dagger a \zeta_{\mathbf{k}} \rangle \\
&= \hbar c \ell^3 \int \mathrm{d}\mathbf{k}\, |\mathbf{k}|\, |F(\mathbf{k})|^2 \\
&= \hbar c \int \mathrm{d}\mathbf{k}\, |\mathbf{k}| |f(\mathbf{k})|^2 \\
&= \mathcal{E}^{\mathrm{cl}}.
\end{aligned}$$

To obtain the last line (2.5) is used.

□

2.1.7 Incoherent fields

Up to now coherent fields $|F\rangle^{\mathrm{c}}$ are considered. It is obvious to postulate that any physically acceptable quantum field is described by a properly normalized element ζ in the domain of the annihilation operator $\hat{a}$. This includes the coherent fields $|F\rangle^{\mathrm{c}}$ because they are properly normalized and belong to the domain of $\hat{a}$ — see Proposition 2.1.4.

If ζ is properly normalized and belongs to the domain of $\hat{a}$ then a classical field $\phi(\mathbf{k})$ is defined by

$$\phi^{\mathrm{cl}}(x) = 2\,\mathrm{Re}\, \ell^3 \int \mathrm{d}\mathbf{k}\, \frac{1}{N_0(\mathbf{k})} e^{-ik_\mu x^\mu} \langle \zeta_{\mathbf{k}} | a \zeta_{\mathbf{k}} \rangle$$

provided that the latter integral is convergent. This classical field is of the form (2.2) with

$$f(\mathbf{k}) = \ell^{3/2} \langle \zeta_{\mathbf{k}} | a \zeta_{\mathbf{k}} \rangle.$$

The corresponding classical energy is (see (2.4))

$$\mathcal{E}^{\mathrm{cl}} = \ell^3 \int \mathrm{d}\mathbf{k}\, \hbar c |\mathbf{k}|\, |\langle \zeta_{\mathbf{k}} | a \zeta_{\mathbf{k}} \rangle|^2 \le +\infty.$$

The quantum expectation $\mathcal{E}^{\mathrm{qu}}$ of the energy is given by

$$\mathcal{E}^{\mathrm{qu}} = \langle \zeta | \hat{H} \zeta \rangle = \langle \hat{H}^{1/2} \zeta | H^{1/2} \zeta \rangle. \tag{2.9}$$

The assumption that ζ belongs to the domain of $\hat{a}$ does not imply that it belongs to the domain of $\hat{H}$, but only to that of $\hat{H}^{1/2}$. Therefore only the latter of the two expressions is used. This gives

$$\mathcal{E}^{\mathrm{qu}} = \ell^3 \int \mathrm{d}\mathbf{k}\, \hbar c |\mathbf{k}|\, ||a \zeta_{\mathbf{k}}||^2 \le +\infty.$$

From Schwarz's inequality follows $\mathcal{E}^{\rm cl} \le \mathcal{E}^{\rm qu}$. This implies that the quantum energy of a free boson field is always larger than or equal to the classical energy of the classical field determined by the quantum field. Equality occurs for coherent fields. One concludes that there exists many quantum fields which all produce the same classical field. If the quantum expectation of the energy is finite then the energy of the coherent field equals the classical energy while incoherent fields have a larger energy.

2.1.8 Correlations

It is tradition to introduce so-called positive frequency and negative frequency parts $\hat\phi^{(\pm)}(x)$ of the field operator $\hat\phi(x)$. They are defined by

$$\phi^{(+)}_{\mathbf{k}}(x) = \frac{1}{N_0(\mathbf{k})} e^{-ik_\mu x^\mu} a \quad \text{and} \quad \phi^{(-)}_{\mathbf{k}}(x) = \frac{1}{N_0(\mathbf{k})} e^{ik_\mu x^\mu} a^\dagger.$$

One has of course

$$\hat\phi(x) = \hat\phi^{(+)}(x) + \hat\phi^{(-)}(x).$$

Fix a field ζ in the domain of $\hat a$. Then a two-point function is formally defined by

$$G_\zeta(x,y) \;=\; \ell^3 \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{k}' \, \langle \zeta_{\mathbf{k}} | \phi^{(-)}_{\mathbf{k}}(x) \phi^{(+)}_{\mathbf{k}'}(y) \zeta_{\mathbf{k}'} \rangle.$$

Consider now the energy density $e(x)$ of the field at space time position x. It is defined by

$$\begin{aligned} e(x) &= \frac{\hbar c}{\ell^2} \frac{\partial}{\partial x^0} \frac{\partial}{\partial y^0} G(x,y)\bigg|_{y=x} + \frac{\hbar c}{\ell^2} \sum_\alpha \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\alpha} G(x,y)\bigg|_{y=x} \\ &= \frac{\hbar c \ell^3}{2(2\pi)^3} \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{k}' \, \frac{|\mathbf{k}||\mathbf{k}'| + \mathbf{k}\cdot\mathbf{k}'}{\sqrt{|\mathbf{k}|\,|\mathbf{k}'|}} \langle \zeta_{\mathbf{k}} | \phi^{(-)}_{\mathbf{k}}(x) \phi^{(+)}_{\mathbf{k}'}(x) \zeta_{\mathbf{k}'} \rangle \end{aligned} \tag{2.10}$$

With this definition the integral of the energy density equals the total energy $\mathcal{E}^{\rm qu}$ as given by (2.9). Indeed,

$$\begin{aligned} \int \mathrm{d}\mathbf{x}\, e(x) &= \frac{\hbar c \ell^3}{2} \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{k}' \, \frac{|\mathbf{k}||\mathbf{k}'| + \mathbf{k}\cdot\mathbf{k}'}{\sqrt{|\mathbf{k}|\,|\mathbf{k}'|}} \delta(\mathbf{k}-\mathbf{k}') \langle \zeta_{\mathbf{k}} | a^\dagger a \zeta_{\mathbf{k}'} \rangle \\ &= \ell^3 \int \mathrm{d}\mathbf{k}\, \hbar c |\mathbf{k}| \langle \zeta_{\mathbf{k}} | a^\dagger a \zeta_{\mathbf{k}} \rangle = \mathcal{E}^{\rm qu}. \end{aligned} \tag{2.11}$$

2.1.9 Lorentz covariance of coherent fields

The Lorentz covariance of the theory is based on the Lorentz invariance of the set of coherent fields.

Fix a complex continuous function $f(\mathbf{k})$ and let $|F(\mathbf{k})\rangle^{\mathrm{c}}$ be the corresponding coherent wave function. Let Λ be a proper Lorentz transformation. The classical field (2.2) transforms as $\phi \to \phi'$ with

$$\begin{aligned} \phi'(x) &= \phi(\Lambda^{-1}x) \\ &= 2\,\mathrm{Re}\int \mathrm{d}\mathbf{k}\, \frac{\ell^{3/2}}{N_0(\mathbf{k})} f(\mathbf{k}) e^{-ik_\mu(\Lambda^{-1}x)^\mu}. \end{aligned} \tag{2.12}$$

If Λ is a spatial rotation R then Λ^{-1} is the hermitian conjugate of Λ. Introduce a new integration variable $\mathbf{k}' = R^{-1}\mathbf{k}$. Because $|\mathbf{k}'| = |\mathbf{k}|$ the Jacobian determinant of the transformation equals 1. There follows that

$$\phi'(x) = 2\,\mathrm{Re}\int \mathrm{d}\mathbf{k}'\, \frac{\ell^{3/2}}{N_0(\mathbf{k}')} f(R^{-1}\mathbf{k}') e^{-ik'_\mu x^\mu}.$$

This shows that the rotation R can be transferred to a rotation of the profile function $f(\mathbf{k})$.

An arbitrary proper Lorentz transformation can be decomposed into a spatial rotation, a Lorentz boost in the third direction, followed again by a rotation. It therefore suffices to consider now a Lorentz boost of the form

$$\Lambda = \begin{pmatrix} \cosh\chi & 0 & 0 & \sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix}. \tag{2.13}$$

Let $k^{1\prime} = k^1$, $k^{2\prime} = k^2$, $k^{3\prime} = k^3\cosh\chi + |\mathbf{k}|\sinh\chi$. Then

$$e^{-ik_\mu(\Lambda^{-1}x)^\mu} = e^{-ik'_\mu x^\mu}.$$

The Jacobian of the transformation from $\mathbf{k}$ to $\mathbf{k}'$ is $|\mathbf{k}|/|\mathbf{k}'|$. Hence (2.12) becomes

$$\phi'(x) = 2\,\mathrm{Re}\int \mathrm{d}\mathbf{k}'\, \frac{\ell^{3/2}}{N_0(\mathbf{k}')} \sqrt{\frac{|\mathbf{k}|}{|\mathbf{k}'|}} f(\mathbf{k}) e^{-ik'_\mu x^\mu}. \tag{2.14}$$

Introduce a transformed profile function f' by

$$f'(\mathbf{k}') = \sqrt{\frac{|\mathbf{k}|}{|\mathbf{k}'|}} f(\mathbf{k}) \quad \text{and hence} \quad F'(\mathbf{k}') = \sqrt{\frac{|\mathbf{k}|}{|\mathbf{k}'|}} F(\mathbf{k}). \tag{2.15}$$

This defined the Lorentz boost $|F'\rangle^{\rm c}$ of the coherent field $|F\rangle^{\rm c}$. Indeed, (2.14) now becomes

$$\phi'(x) \quad = \quad 2\,\mathrm{Re}\int \mathrm{d}\mathbf{k}'\,\frac{\ell^{3/2}}{N_0(\mathbf{k}')}f'(\mathbf{k}')e^{-ik'_\mu x^\mu}.$$

This is of the same form as (2.2). Note that $\mathbf{k}' \mapsto F'(\mathbf{k}')$ is a continuous function on $\mathbb{R}^3_{\rm o}$ so that $|F'\rangle^{\rm c}$ is a properly normalized element of Γ. One concludes the following.

Theorem 2.1.9 *The set of coherent fields is invariant under proper Lorentz transformations.*

2.1.10 Lorentz covariance of non-coherent fields

The Lorentz transformation of a coherent wave function is not a linear operation. This is a consequence of the normalization of these wave functions. Introduce therefore denormalized coherent wave functions defined by

$$|z\rangle^\bullet = e^{-z}e^{\frac{1}{2}|z|^2}|z\rangle^{\rm c} = e^{-a}\sum_{n=0}^\infty \frac{1}{\sqrt{n!}}z^n|n\rangle.$$

They all lie in the plane

$$\{\zeta\,:\,{}^\bullet\langle 1|\zeta\rangle = e^{-1}\}.$$

Any wave function ζ can be decomposed into coherent wave functions by

$$\zeta = \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}z\,|z\rangle^{\rm c}\,{}^{\rm c}\langle z|\zeta\rangle.$$

See Section 3.2.2 of [6]. Denormalize ζ to become a point $\zeta^\bullet$ in the plane (2.16). This gives

$$\begin{aligned}\zeta^\bullet &= \frac{1}{{}^\bullet\langle 1|\zeta\rangle e}\zeta\\ &= \frac{1}{{}^\bullet\langle 1|\zeta\rangle \pi e}\int_{\mathbb{C}}\mathrm{d}z\,|z\rangle^{\rm c}\,{}^{\rm c}\langle z|\zeta\rangle\\ &= \frac{1}{{}^\bullet\langle 1|\zeta\rangle \pi e}\int_{\mathbb{C}}\mathrm{d}z\,\left[e^{z}e^{-\frac{1}{2}|z|^2}\,{}^{\rm c}\langle z|\zeta\rangle\right]\,|z\rangle^\bullet.\end{aligned}$$

This is the decomposition of $\zeta^\bullet$ into denormalized coherent wave functions.

Let us now generalize (2.16) by linearity in the plane of denormalized wave functions. Expression (2.15) can be written as

$$|F'(\mathbf{k}')\rangle^\bullet \quad = \quad e^{-a}\, T\left(\sqrt{\frac{|\mathbf{k}|}{|\mathbf{k}'|}}\right) e^a |F(\mathbf{k})\rangle^\bullet, \tag{2.16}$$

where T is the linear operator defined by

$$T(\lambda) = \sum_{n=0}^{\infty} \lambda^n |n\rangle\, \langle n| = \lambda^{a^\dagger a}.$$

The obvious generalization is

$$\zeta^{\bullet\prime}_{\mathbf{k}'} \quad = \quad e^{-a}\, T\left(\sqrt{\frac{|\mathbf{k}|}{|\mathbf{k}'|}}\right) e^a \zeta^\bullet_{\mathbf{k}}.$$

This defines the Lorentz transformation in the plane (2.16). By normalization one obtains the Lorentz transform θ of the field ζ. It is given by

$$\theta_{\mathbf{k}} \quad = \quad \frac{1}{||\zeta^{\bullet\prime}_{\mathbf{k}}||}\zeta^{\bullet\prime}_{\mathbf{k}}$$

2.2 Electromagnetic fields

The vector potential $A_\mu(x)$ of classical electromagnetism has a so-called gauge freedom. This means that it is not fully determined by the physical quantities which are the electric and magnetic forces. It is tradition to fix this freedom by use of the Lorentz gauge. It has the advantage of leading to a theory which is manifestly Lorentz covariant. However, it does not eliminate all freedom of choice. The description of free electromagnetic fields is most convenient in the so-called transverse gauge. It limits the number of degrees of freedom to two transversely propagating electromagnetic waves. The discussion on other degrees of freedom in the case of interacting electromagnetic fields is postponed to Part II of this work.

2.2.1 The classical vector potential

An electromagnetic wave traveling in direction 3 with electric component in direction 1 can be described by the vector potential

$$A^{\rm cl}(x) \quad \sim \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cos(k^{\rm ph}(x^3 - x^0)).$$

Here $k^{\rm ph}$ is the wave vector. The index 'ph' is used to label wave vectors of the electromagnetic field. The electric and magnetic fields can be derived from the vector potential $A^{\rm cl}(x)$ by

$$\begin{aligned} E^{\rm cl}_\alpha &= -\frac{\partial A^{\rm cl}_\alpha}{\partial t} - c\frac{\partial A^{\rm cl}_0}{\partial x^\alpha}, \\ B^{\rm cl}_\alpha &= \sum_{\beta,\gamma} \epsilon_{\alpha,\beta,\gamma} \frac{\partial A^{\rm cl}_\gamma}{\partial x^\beta}. \end{aligned}$$

One then finds

$$E^{\rm cl}_1 \sim ck\sin(k^{\rm ph}(x^3 - x^0)),$$

and $B^{\rm cl}_2 = -\frac{1}{c}E^{\rm cl}_1$ and $E^{\rm cl}_2 = E^{\rm cl}_3 = cB^{\rm cl}_1 = cB^{\rm cl}_3 = 0$.

Now let $\Xi(\mathbf{k}^{\rm ph})$ be a rotation matrix which rotates the arbitrary wave vector $\mathbf{k}^{\rm ph} \in \mathbb{R}^3_{\rm o}$ into the positive z-direction. An explicit choice is found in the Appendix A. Then an electromagnetic wave with wave vector $\mathbf{k}^{\rm ph}$ is described by the vector potential with components $A^{\rm cl}_0(x) = 0$ and

$$A^{\rm cl}_\alpha(x) \sim \mathrm{Re}\,\Xi_{1,\alpha}(\mathbf{k}^{\rm ph})e^{-ik^{\rm ph}_\mu x^\mu}.$$

After smearing out with a complex weight function $f(\mathbf{k}^{\rm ph})$, and inserting a normalization factor as before (see (2.3)), this becomes

$$A^{\rm cl}_\alpha(x) = \mathrm{Re}\int \mathrm{d}\mathbf{k}^{\rm ph}\,\frac{\lambda\ell^{3/2}}{N_0(\mathbf{k}^{\rm ph})} f(\mathbf{k}^{\rm ph})\Xi_{1,\alpha}(\mathbf{k}^{\rm ph})e^{-ik^{\rm ph}_\mu x^\mu}.$$

The parameter λ could be absorbed into the weight function $f(\mathbf{k}^{\rm ph})$. However, it is kept for dimensional reasons.

The free electromagnetic wave has two possible polarizations. The second linear polarization is obtained by replacing $\Xi_{1,\alpha}(\mathbf{k}^{\rm ph})$ by $\Xi_{2,\alpha}(\mathbf{k}^{\rm ph})$ in the previous expression. In addition, the two polarizations can be combined by adding up the corresponding vector potentials. The general expression is of the form

$$A^{\rm cl}_\alpha(x) = \mathrm{Re}\int \mathrm{d}\mathbf{k}^{\rm ph}\,\frac{\lambda\ell^{3/2}}{N_0(\mathbf{k}^{\rm ph})} \sum_{\beta=1,2} f_\beta(\mathbf{k}^{\rm ph})\Xi_{\beta,\alpha}(\mathbf{k}^{\rm ph})e^{-ik^{\rm ph}_\mu x^\mu}. \tag{2.17}$$

2.2.2 Field operators

Because the electromagnetic wave has two polarizations it is obvious to consider a 2-dimensional quantum harmonic oscillator instead of the single oscillator used in Section 2.1 on scalar bosons.

Let $a_{\rm H}$ and $a_{\rm V}$ be the annihilation operators for a photon with horizontal respectively vertical polarization. The free-field Hamiltonian of the quantized electromagnetic field $\hat{H}^{\rm ph}$ is the diagonal operator defined by

$$H^{\rm ph}_{\mathbf{k}^{\rm ph}} = \hbar c|\mathbf{k}^{\rm ph}|\left(a^\dagger_{\rm H}a_{\rm H} + a^\dagger_{\rm V}a_{\rm V}\right). \tag{2.18}$$

Field operators $\hat{A}_\alpha(x)$ are defined by

$$\begin{aligned} A_{\alpha,\mathbf{k}^{\rm ph}}(x) &= \frac{\lambda}{2N_0(\mathbf{k}^{\rm ph})}\varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph})\left[e^{-ik^{\rm ph}_\mu x^\mu}a_{\rm H} + e^{ik^{\rm ph}_\mu x^\mu}a^\dagger_{\rm H}\right] \\ &\quad + \frac{\lambda}{2N_0(\mathbf{k}^{\rm ph})}\varepsilon^{(V)}_\alpha(\mathbf{k}^{\rm ph})\left[e^{-ik^{\rm ph}_\mu x^\mu}a_{\rm V} + e^{ik^{\rm ph}_\mu x^\mu}a^\dagger_{\rm V}\right], \end{aligned} \tag{2.19}$$

with polarization vectors $\varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph})$ and $\varepsilon^{(V)}_\alpha(\mathbf{k}^{\rm ph})$ given by two rows of the rotation matrix Ξ

$$\varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph}) = \Xi_{1,\alpha}(\mathbf{k}^{\rm ph}) \quad \text{and} \quad \varepsilon^{(V)}_\alpha(\mathbf{k}^{\rm ph}) = \Xi_{2,\alpha}(\mathbf{k}^{\rm ph}).$$

Note that $a_{\rm H}$ and $a_{\rm V}$ commute and that $[a_{\rm H}, a^\dagger_{\rm H}] = \mathbb{I}$ and $[a_{\rm V}, a^\dagger_{\rm V}] = \mathbb{I}$. This can be used to verify that the field operators $\hat{A}_\alpha(x)$ satisfy Heisenberg's equation of motion

$$i\hbar c\partial_0\hat{A}_\alpha(x) = \left[\hat{A}_\alpha(x), \hat{H}^{\rm ph}\right]_- .$$

Fix a properly normalized field ζ in Γ. The quantum expectation of the field operators becomes

$$\begin{aligned} A^{\rm cl}_\alpha(x) &= \ell^3\int \mathrm{d}\mathbf{k}^{\rm ph}\,(\zeta, |A_\alpha(x)\zeta)_{\mathbf{k}^{\rm ph}} \\ &= \ell^3\,\mathrm{Re}\int \mathrm{d}\mathbf{k}^{\rm ph}\frac{\lambda}{N_0(\mathbf{k}^{\rm ph})}e^{-ik^{\rm ph}_\mu x^\mu} \\ &\quad\times\left[\varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph})\langle\zeta_{\mathbf{k}^{\rm ph}}|a_{\rm H}\zeta_{\mathbf{k}^{\rm ph}}\rangle + \varepsilon^{(V)}_\alpha(\mathbf{k}^{\rm ph})\langle\zeta_{\mathbf{k}^{\rm ph}}|a_{\rm V}\zeta_{\mathbf{k}^{\rm ph}}\rangle\right]. \end{aligned} \tag{2.20}$$

This is of the form (2.17) with

$$f_1(\mathbf{k}^{\rm ph}) = \ell^{3/2}\langle\zeta_{\mathbf{k}^{\rm ph}}|a_{\rm H}\zeta_{\mathbf{k}^{\rm ph}}\rangle \quad \text{and} \quad f_2(\mathbf{k}^{\rm ph}) = \ell^{3/2}\langle\zeta_{\mathbf{k}^{\rm ph}}|a_{\rm V}\zeta_{\mathbf{k}^{\rm ph}}\rangle.$$

Operator-valued electric and magnetic fields are defined by

$$\begin{aligned} \hat{E}_\alpha &= -c\partial_0\hat{A}_\alpha, \\ \hat{B}_\alpha &= \sum_{\beta,\gamma}\epsilon_{\alpha,\beta,\gamma}\frac{\partial}{\partial\mathbf{x}_\beta}\hat{A}_\gamma. \end{aligned}$$

Gauss's law in absence of charges is satisfied. Indeed, the divergence of the electric field operators vanishes, as follows from

$$\begin{aligned}\sum_\alpha \partial_\alpha E_{\alpha,\mathbf{k}^{\rm ph}} &= \frac{1}{2N_0(\mathbf{k}^{\rm ph})}\lambda c|\mathbf{k}^{\rm ph}|\left(\sum_\alpha k^{\rm ph}_\alpha \varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph})\right)\left[e^{-ik^{\rm ph}_\mu x^\mu}a_{\rm H}+e^{ik^{\rm ph}_\mu x^\mu}a^\dagger_{\rm H}\right]\\ &\quad +\frac{1}{2N_0(\mathbf{k}^{\rm ph})}\lambda c|\mathbf{k}^{\rm ph}|\left(\sum_\alpha k^{\rm ph}_\alpha \varepsilon^{(V)}_\alpha(\mathbf{k}^{\rm ph})\right)\left[e^{-ik^{\rm ph}_\mu x^\mu}a_{\rm V}+e^{ik^{\rm ph}_\mu x^\mu}a^\dagger_{\rm V}\right]\\ &= 0, \end{aligned}\tag{2.21}$$

because

$$\sum_\alpha k^{\rm ph}_\alpha \varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph}) = \left(\Xi(\mathbf{k}^{\rm ph})\mathbf{k}^{\rm ph}\right)_1 = |\mathbf{k}^{\rm ph}|(\mathbf{e}_3)_1 \tag{2.22}$$

vanishes, as well as a similar expression for the vertical polarization.

Finally let us calculate the commutation relations

$$\begin{aligned}\left[A_{\alpha,\mathbf{k}^{\rm ph}}(x), A_{\beta,\mathbf{k}^{\rm ph}}(y)\right]_- &= -\frac{i}{(2\pi)^3 4|\mathbf{k}^{\rm ph}|\ell}\lambda^2\sin(k^{\rm ph}_\mu(x-y)^\mu)\\ &\quad\times\left(\varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph})\varepsilon^{(H)}_\beta(\mathbf{k}^{\rm ph})+\varepsilon^{(V}_\alpha(\mathbf{k}^{\rm ph})\varepsilon^{(V)}_\beta(\mathbf{k}^{\rm ph})\right).\end{aligned}$$

These commutation relations differ from the standard ones in the first place because the integration over the $\mathbf{k}^{\rm ph}$ vector, found in the standard theory, is missing.

2.2.3 Coherent fields

A pair of complex numbers z, w determines a coherent state of the two-dimensional quantum harmonic oscillator. It satisfies $a_{\rm H}|z,w\rangle^{\rm c} = z|z,w\rangle^{\rm c}$ and $a_{\rm V}|z,w\rangle^{\rm c} = w|z,w\rangle^{\rm c}$. Given two complex continuous functions $f_1(\mathbf{k}^{\rm ph})$, $f_2(\mathbf{k}^{\rm ph})$ a properly normalized field ζ in Γ is defined by

$$\zeta_{\mathbf{k}^{\rm ph}} = |F_1(\mathbf{k}^{\rm ph}), F_2(\mathbf{k}^{\rm ph})\rangle^{\rm c}, \qquad \mathbf{k}^{\rm ph}\in\mathbb{R}^3_{\rm o},$$

with $F_i(\mathbf{k}^{\rm ph}) = \ell^{-3/2}f_i(\mathbf{k}^{\rm ph})$, $i=1,2$. This field ζ describes a coherent electromagnetic field. It belongs to the domain of the free Hamiltonian $\hat{H}^{\rm ph}$. The quantum expectation of the latter equals

$$\begin{aligned}\mathcal{E}^{\rm qu} &= \ell^3\int {\rm d}\mathbf{k}\,(\zeta,\hat{H}\zeta)_{\mathbf{k}}\\ &= \int {\rm d}\mathbf{k}\,\hbar c|\mathbf{k}^{\rm ph}|\left(|f_1(\mathbf{k}^{\rm ph})|^2+|f_2(\mathbf{k}^{\rm ph})|^2\right)\leq+\infty.\end{aligned}$$

The interpretation is obvious: $|f_1(\mathbf{k}^{\rm ph})|^2$ and $|f_2(\mathbf{k}^{\rm ph})|^2$ are the expected densities for horizontally, respectively vertically poarized photons with kinetic energy $\hbar c|\mathbf{k}^{\rm ph}|$.

2.2.4 Single photon states

An important example of an incoherent field is the electromagnetic field produced by a single photon. In the present formalism this field requires a wave vector-dependent superposition of the single photon wave function, say $|1,0\rangle$ for a horizontally polarized photon, with the ground state $|0,0\rangle$ of the two-dimensional harmonic oscillator. This superposition can be written as

$$\zeta_{\mathbf{k}^{\rm ph}} = \sqrt{\rho(\mathbf{k}^{\rm ph})}e^{i\phi(\mathbf{k}^{\rm ph})}|1,0\rangle + \sqrt{1-\rho(\mathbf{k}^{\rm ph})}|0,0\rangle.$$

The energy of the electromagnetic wave equals

$$\mathcal{E}^{\rm qu} = \hbar c \int \mathrm{d}\mathbf{k}^{\rm ph}\, |\mathbf{k}^{\rm ph}|\rho(\mathbf{k}^{\rm ph}).$$

The wave vector distribution $|\mathbf{k}^{\rm ph}|\rho(\mathbf{k}^{\rm ph})$ must be integrable to keep the total energy finite. In particular, $\rho(\mathbf{k}^{\rm ph})$ cannot be taken constant. Therefore, the superposition of the one-photon wave function and the ground state wave function is a necessity.

The quantum expectation of the vector potential evaluates to

$$A^{\rm cl}_\alpha(x) = \lambda \int \mathrm{d}\mathbf{k}^{\rm ph}\, \frac{\ell^{5/2}}{N_0(\mathbf{k}^{\rm ph})} \sqrt{\rho(\mathbf{k}^{\rm ph})(1-\rho(\mathbf{k}^{\rm ph}))}\varepsilon^{(H)}_\alpha(\mathbf{k}^{\rm ph})\,\mathrm{Re}\, e^{i\phi(\mathbf{k}^{\rm ph})}e^{-ik^{\rm ph}_\mu x^\mu}.$$

Note that the contribution to the classical electromagnetic field comes from the region where the overlap with the ground state is neither 0 nor 1.

The one-photon field discussed above is linearly polarized. An example of circularly polarized one-photon field is obtained by choosing

$$\zeta_{\mathbf{k}^{\rm ph}} = \sqrt{\rho(\mathbf{k}^{\rm ph})}e^{i\phi(\mathbf{k}^{\rm ph})}\frac{1}{\sqrt{2}}(|1,0\rangle \pm i|0,1\rangle) + \sqrt{1-\rho(\mathbf{k}^{\rm ph})}|0,0\rangle.$$

The spin is given by $S_2 = \pm\ell^3 \int \mathrm{d}\mathbf{k}^{\rm ph}\, \rho(\mathbf{k}^{\rm ph})$. Note that $\frac{1}{\sqrt{2}}(|1,0\rangle \pm i|0,1\rangle)$ is a wave function of the 2-dimensional harmonic oscillator, just like $|1,0\rangle$ or $|0,1\rangle$. Both linearly and circularly polarized one-photon fields exist in the present theory.

2.2.5 SU(2) gauge symmetry

The Hamiltonian (2.18) of the two-dimensional quantum harmonic oscillator is invariant for certain unitary transformations $U^{\rm ph}$ of the Hilbert space $\mathcal{H}$, constructed as follows. Fix a unitary 2-by-2 matrix

$$\Pi = \begin{pmatrix} \Pi_{\rm HH} & \Pi_{\rm HV} \\ \Pi_{\rm VH} & \Pi_{\rm VV} \end{pmatrix}.$$

If Π belongs to $SU(2)$ then the operators

$$\begin{aligned} b_{\mathrm{H}} &= \Pi_{\mathrm{HH}} a_{\mathrm{H}} + \Pi_{\mathrm{HV}} a_{\mathrm{V}}, \\ b_{\mathrm{V}} &= \Pi_{\mathrm{VH}} a_{\mathrm{H}} + \Pi_{\mathrm{VV}} a_{\mathrm{V}}. \end{aligned}$$

and their conjugates $b^\dagger_{\mathrm{H}}$, $b^\dagger_{\mathrm{V}}$ satisfy the same canonical commutation relations as the original creation and annihilation operators. They define a unitary operator $U^{\mathrm{ph}}(\Pi)$ by linear extension of

$$|m,n\rangle = \frac{1}{\sqrt{m!n!}}(a^\dagger_{\mathrm{H}})^m (a^\dagger_{\mathrm{V}})^n |\emptyset\rangle \mapsto U^{\mathrm{ph}}(\Pi)|m,n\rangle = \frac{1}{\sqrt{m!n!}}(b^\dagger_{\mathrm{H}})^m (b^\dagger_{\mathrm{V}})^n |\emptyset\rangle.$$

It satisfies

$$U^{\mathrm{ph}}(\Pi) a_{\mathrm{H}} U^{\mathrm{ph}\,-1}(\Pi) = b_{\mathrm{H}} \quad \text{and} \quad U^{\mathrm{ph}}(\Pi) a_{\mathrm{V}} U^{\mathrm{ph}\,-1}(\Pi) = b_{\mathrm{V}}.$$

In particular, it commutes with the Hamiltonian (2.18)

$$\begin{aligned} U^{\mathrm{ph}}(\Pi) H^{\mathrm{ph}}_{\mathbf{k}^{\mathrm{ph}}} U^{\mathrm{ph}\,-1}(\Pi) &= \hbar c |\mathbf{k}^{\mathrm{ph}}| \left(b^\dagger_{\mathrm{H}} b_{\mathrm{H}} + b^\dagger_{\mathrm{V}} b_{\mathrm{V}} \right) \\ &= \hbar c |\mathbf{k}^{\mathrm{ph}}| \Big(\left[\overline{\Pi_{\mathrm{HH}}} a^\dagger_{\mathrm{H}} + \overline{\Pi_{\mathrm{HV}}} a^\dagger_{\mathrm{V}} \right] \left[\Pi_{\mathrm{HH}} a_{\mathrm{H}} + \Pi_{\mathrm{HV}} a_{\mathrm{V}} \right] \\ &\qquad + \left[\overline{\Pi_{\mathrm{VH}}} a^\dagger_{\mathrm{H}} + \overline{\Pi_{\mathrm{VV}}} a^\dagger_{\mathrm{V}} \right] \left[\Pi_{\mathrm{VH}} a_{\mathrm{H}} + \Pi_{\mathrm{VV}} a_{\mathrm{V}} \right] \Big) \\ &= \hbar c |\mathbf{k}^{\mathrm{ph}}| \left(a^\dagger_{\mathrm{H}} a_{\mathrm{H}} + a^\dagger_{\mathrm{V}} a_{\mathrm{V}} \right) \\ &= H^{\mathrm{ph}}_{\mathbf{k}^{\mathrm{ph}}}. \end{aligned}$$

The effect of the unitary transformation on the field operators is

$$\begin{aligned} & U^{\mathrm{ph}}(\Pi) A_{\alpha,\mathbf{k}^{\mathrm{ph}}}(x) U^{\mathrm{ph}\,-1}(\Pi) \\ = & \frac{\lambda}{2N_0(\mathbf{k}^{\mathrm{ph}})} \varepsilon^{(H)}_\alpha(\mathbf{k}^{\mathrm{ph}}) \left[e^{-ik^{\mathrm{ph}}_\mu x^\mu} b_{\mathrm{H}} + e^{ik^{\mathrm{ph}}_\mu x^\mu} b^\dagger_{\mathrm{H}} \right] \\ & + \frac{\lambda}{2N_0(\mathbf{k}^{\mathrm{ph}})} \varepsilon^{(V)}_\alpha(\mathbf{k}^{\mathrm{ph}}) \left[e^{-ik^{\mathrm{ph}}_\mu x^\mu} b_{\mathrm{V}} + e^{ik^{\mathrm{ph}}_\mu x^\mu} b^\dagger_{\mathrm{V}} \right] \\ = & \frac{\lambda}{2N_0(\mathbf{k}^{\mathrm{ph}})} \left[\varepsilon^{(H)}_\alpha(\mathbf{k}^{\mathrm{ph}}) \Pi_{\mathrm{HH}} + \varepsilon^{(V)}_\alpha(\mathbf{k}^{\mathrm{ph}}) \Pi_{\mathrm{VH}} \right] e^{-ik^{\mathrm{ph}}_\mu x^\mu} a_{\mathrm{H}} \\ + & \frac{\lambda}{2N_0(\mathbf{k}^{\mathrm{ph}})} \left[\varepsilon^{(H)}_\alpha(\mathbf{k}^{\mathrm{ph}}) \Pi_{\mathrm{HV}} + \varepsilon^{(V)}_\alpha(\mathbf{k}^{\mathrm{ph}}) \Pi_{\mathrm{VV}} \right] e^{ik^{\mathrm{ph}}_\mu x^\mu} a_{\mathrm{V}} \\ + & \ \text{h.c.} \end{aligned}$$

At this point it becomes clear that the field operators, as defined by (2.19), are not in their most general form. The unitary transformation $U^{\mathrm{ph}}(\Pi)$ is a gauge transformation not affecting the physical content of the theory. This gauge freedom has been removed by choosing a specific expression for the field operators $A_{\alpha,\mathbf{k}^{\rm ph}}(x)$, which is favoring orthogonally polarized waves.

Let us continue under the assumption that the matrix Π is an orthogonal matrix, i.e. it is a 2-dimensional rotation

$$\Pi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

The effect of the transformation $U^{\rm ph}(\Pi)$ is a rotation of the polarization vectors $\varepsilon_\alpha^{(H)}(\mathbf{k}^{\rm ph})$ and $\varepsilon_\alpha^{(V)}(\mathbf{k}^{\rm ph})$. It is straightforward to verify that, in vector notation,

$$U^{\rm ph}(-\phi)A_{\mathbf{k}^{\rm ph}}(x)U^{\rm ph}(\phi) = \Xi^{\rm T}(\mathbf{k}^{\rm ph})M(\phi)\Xi(\mathbf{k}^{\rm ph})A_{\mathbf{k}^{\rm ph}}(x), \tag{2.23}$$

with the matrix $M(\phi)$ given by

$$M(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed, use

$$\begin{aligned} \sum_\alpha \Xi^\dagger_{\alpha,1}A_{\alpha,\mathbf{k}^{\rm ph}}(x) &= \frac{\lambda}{2N_0(\mathbf{k}^{\rm ph})}\left[e^{-ik^{\rm ph}_\mu x^\mu}a_{\rm H} + e^{ik^{\rm ph}_\mu x^\mu}a^\dagger_{\rm H}\right], \\ \sum_\alpha \Xi^\dagger_{\alpha,2}A_{\alpha,\mathbf{k}^{\rm ph}}(x) &= \frac{\lambda}{2N_0(\mathbf{k}^{\rm ph})}\left[e^{-ik^{\rm ph}_\mu x^\mu}a_{\rm V} + e^{ik^{\rm ph}_\mu x^\mu}a^\dagger_{\rm V}\right], \end{aligned}$$

to obtain

$$\begin{aligned} & U^{\rm ph}(\Pi)A_{\alpha,\mathbf{k}^{\rm ph}}(x)U^{\rm ph\,-1}(\Pi) \\ = & \sum_\beta\Big\{\left[\Xi_{1,\alpha}(\mathbf{k}^{\rm ph})\cos(\phi) + \Xi_{2,\alpha}(\mathbf{k}^{\rm ph})\sin(\phi)\right]\Xi^\dagger_{\beta,1}(\mathbf{k}^{\rm ph}) \\ + & \left[-\Xi_{1,\alpha}(\mathbf{k}^{\rm ph})\sin(\phi) + \Xi_{2,\alpha}(\mathbf{k}^{\rm ph})\cos(\phi)\right]\Xi^\dagger_{\beta,2}(\mathbf{k}^{\rm ph})\Big\}A_{\beta,\mathbf{k}^{\rm ph}}(x) \\ = & \sum_\beta[\Xi^{\rm T}(\mathbf{k}^{\rm ph})M(\phi)\Xi(\mathbf{k}^{\rm ph})]_{\alpha,\beta}A_{\beta,\mathbf{k}^{\rm ph}}(x). \end{aligned}$$

To obtain the last line $\sum_\beta \Xi_{3,\beta}A_{\beta,\mathbf{k}^{\rm ph}}(x) = 0$ is used.

Finally consider the action of $U^{\rm ph}(\Pi)$ on coherent fields. From

$$|z,w\rangle^{\rm c} = e^{za^\dagger_{\rm H}-\overline{z}a_{\rm H}}e^{wa^\dagger_{\rm V}-\overline{w}a_{\rm V}}|0,0\rangle$$

follows

$$
\begin{aligned}
& U^{\mathrm{ph}}(\Pi)|z,w\rangle^{\mathrm{c}} \\
= \;& e^{zb_{\mathrm{H}}^{\dagger}-\overline{z}b_{\mathrm{H}}}e^{wb_{\mathrm{V}}^{\dagger}-\overline{w}b_{\mathrm{V}}}|0,0\rangle \\
= \;& \exp\big(z(\Pi_{\mathrm{HH}}a_{\mathrm{H}}+\Pi_{\mathrm{HV}}a_{\mathrm{V}})^{\dagger}-\overline{z}(\Pi_{\mathrm{HH}}a_{\mathrm{H}}+\Pi_{\mathrm{HV}}a_{\mathrm{V}})\big) \\
& \times\exp\big(w(\Pi_{\mathrm{VH}}a_{\mathrm{H}}+\Pi_{\mathrm{VV}}a_{\mathrm{V}})^{\dagger}-\overline{w}(\Pi_{\mathrm{VH}}a_{\mathrm{H}}+\Pi_{\mathrm{VV}}a_{\mathrm{V}})\big) \\
= \;& \exp\left(\frac{1}{2}\Big[z(\Pi_{\mathrm{HH}}a_{\mathrm{H}}+\Pi_{\mathrm{HV}}a_{\mathrm{V}})^{\dagger}-\overline{z}(\Pi_{\mathrm{HH}}a_{\mathrm{H}}+\Pi_{\mathrm{HV}}a_{\mathrm{V}}),\right. \\
& \qquad \left. w(\Pi_{\mathrm{VH}}a_{\mathrm{H}}+\Pi_{\mathrm{VV}}a_{\mathrm{V}})^{\dagger}-\overline{w}(\Pi_{\mathrm{VH}}a_{\mathrm{H}}+\Pi_{\mathrm{VV}}a_{\mathrm{V}})\Big]_{-}\right) \\
& \times\exp\Big(z(\Pi_{\mathrm{HH}}a_{\mathrm{H}}+\Pi_{\mathrm{HV}}a_{\mathrm{V}})^{\dagger}-\overline{z}(\Pi_{\mathrm{HH}}a_{\mathrm{H}}+\Pi_{\mathrm{HV}}a_{\mathrm{V}}) \\
& \quad +w(\Pi_{\mathrm{VH}}a_{\mathrm{H}}+\Pi_{\mathrm{VV}}a_{\mathrm{V}})^{\dagger}-\overline{w}(\Pi_{\mathrm{VH}}a_{\mathrm{H}}+\Pi_{\mathrm{VV}}a_{\mathrm{V}})\Big).
\end{aligned}
$$

The commutator in the first exponential vanishes because Π belongs to $\mathrm{SU}(2)$. The result is

$$
U^{\mathrm{ph}}(\Pi)|z,w\rangle^{\mathrm{c}} \;=\; |z',w'\rangle^{\mathrm{c}}
$$

with

$$
\begin{pmatrix} z' \\ w' \end{pmatrix} \;=\; \Pi^{\dagger}\begin{pmatrix} z \\ w \end{pmatrix}.
$$

This shows that the set of coherent fields is invariant under the unitary transformations $U^{\mathrm{ph}}(\Pi)$, where Π is any element of $\mathrm{SU}(2)$.

Chapter 3

Fermionic Fields

3.1 Scalar fermions

3.1.1 The Klein-Gordon equation

This section concerns the quantum field description of fermions with a rest mass $m > 0$. The appropriate wave equation is the Klein-Gordon equation

$$(\Box + \kappa^2)\phi(x) = 0 \quad \text{with } \kappa = \frac{mc}{\hbar}. \tag{3.1}$$

For $m = 0$ it reduces to the d'Alembert equation $\Box\phi = 0$, discussed in Section 2.1.5. Propagating wave solutions are of the same form as in Section 2.1.5

$$\phi(x) \quad = \quad 2\,\mathrm{Re}\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\,\frac{\ell^{3/2}}{N_\kappa(\mathbf{k})} f(\mathbf{k}) e^{-ik_\mu x^\mu}, \tag{3.2}$$

but with a dispersion relation given by the positive square root

$$k^0 = \omega(\mathbf{k})/c \quad \text{with} \quad \omega(\mathbf{k}) = c\sqrt{\kappa^2 + |\mathbf{k}|^2},$$

and a corresponding normalization

$$N_\kappa(\mathbf{k}) = \sqrt{(2\pi)^3 2\ell\omega(\mathbf{k})/c}.$$

The constant ℓ is inserted in (3.2) for dimensional reasons. It makes $|f(\mathbf{k})|^2$ into a density.

3.1.2 Larmor precession

We use the harmonic oscillator in the description of bosons because it exhibits periodic motion. An alternative model exhibiting periodic motion is that of

Larmor precession. It involves the Pauli matrices σ_α, $\alpha = 1, 2, 3$. The time evolution is

$$\begin{aligned} \sigma_1(t) &= \sigma_1 \cos(\omega t) + \sigma_2 \sin(\omega t), && (3.3)\\ \sigma_2(t) &= \sigma_2 \cos(\omega t) - \sigma_1 \sin(\omega t), && (3.4)\\ \sigma_3(t) &= \sigma_3. \end{aligned}$$

The Hamiltonian reads

$$H = -\frac{1}{2}\hbar\omega\sigma_3. \tag{3.5}$$

Note that

$$\sigma_\pm(t) = \sigma_\pm e^{\mp i\omega t},$$

where

$$\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2).$$

A wave function $|z\rangle$ describing the state of the system consists of two complex numbers z_+, z_- which satisfy the normalization condition $|z_+|^2 + |z_-|^2 = 1$. The quantum expectation of the Hamiltonian (3.5) equals

$$\begin{aligned} \langle z|\hat{H}|z\rangle &= -\frac{1}{2}\hbar\omega(\mathbf{k})(\overline{z_1} \quad \overline{z_2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= -\frac{1}{2}\hbar\omega(|z_1|^2 - |z_2|^2). \end{aligned}$$

The quantum expectation of the matrices $\hat{\sigma}_\pm(t)$ is given by

$$\langle z|\hat{\sigma}_+(t)|z\rangle = \frac{1}{2}\overline{z_1}z_2 e^{-i\omega t} \quad \text{and} \quad \langle z|\hat{\sigma}_-(t)|z\rangle = \frac{1}{2}\overline{z_2}z_1 e^{i\omega t}.$$

3.1.3 Fermionic state space

Let Γ_2 denote the linear space of continuous fields $\zeta : \mathbf{k} \in \mathbb{R}^3 \mapsto \zeta_\mathbf{k} \in \mathbb{C}^2$. Like in the case of bosonic fields it is a locally convex Hausdorff space. However, because the Hilbert space $\mathbb{C}^2$ is finite-dimensional it is also a Banach space. An element ζ of Γ_2 is said to be properly normalized if $||\zeta_\mathbf{k}|| = 1$ for all $\mathbf{k}$. States of the fermionic quantum field theory are represented by properly normalized fields.

Note that any properly normalize field ζ of Γ_2 can be written into the form

$$\zeta_\mathbf{k} = \begin{pmatrix} \sqrt{1-\rho(\mathbf{k})}e^{i\chi(\mathbf{k})} \\ \sqrt{\rho(\mathbf{k})}e^{i\xi(\mathbf{k})} \end{pmatrix}, \tag{3.6}$$

where ρ, χ and ξ are real-valued functions of $\mathbf{k} \in \mathbb{R}^3$. By adopting this way of writing one tacitly assumes that $(1,0)^{\mathrm{T}}$ means absence of the fermion, while $(0,1)^{\mathrm{T}}$ means presence of the fermion. Note the analogy with single photon states as described in Section 2.2.4. With this interpretation $\rho(\mathbf{k})$ becomes the density of the fermion field with wave vector $\mathbf{k}$.

The Hamiltonian (3.5) of Larmor precession defines a diagonal operator $\hat{H}^{\mathrm{el}}$ by $[\hat{H}^{\mathrm{el}}\zeta]_{\mathbf{k}} = H^{\mathrm{el}}_{\mathbf{k}}\zeta_{\mathbf{k}}$ with

$$H^{\mathrm{el}}_{\mathbf{k}} = \frac{1}{2}\hbar\omega(\mathbf{k})(\mathbb{I} - \sigma_3). \tag{3.7}$$

A constant matrix has been added to make the Hamiltonian non-negative. This does not change the dynamics of the Larmor precession. The domain of definition of $\hat{H}^{\mathrm{el}}$ is all of Γ_2.

With the help of (3.6) the quantum expectation of the Hamiltonian becomes

$$\begin{aligned} \langle \hat{H}^{\mathrm{el}} \rangle &= \ell^3 \int \mathrm{d}\mathbf{k}\, \left(\zeta, \hat{H}\zeta\right)_{\mathbf{k}} \\ &= \ell^3 \int \mathrm{d}\mathbf{k}\, \hbar\omega(\mathbf{k})\rho(\mathbf{k}). \end{aligned} \tag{3.8}$$

This reveals that $\rho(\mathbf{k})$ is a distribution of quantum particles with dispersion relation $\omega(\mathbf{k})$. It is restricted by the condition that $0 \le \rho(\mathbf{k}) \le 1$ for all $\mathbf{k}$. Because the energy must remain finite the distribution $\rho(\mathbf{k})$ should go to 0 fast enough for large values of the wave vector $|\mathbf{k}|$.

3.1.4 Field operator

Introduce now the field operator $\hat{\phi}(x)$ defined by $[\hat{\phi}(x)\zeta]_{\mathbf{k}} = \phi_{\mathbf{k}}\zeta_{\mathbf{k}}$ with

$$\phi_{\mathbf{k}}(x) = \frac{1}{N_\kappa(\mathbf{k})}\left[\sigma_+(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \sigma_-(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{3.9}$$

It is tradition to decompose this field operator into so-called positive-frequency and negative-frequency operators

$$\begin{aligned} \hat{\phi}(x) &= \hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x), \qquad \text{with} \\ \phi^{(+)}_{\mathbf{k}}(x) &= \frac{1}{N_\kappa(\mathbf{k})}\sigma_+(t)e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{N_\kappa(\mathbf{k})}\sigma_+ e^{-ik_\nu x^\nu}, \\ \phi^{(-)}_{\mathbf{k}}(x) &= \frac{1}{N_\kappa(\mathbf{k})}\sigma_-(t)e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{N_\kappa(\mathbf{k})}\sigma_- e^{ik_\nu x^\nu}. \end{aligned}$$

They satisfy the anti-commutation relations

$$\hat{\phi}^{(+)}(x)\hat{\phi}^{(+)}(y) = 0,$$

$$\{\hat{\phi}^{(+)}(x), \hat{\phi}^{(-)}(y)\}_+ = \left(\mathbf{k} \mapsto \frac{c}{2(2\pi)^3\ell\omega(\mathbf{k})} e^{-ik_\mu(x-y)^\mu}\right). \tag{3.10}$$

These anti-commutation relations are non-canonical. Note that

$$\hat{\phi}^{(-)}(x) = \left(\hat{\phi}^{(+)}(x)\right)^\dagger.$$

The classical field $\phi^{\rm cl}(x)$ corresponding with the field operator $\hat{\phi}(x)$ equals

$$\begin{aligned}
\phi^{\rm cl}(x) &= \ell^3 \int \mathrm{d}\mathbf{k} \left(\zeta, \hat{\phi}(x)\zeta\right)_\mathbf{k} \\
&= \ell^3 \int \mathrm{d}\mathbf{k} \, \langle\zeta_\mathbf{k}| \frac{1}{N(k)} \left[\sigma_+ e^{-ik_\mu x^\nu} + \sigma_- e^{ik_\mu x^\nu}\right] \zeta_\mathbf{k}\rangle \\
&= \int \mathrm{d}\mathbf{k} \, \frac{\ell^3}{N_\kappa(\mathbf{k})} \sqrt{\rho(\mathbf{k})(1-\rho(\mathbf{k}))} 2 \,\mathrm{Re}\, e^{-i(\chi(\mathbf{k})-\xi(\mathbf{k}))} e^{-ik_\mu x^\mu}. \qquad (3.11)
\end{aligned}$$

The integral converges provided that the density $\rho(\mathbf{k})$ tends fast enough either to 0 or to 1 for large values of $|\mathbf{k}|$. The expression (3.11) is of the form (3.2) with

$$f(\mathbf{k}) = \ell^{3/2} \sqrt{\rho(\mathbf{k})(1-\rho(\mathbf{k}))} e^{-i(\chi(\mathbf{k})-\xi(\mathbf{k}))}.$$

3.2 The free Dirac equation

3.2.1 The algebra of creation and annihilation operators

The electron wave is fermionic. It has two polarizations, which are related to the spin of the electron. In addition, the electron has an anti-particle, which is the positron. This means that the electron field has 4 internal degrees of freedom and that we need 4 copies of the spin matrices $\sigma_\pm$ instead of the single copy introduced in Section 3.1.2. The corresponding matrices are denoted $\sigma_s^{(\pm)}$, with the index s running from 1 to 4. They satisfy the anti-commutation relations

$$\begin{aligned}
\left\{\sigma_s^{(+)}, \sigma_t^{(+)}\right\}_+ &= 0, \\
\left\{\sigma_s^{(+)}, \sigma_t^{(-)}\right\}_+ &= \delta_{s,t}.
\end{aligned}$$

The hermitian conjugate of $\sigma_s^{(+)}$ is $\sigma_s^{(-)}$. Together they generate an algebra known as a Clifford algebra. An explicit representation of the operators as

16-by-16 matrices is easily constructed (see for instance Section 3-9 of [7]). However, it is not needed in the sequel.

Basis vectors of the 16-dimensional Hilbert space $\mathcal{H}_{16}$ are specified by subsets $\Lambda \subset \{1,2,3,4\}$ and are given by

$$|\Lambda\rangle = [\sigma_4^{(-)}]^{\mathbb{I}_{4\in\Lambda}}[\sigma_3^{(-)}]^{\mathbb{I}_{3\in\Lambda}}[\sigma_2^{(-)}]^{\mathbb{I}_{2\in\Lambda}}[\sigma_1^{(-)}]^{\mathbb{I}_{1\in\Lambda}}|\emptyset\rangle.$$

For instance, if $\Lambda = \{1,3\}$ then $|\{1,3\}\rangle = \sigma_3^{(-)}\sigma_1^{(-)}|\emptyset\rangle$.

The field operators $\hat{\phi}_s(x)$ are diagonal operators on Γ_2 defined by matrices $\phi_{s,\mathbf{k}}(x)$. The latter can be written in the form (3.9). They satisfy the anti-commutation relations

$$\begin{aligned} \{\phi_{s,\mathbf{k}}^{(+)}(x), \phi_{t,\mathbf{k}'}^{(+)}(y)\}_+ &= 0 \quad \text{and} \\ \{\phi_{s,\mathbf{k}}^{(+)}(x), \phi_{t,\mathbf{k}'}^{(-)}(y)\}_+ &= \frac{c}{2(2\pi)^3\ell\omega(\mathbf{k})}\delta_{s,t}e^{-ik_\mu x^\mu}e^{ik'_\mu y^\mu}. \end{aligned} \tag{3.12}$$

Note that (3.9) implies that

$$\phi_{s,\mathbf{k}}^{(+)}(x) = \frac{1}{N_\kappa(\mathbf{k})}e^{-ik_\mu x^\mu}\sigma_s^{(+)}.$$

A familiar notation for these operators, evaluated at $x = 0$, is

$$b_\uparrow = \sigma_1^{(+)}, \quad b_\downarrow = \sigma_2^{(+)}, \quad d_\downarrow = \sigma_3^{(+)}, \quad d_\uparrow = \sigma_4^{(+)}.$$

This alternative notation is not used here.

3.2.2 The Hamiltonian

The Hamiltonian of the electron field $\hat{H}^{\mathrm{el}}$ is the sum of 4 copies of the scalar Hamiltonian (3.7). It is defined by $[\hat{H}^{\mathrm{el}}\zeta]_{\mathbf{k}} = H^{\mathrm{el}}_{\mathbf{k}}\zeta_{\mathbf{k}}$ with

$$H^{\mathrm{el}}_{\mathbf{k}} = \frac{1}{2}\hbar\omega(\mathbf{k})\sum_{s=1}^{4}(\mathbb{I} - \sigma_s^3). \tag{3.13}$$

Note that the Hamiltonian is positive. It is tradition to assign negative energies to positrons and positive energies to electrons. This tradition is not followed here because it does not make sense. It is a remainder of Dirac's interpretation of positrons as holes in a sea of electrons. The alternative treatment assigns the vacuum state to one of the eigenstates of σ_3 instead of assigning a particle/anti-particle pair to the two eigenstates. The dimension of the Hilbert space goes up from 4 (the number of components of a Dirac spinor) to 16. This is meaningful because the Dirac equation considered here is an equation for field operators and not the original one which holds for classical field spinors (see (3.26) below).

Number operators N_s are defined by

$$N_s = \sigma_s^{(-)}\sigma_s^{(+)} \quad s = 1, 2, 3, 4. \tag{3.14}$$

They appear in the Hamiltonian

$$H_{\mathbf{k}}^{\mathrm{el}} = \hbar\omega(\mathbf{k}) \sum_{s=1}^{4} N_s. \tag{3.15}$$

The field operators $\hat{\phi}_s^{(+)}(x)$ satisfy Heisenberg's equations of motion

$$i\hbar c \partial_0 \hat{\phi}_s^{(+)}(x) = \left[\hat{\phi}_s^{(+)}(x), \hat{H}^{\mathrm{el}}\right]_- .$$

In principle, this is all that is needed for a description of electron/positron fields. However, in the next part of this work the notion of electric current is needed for a description of the interactions between the electromagnetic field and the electron/positron field. The actual expression which will be used is given by (3.38, 3.39) in the next section. The derivation of this result is far from trivial and follows the approach initiated by Dirac. Let us start by introducing Dirac's equation for quantum field operators.

3.2.3 Dirac's equation

Introduce the gamma matrices. In the standard representation they read

$$\gamma_0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \quad \text{and} \quad \gamma_\alpha = \begin{pmatrix} 0 & -\sigma_\alpha \\ \sigma_\alpha & 0 \end{pmatrix}.$$

Next introduce auxiliary field operators $\hat{\psi}_r$, $r = 1, 2, 3, 4$. They are called the *Dirac field operators* and are defined by [8]

$$\begin{aligned} \psi_{r,\mathbf{k}}(x) &= \sqrt{2\ell k_0} \left[\sum_{s=1,2} u_r^{(s)}(\mathbf{k}) \phi_{s,\mathbf{k}}^{(+)}(x) + \sum_{t=3,4} v_r^{(t)}(\mathbf{k}) \phi_{t,\mathbf{k}}^{(-)}(x) \right] \\ &= \frac{1}{\sqrt{(2\pi)^3}} \left[\sum_{s=1,2} u_r^{(s)}(\mathbf{k}) e^{-ik_\mu x^\mu} \sigma_s^{(+)} + \sum_{t=3,4} v_r^{(t)}(\mathbf{k}) e^{ik_\mu x^\mu} \sigma_t^{(-)} \right]. \end{aligned} \tag{3.16}$$

The vectors $u^{(1)}, u^{(2)}, v^{(3)}, v^{(4)}$ are the analogues of the polarization vectors of the photon. They are partly fixed by the requirement that the vector with components $\hat{\psi}_r$ satisfies Dirac's equation

$$i\gamma^\mu \partial_\mu \hat{\psi}(x) = \kappa \hat{\psi}(x). \tag{3.17}$$

Indeed, using

$$\partial_\mu \phi^{(\pm)}_{s,\mathbf{k}} = \mp i k_\mu \phi^{(\pm)}_{s,\mathbf{k}}$$

one finds that a sufficient condition for (3.17) to hold is

$$\gamma^\mu k_\mu u^{(s)} = \kappa u^{(s)} \quad \text{and} \quad \gamma^\mu k_\mu v^{(t)} = -\kappa v^{(t)}.$$

Each of these two equations has two independent solutions. See Appendix B. They can be chosen to satisfy the orthogonality relations

$$\begin{aligned} \sum_r \overline{u^{(s)}_r}(\mathbf{k}) u^{(s')}_r(\mathbf{k}) &= \delta_{s,s'}, \\ \sum_r \overline{v^{(t)}_r}(\mathbf{k}) v^{(t')}_r(\mathbf{k}) &= \delta_{t,t'}, \\ \sum_r \overline{u^{(s)}_r}(\mathbf{k}) v^{(t)}_r(-\mathbf{k}) &= 0. \end{aligned} \tag{3.18}$$

Let $T(\mathbf{k})$ be the matrix which maps the orthonormal bazis vectors $u^{(s)}(0)$ and $v^{(t)}(0)$, defined at wave vector $\mathbf{k} = 0$, onto their values at arbitrary k

$$\begin{aligned} T(\mathbf{k}) u^{(s)}(\mathbf{k} = 0) &= u^{(s)}(\mathbf{k}), \quad s = 1, 2, \\ T(\mathbf{k}) v^{(t)}(\mathbf{k} = 0) &= v^{(t)}(\mathbf{k}), \quad t = 3, 4. \end{aligned} \tag{3.19}$$

Clearly is $T(0) = \text{id}$. It is shown in Appendix C that this matrix is hermitian and that its determinant equals $\det T(\mathbf{k}) = \kappa/k_0$. In particular, this implies that the matrix $T(\mathbf{k})$ is non-singular. A continuity argument then shows that the matrix is positive-definite. An explicit expression for this matrix is

$$T(\mathbf{k}) = \frac{2\kappa}{\sqrt{2k_0(k_0 + \kappa)}} (k_\nu \gamma^\nu \gamma^0 + \kappa). \tag{3.20}$$

For further use note the inverse relations

$$\sum_r \overline{u^{(s)}_r(-\mathbf{k})} \psi_{r,\mathbf{k}}(x) = \frac{\kappa}{k_0} \sqrt{2\ell k_0} \phi^{(+)}_{s,\mathbf{k}}(x), \qquad s = 1, 2, \tag{3.21}$$

$$\sum_r \overline{v^{(t)}_r(-\mathbf{k})} \psi_{r,\mathbf{k}}(x) = \frac{\kappa}{k_0} \sqrt{2\ell k_0} \phi^{(-)}_{t,\mathbf{k}}(x), \qquad t = 3, 4. \tag{3.22}$$

Some further properties are (see Appendix D)

$$\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(s')}(\mathbf{k})\rangle = \frac{k^\mu}{k_0}\delta_{s,s'}, \quad s,s'=1,2, \tag{3.23}$$

and

$$\langle v^{(t)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t')}(\mathbf{k})\rangle = \frac{k^\mu}{k_0}\delta_{t,t'}, \quad t,t'=3,4. \tag{3.24}$$

It is easy to see, using the anti-commutation relations, that

$$\{\psi_{r,\mathbf{k}}(x),\psi_{r',\mathbf{k}'}(y)\}_+ = 0$$

holds for any choice of parameters and indices. On the other hand,

$$\left\{\psi_{r,\mathbf{k}}(x),\psi^\dagger_{r',\mathbf{k}}(x)\right\}_+ = \frac{1}{(2\pi)^3}\delta_{r,r'}$$

holds only for equal positions and momenta. This relation can be written as

$$\left\{\hat\psi_r(x),\hat\psi^\dagger_{r'}(x)\right\}_+ = \frac{1}{(2\pi)^3}\delta_{r,r'}$$

An electron/positron field is now determined by a properly normalized field ζ of the form

$$\zeta_\mathbf{k} = \sum_{\Lambda\subset\{1,2,3,4\}\}} z^\Lambda_\mathbf{k}|\Lambda\rangle,$$

with complex coefficients $z^\Lambda_\mathbf{k}$ satisfying

$$\sum_{\Lambda\subset\{1,2,3,4\}} |z^\Lambda_\mathbf{k}|^2 = 1, \qquad \text{for all } \mathbf{k}.$$

It defines a Dirac spinor containing classical fields by

$$\phi^{\mathrm{cl}}{}_r(x) = \ell^3\int \mathrm{d}\mathbf{k}\,\langle\zeta_\mathbf{k}|\psi_{r,\mathbf{k}}\zeta_\mathbf{k}\rangle, \tag{3.25}$$

whenever the integral converges. This Dirac spinor $\phi^{\mathrm{cl}}(x)$ with 4 components satisfies the Dirac equation

$$i\gamma^\mu\partial_\mu\phi^{\mathrm{cl}}(x) = \kappa\phi^{\mathrm{cl}}(x). \tag{3.26}$$

Finally note that each of the Dirac field operators, as constructed here, is not only a solution of the Klein-Gordon equation

$$[\Box + \kappa]\psi_{r,\mathbf{k}} = 0 \tag{3.27}$$

but also of the partial equations

$$\begin{aligned}\left[c^2\partial_0^2 + [\hbar\omega(\mathbf{k})]^2\right]\psi_{r,\mathbf{k}}(x) &= 0,\\ \left[\Delta + |\mathbf{k}|^2\right]\psi_{r,\mathbf{k}}(x) &= 0.\end{aligned}$$

A Lorentz transformation can mix up these two equations. See Section 3.4 below for a further discussion of these matters.

3.2.4 The adjoint equation

The so-called adjoint spinor is defined by

$$\hat{\psi}^{\mathrm{a}}{}_r(x) = \sum_{r'} \hat{\psi}^\dagger_{r'}(x)\gamma^0_{r',r}.$$

It satisfies the adjoint equation

$$-i\partial_\mu \sum_r \hat{\psi}^{\mathrm{a}}{}_r(x)\gamma^\mu_{r,r'} = \kappa\hat{\psi}^{\mathrm{a}}{}_{r'}(x). \tag{3.28}$$

To prove this take the adjoint of the Dirac equation and multiply with γ^0 from the right. This gives

$$-i\partial_\mu \sum_{r,r'} \hat{\psi}^\dagger_{r'}(x)(\gamma^\mu)^\dagger_{r',r}\gamma^0_{r,r''} = \kappa\hat{\psi}^{\mathrm{a}}{}_{r''}(x).$$

Next use that $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$ to obtain (3.28).

3.2.5 Charge conjugation

The charge conjugation matrix C is defined by

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^{\mathrm{T}}.$$

Using the standard representation of the gamma matrices it equals $C = i\gamma^2\gamma^0$. See for instance Section 10.3.2 of [9]. In explicit form is

$$C = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The main properties of the matrix C are

- $C^{-1} = C^\dagger = C^{\mathrm{T}} = -C$;
- $\sum_{r'} C_{r,r'} \overline{u^{(1)}_{r'}(\mathbf{k})} = v^{(4)}_r(-\mathbf{k})$;
- $\sum_{r'} C_{r,r'} \overline{u^{(2)}_{r'}(\mathbf{k})} = v^{(3)}_r(-\mathbf{k})$.

The charge conjugation operator C_{c} is a linear operator on $\mathcal{H}_{16}$ with the properties that $C_{\mathrm{c}}^{-1} = C_{\mathrm{c}}^\dagger = -C_{\mathrm{c}}$ and

$$\begin{aligned} C_{\mathrm{c}} \psi_{r,\mathbf{k}}(x) C_{\mathrm{c}}^{-1} &= -\sum_{r'} C_{r,r'} \psi^{\mathrm{a}}_{r',\mathbf{k}}(x), \\ C_{\mathrm{c}} \psi^{\mathrm{a}}_{r,\mathbf{k}}(x) C_{\mathrm{c}}^{-1} &= \sum_{r'} C_{r,r'} \psi_{r',\mathbf{k}}(x). \end{aligned} \tag{3.29}$$

An explicit definition of the operator C_{c} is given in Appendix E

3.3 The Dirac current

3.3.1 Two-point correlations

Fix a properly normalized electron field ζ. A two-point correlation function for the Dirac field operators $\hat{\psi}_r(x)$ is defined by (compare with that introduced in Section 2.1.8)

$$G_{r',r}(x,x') = \ell^3 \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{k}' \, \langle \zeta_{\mathbf{k}} | \psi^{\mathrm{a}}_{r,\mathbf{k}}(x) \psi_{r',\mathbf{k}'}(x') \zeta_{\mathbf{k}'} \rangle, \tag{3.30}$$

whenever the integrals converge. Note the order of the indices r, r'. By use of Dirac's equation there follows

$$i\kappa G_{r',r}(x,x') = -\frac{\partial}{\partial x'^\mu} \sum_{r''} \gamma^\mu_{r',r''} G_{r'',r}(x,x'). \tag{3.31}$$

On the other hand, using the adjoint equation, one obtains

$$i\kappa G_{r',r}(x,x') = \frac{\partial}{\partial x^\mu} \sum_{r''} G_{r',r''}(x,x') \gamma^\mu_{r'',r}. \tag{3.32}$$

Subtracting one expression from the other yields

$$0 = \frac{\partial}{\partial x'^\mu} \sum_{r''} \gamma^\mu_{r',r''} G_{r'',r}(x,x') + \frac{\partial}{\partial x^\mu} \sum_{r''} G_{r',r''}(x,x') \gamma^\mu_{r'',r}.$$

Now take $r = r'$ and sum over r. There follows

$$0 = \frac{\partial}{\partial x'^\mu} \operatorname{Tr} \gamma^\mu G(x,x') + \frac{\partial}{\partial x^\mu} \operatorname{Tr} G(x,x')\gamma^\mu.$$

In particular, one has

$$0 = \frac{\partial}{\partial x^\mu} \operatorname{Tr} \gamma^\mu G(x,x). \tag{3.33}$$

This result shows that the vector $r(x)$ with 4 components

$$r^\mu(x) = \operatorname{Tr} \gamma^\mu G(x,x)$$

satisfies the continuity equation.

3.3.2 Properties of the particle current

The vector $r(x)$, introduced above, describes a current, which however is not yet the electric current. The latter is introduced in the next section. It is tempting to interpret $r(x)$ as the particle current. However, this interpretation has some difficulties. The integration over space of its zeroth component, which should be the total number of particles, is usually divergent. The latter is anyhow not a very interesting quantity once the interaction with the electromagnetic field is turned on because it is not conserved. An electron and a positron may annihilate each other or may be created by a pair of photons. When doing so the total number of electrons plus positrons is changed. On the other hand the total charge remains conserved in the presence of interactions. It is therefore the quantity of interest.

The components of $r(x)$ are real numbers. Indeed, using $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$ one verifies that

$$\begin{aligned}
\overline{r^\mu(x)} &= \overline{\operatorname{Tr} \gamma^\mu G(x,x)} \\
&= \operatorname{Tr} G^\dagger(x,x)\gamma^0\gamma^\mu\gamma^0 \\
&= \sum_{r,r'} \overline{\langle\zeta_{\mathbf{k}}|[\psi_{r,\mathbf{k}}(x)]^\dagger\psi_{r',\mathbf{k}'}(x)\zeta_{\mathbf{k}'}\rangle}\gamma^0_{r,r}[\gamma^0\gamma^\mu\gamma^0]_{r',r} \\
&= \sum_{r,r'} \ell^3 \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{k}' \, \langle\zeta_{\mathbf{k}'}|[\psi_{r',\mathbf{k}'}(x)]^\dagger\psi_{r,\mathbf{k}}(x)\zeta_{\mathbf{k}}\rangle[\gamma^0\gamma^\mu]_{r',r} \\
&= \sum_{r,r'} G_{r',r}(x,x)\gamma^\mu_{r,r'} \\
&= \operatorname{Tr} G(x,x)\gamma^\mu \\
&= r^\mu(x).
\end{aligned}$$

The current operator $\hat{R}(x)$ corresponding to the classical current $r(x)$ is given by the kernel

$$R^{\mu}_{\mathbf{k},\mathbf{k}'}(x) = \sum_{r,r'} \psi^{\mathrm{a}}_{r,\mathbf{k}}(x)\gamma^{\mu}_{r,r'}\psi_{r',\mathbf{k}'}(x).$$

Indeed, one can write

$$\begin{aligned} r^{\mu}(x) &= \ell^3 \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{k}' \, \langle \zeta_{\mathbf{k}} | R^{\mu}_{\mathbf{k},\mathbf{k}'}(x)\zeta_{\mathbf{k}'}\rangle \\ &= \ell^3 \int \mathrm{d}\mathbf{k} \, \langle \zeta_{\mathbf{k}} | [\hat{R}^{\mu}(x)\zeta]_{\mathbf{k}}\rangle \\ &= \langle \zeta | \hat{R}^{\mu}(x)\zeta\rangle. \end{aligned}$$

The zeroth component of the current operator $\hat{R}(x)$ is a density operator. Its kernel simplifies to

$$R^{0}_{\mathbf{k},\mathbf{k}'}(x) = \sum_{r} \left(\psi_{r,\mathbf{k}}(x)\right)^{\dagger} \psi_{r,\mathbf{k}'}(x). \tag{3.34}$$

This is a positive operator, in the sense that for any field ζ one has $r^0(x) \geq 0$. The integral over space is a constant of the motion. However, it turns out that the integral diverges for wave functions of interest.

For further use let us show that the kernels of the integral operators $\hat{R}^{\mu}(x)$ are symmetric, this is, they satisfy

$$\left(R^{\mu}_{\mathbf{k},\mathbf{k}'}(x)\right)^{\dagger} = R^{\mu}_{\mathbf{k}',\mathbf{k}}(x). \tag{3.35}$$

Indeed, one calculates, using $\gamma^0[\gamma^{\mu}]^{\dagger} = \gamma^{\mu}\gamma^0$,

$$\begin{aligned} \left(R^{\mu}_{\mathbf{k},\mathbf{k}'}(x)\right)^{\dagger} &= \sum_{r,r'} \overline{\gamma^{\mu}_{r,r'}} \left(\psi^{\mathrm{a}}_{r',\mathbf{k}'}(x)\psi_{r,\mathbf{k}}(x)\right)^{\dagger} \\ &= \sum_{r,r'} [\gamma^{\mu}]^{\dagger}_{r',r} \left(\psi_{r,\mathbf{k}}(x)\right)^{\dagger} \sum_{r''} \overline{\gamma^{0}_{r'',r'}} \psi_{r'',\mathbf{k}'}(x) \\ &= \sum_{r,r',r''} \gamma^{0}_{r',r} \left(\psi_{r,\mathbf{k}}(x)\right)^{\dagger} \gamma^{\mu}_{r'',r'}\psi_{r'',\mathbf{k}'}(x) \\ &= \sum_{r',r''} \psi^{\mathrm{a}}_{r',\mathbf{k}}(x)\gamma^{\mu}_{r'',r'}\psi_{r'',\mathbf{k}'}(x) \\ &= R^{\mu}_{\mathbf{k}',\mathbf{k}}(x). \end{aligned}$$

This proves (3.35).

3.3.3 The electric current

The electric current operators $\hat{J}^\mu(x)$ are integral operators defined by the symmetric kernels

$$J^\mu_{\mathbf{k},\mathbf{k}'}(x) \quad = \quad \frac{1}{2}qc\left(R^\mu_{\mathbf{k},\mathbf{k}'}(x) - C_{\text{c}}R^\mu_{\mathbf{k},\mathbf{k}'}(x)C_{\text{c}}^{-1}\right). \tag{3.36}$$

Here, q is a unit of charge. The domain of definition of $\hat{J}^\mu(x)$ consists of the fields $\zeta \in \Gamma_2$ for which the integrals

$$\int \mathrm{d}\mathbf{k}'\, J^\mu_{\mathbf{k},\mathbf{k}'}(x)\zeta_{\mathbf{k}'}$$

are absolutely convergent. Because $\hat{R}$ satisfies the continuity equation also $\hat{J}$ does.

It is shown in the Appendix F that

$$J^\mu_{\mathbf{k},\mathbf{k}'}(x) \quad = \quad \frac{1}{2}qc\sum_{r,r'}\gamma^\mu_{r,r'}\psi^{\text{a}}_{r,\mathbf{k}}(x)\psi_{r',\mathbf{k}'}(x) - \frac{1}{2}qc\sum_{r,r'}\gamma^\mu_{r',r}\psi_{r,\mathbf{k}}(x)\psi^{\text{a}}_{r',\mathbf{k}'}(x). \tag{3.37}$$

This is a well-known expression for the Dirac current, adapted to the present context. Using the definitions of $\hat{\psi}$ and $\hat{\psi}^{\text{a}}$ it can be further evaluated. Let us split the expression into two contributions, one which commutes with the number operator N, and another, called the off-diagonal part [10], which consists of terms creating or annihilating electron-positron pairs. It is shown in the same Appendix F that

$$J^\mu_{\mathbf{k},\mathbf{k}'}(x) \quad = \quad J^{\text{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x) + J^{\text{off},\mu}_{\mathbf{k},\mathbf{k}'}(x)$$

where

$$\begin{aligned} J^{\text{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x) \quad = \quad & \frac{qc}{2(2\pi)^3}e^{i(k_\nu-k'_\nu)x^\nu}\sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\sigma^{(-)}_s\sigma^{(+)}_t \\ & -\frac{qc}{2(2\pi)^3}e^{i(k_\nu-k'_\nu)x^\nu}\sum_{s,t=3,4}\langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle\sigma^{(-)}_t\sigma^{(+)}_s \\ & +(\mathbf{k}\leftrightarrow\mathbf{k}') \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} J^{\text{off},\mu}_{\mathbf{k},\mathbf{k}'}(x) \quad = \quad & \frac{qc}{2(2\pi)^3}e^{i(k_\nu+k'_\nu)x^\nu}\sum_{s=1,2}\sum_{t=3,4}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle\,\sigma^{(-)}_s\sigma^{(-)}_t \\ & +\frac{qc}{2(2\pi)^3}e^{-i(k_\nu+k'_\nu)x^\nu}\sum_{s=3,4}\sum_{t=1,2}\langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\,\sigma^{(+)}_s\sigma^{(+)}_t \end{aligned}$$

$$+(\mathbf{k} \leftrightarrow \mathbf{k}'). \tag{3.39}$$

Note that these are normal-ordered expressions, with creation operators $\sigma^{(-)}$ left of annihilation operators $\sigma^{(+)}$, whenever both types occur simultaneously.

One verifies that each of the two current operators $\hat{J}^{\text{diag},\mu}(x)$ and $\hat{J}^{\text{off},\mu}(x)$ satisfies the continuity equation

$$i\partial_\mu J^{\text{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x) = 0, \quad \text{respectively} \quad i\partial_\mu J^{\text{off},\mu}_{\mathbf{k},\mathbf{k}'}(x) = 0. \tag{3.40}$$

See the Appendix G. In addition they satisfy

$$k_\mu J^{\text{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x) = k'_\mu J^{\text{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x) \quad \text{and} \quad k_\mu J^{\text{off},\mu}_{\mathbf{k},\mathbf{k}'}(x) = -k'_\mu J^{\text{off},\mu}_{\mathbf{k},\mathbf{k}'}(x). \tag{3.41}$$

Let us now calculate the total charge $\hat{Q}$, which is the diagonal operator satisfying

$$\frac{1}{c}\int \mathrm{d}\mathbf{x}\, J^0_{\mathbf{k},\mathbf{k}'}(x) = \delta(\mathbf{k}-\mathbf{k}')Q_{\mathbf{k}}. \tag{3.42}$$

Using the orthogonality relations (3.23, 3.24) one finds

$$\begin{aligned}
\int \mathrm{d}\mathbf{x}\, J^{\text{diag},0}_{\mathbf{k},\mathbf{k}'}(x) &= \frac{qc}{2(2\pi)^3}\int \mathrm{d}\mathbf{x}\, e^{i(k_\nu-k'_\nu)x^\nu} \sum_{s,t=1,2} \langle u^{(s)}(\mathbf{k})|u^{(t)}(\mathbf{k}')\rangle \sigma^{(-)}_s\sigma^{(+)}_t \\
&\quad -\frac{qc}{2(2\pi)^3}\int \mathrm{d}\mathbf{x}\, e^{i(k_\nu-k'_\nu)x^\nu} \sum_{s,t=3,4} \langle v^{(s)}(\mathbf{k}')|v^{(t)}(\mathbf{k})\rangle \sigma^{(-)}_t\sigma^{(+)}_s \\
&\quad +(\mathbf{k} \leftrightarrow \mathbf{k}') \\
&= qc\delta(\mathbf{k}-\mathbf{k}') \sum_{s,t=1,2} \langle u^{(s)}(\mathbf{k})|u^{(t)}(\mathbf{k})\rangle \sigma^{(-)}_s\sigma^{(+)}_t \\
&\quad -qc\delta(\mathbf{k}-\mathbf{k}') \sum_{s,t=3,4} \langle v^{(s)}(\mathbf{k})|v^{(t)}(\mathbf{k})\rangle \sigma^{(-)}_t\sigma^{(+)}_s \\
&= qc\delta(\mathbf{k}-\mathbf{k}') \left[\sum_{s=1,2} \sigma^{(-)}_s\sigma^{(+)}_s - \sum_{s=3,4} \sigma^{(-)}_s\sigma^{(+)}_s\right] \\
&= qc\delta(\mathbf{k}-\mathbf{k}') \left(N_1+N_2-N_3-N_4\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int \mathrm{d}\mathbf{x}\, J^{\text{off},0}_{\mathbf{k},\mathbf{k}'}(x) &= \frac{qc}{2(2\pi)^3}\int \mathrm{d}\mathbf{x}\, e^{i(k_\mu+k'_\mu)x^\mu} \sum_{s=1,2}\sum_{t=3,4} \langle u^{(s)}(\mathbf{k})|v^{(t)}(\mathbf{k}')\rangle\, \sigma^{(-)}_s\sigma^{(-)}_t \\
&\quad +\frac{qc}{2(2\pi)^3}\int \mathrm{d}\mathbf{x}\, e^{-i(k_\mu+k'_\mu)x^\mu} \sum_{s=3,4}\sum_{t=1,2} \langle v^{(s)}(\mathbf{k})|u^{(t)}(\mathbf{k}')\rangle\, \sigma^{(+)}_s\sigma^{(+)}_t \\
&\quad +(\mathbf{k} \leftrightarrow \mathbf{k}')
\end{aligned}$$

$$\begin{aligned}
&= qc\delta(\mathbf{k}+\mathbf{k}')\sum_{s=1,2}\sum_{t=3,4}\langle u^{(s)}(\mathbf{k})|v^{(t)}(-\mathbf{k})\rangle\,\sigma_s^{(-)}\sigma_t^{(-)}\\
&\quad +qc\delta(\mathbf{k}+\mathbf{k}')\sum_{s=3,4}\sum_{t=1,2}\langle v^{(s)}(\mathbf{k})|u^{(t)}(-\mathbf{k})\rangle\,\sigma_s^{(+)}\sigma_t^{(+)}\\
&= 0.
\end{aligned}$$

One concludes that

$$\frac{1}{c}\int \mathrm{d}\mathbf{x}\, J^0_{\mathbf{k},\mathbf{k}'}(x) \;=\; q\delta(\mathbf{k}-\mathbf{k}')\,(N_1+N_2-N_3-N_4)\,.$$

This implies

$$\hat{Q} \;=\; q\,(N_1+N_2-N_3-N_4)\,.$$

The obvious interpretation is that the components 1 and 2 of the field describe an electron with charge q, and that components 3 and 4 describe a positron with charge $-q$.

Note that

$$\begin{aligned}
[\phi^{(+)}_{s,\mathbf{k}},Q]_- &= q\phi^{(+)}_{s,\mathbf{k}}, \qquad s=1,2,\\
[\phi^{(-)}_{s,\mathbf{k}},Q]_- &= q\phi^{(-)}_{s,\mathbf{k}}, \qquad s=3,4.
\end{aligned}$$

This implies

$$[\psi_{r,\mathbf{k}}(x),Q]_- \;=\; q\psi_{r,\mathbf{k}}(x), \qquad r=1,2,3,4.$$

By taking the hermitian conjugate one obtains

$$\left[\psi^\dagger_{r,\mathbf{k}}(x),Q\right]_- \;=\; -q\psi^\dagger_{r,\mathbf{k}}(x), \qquad r=1,2,3,4.$$

From (3.37) then follows that

$$\left[J^\mu_{\mathbf{k},\mathbf{k}'}(x),Q\right]_- \;=\; 0. \tag{3.43}$$

3.3.4 Example of a polarized electron field

Assume a field ζ with wave functions of the form

$$\zeta_\mathbf{k} \;=\; e^{i\chi(\mathbf{k})}\sqrt{1-\rho(\mathbf{k})}|\emptyset\rangle + e^{i\xi(\mathbf{k})}\sqrt{\rho(\mathbf{k})}|\{1\}\rangle.$$

It describes an electron field polarized with spin up. Take

$$\rho(\mathbf{k}) = \frac{1}{\cosh^2(a|\mathbf{k}|)} \quad \text{and} \quad \xi(\mathbf{k}) = \chi(\mathbf{k}) = 0. \tag{3.44}$$

This gives

$$\zeta_{\mathbf{k}} = \tanh(a|\mathbf{k}|)|\emptyset\rangle + \frac{1}{\cosh(a|\mathbf{k}|)}|\{1\}\rangle.$$

The Dirac field operators $\hat{\psi}_r$ acting on this field ζ yield

$$\begin{aligned}
&\psi_{r,\mathbf{k}}\zeta_{\mathbf{k}} \\
&= \frac{1}{\sqrt{(2\pi)^3}}\left[u_r^{(1)}(\mathbf{k})e^{-ik_\mu x^\mu}\sigma_1^{(+)}\frac{1}{\cosh(a|\mathbf{k}|)}|\{1\}\rangle + \sum_{s=3,4} v_r^{(s)}(\mathbf{k})e^{ik_\mu x^\mu}\sigma_s^{(-)}\zeta_{\mathbf{k}}\right] \\
&= \frac{1}{\sqrt{(2\pi)^3}}\left[u_r^{(1)}(\mathbf{k})e^{-ik_\mu x^\mu}\frac{1}{\cosh(a|\mathbf{k}|)}|\emptyset\rangle + \sum_{s=3,4} v_r^{(s)}(\mathbf{k})e^{ik_\mu x^\mu}\sigma_s^{(-)}\zeta_{\mathbf{k}}\right].
\end{aligned}$$

The quantum expectation equals

$$\langle\zeta_{\mathbf{k}}|\psi_{r,\mathbf{k}}\zeta_{\mathbf{k}}\rangle = \frac{1}{\sqrt{(2\pi)^3}}u_r^{(1)}(\mathbf{k})e^{-ik_\mu x^\mu}\frac{\tanh(a|\mathbf{k}|)}{\cosh(a|\mathbf{k}|)}.$$

The classical Dirac spinor has components

$$\begin{aligned}
{\phi^{\mathrm{cl}}}_r(x) &= \ell^3\int \mathrm{d}\mathbf{k}\,\langle\zeta_{\mathbf{k}}|\psi_{r,\mathbf{k}}\zeta_{\mathbf{k}}\rangle \\
&= \frac{\ell^3}{\sqrt{(2\pi)^3}}\int \mathrm{d}\mathbf{k}\,u_r^{(1)}(\mathbf{k})e^{-ik_\mu x^\mu}\frac{\tanh(a|\mathbf{k}|)}{\cosh(a|\mathbf{k}|)}.
\end{aligned}$$

The quantum energy of the field is given by

$$\begin{aligned}
\langle\hat{H}\rangle &= \ell^3\int \mathrm{d}\mathbf{k}\,\hbar\omega(\mathbf{k})\langle\zeta_{\mathbf{k}}|N_1\zeta_{\mathbf{k}}\rangle \\
&= \ell^3\int \mathrm{d}\mathbf{k}\,\hbar\omega(\mathbf{k})\rho(\mathbf{k}) \\
&= \ell^3\hbar c\int \mathrm{d}\mathbf{k}\,\frac{\sqrt{\kappa^2+|\mathbf{k}|^2}}{\cosh^2(a|\mathbf{k}|)} \\
&= \frac{4\pi\hbar c\ell^3}{a^3}\int_0^\infty \mathrm{d}r\,r^2\sqrt{\kappa^2+\left(\frac{r}{a}\right)^2}\frac{1}{\cosh^2(r)}.
\end{aligned} \tag{3.45}$$

The charge is given by

$$\begin{aligned}
\langle\hat{Q}\rangle &= \ell^3 q\int \mathrm{d}\mathbf{k}\,\langle\zeta_{\mathbf{k}}|N_1\zeta_{\mathbf{k}}\rangle \\
&= \ell^3 q\int \mathrm{d}\mathbf{k}\,\rho(\mathbf{k})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q4\pi\ell^3}{a^3}\int_0^\infty \mathrm{d}r\, r^2 \frac{1}{\cosh^2(r)} \\
&= \frac{q\pi^3\ell^3}{3a^3}. \qquad (3.46)
\end{aligned}
$$

In the limit of large a one obtains from the combination of (3.45) and (3.46)

$$
E \simeq mc^2 \frac{Q}{q}.
$$

If the field contains exactly one electron then the total charge Q equals the elementary charge q and the total energy in the long-wavelength limit is the rest mass energy of a single electron.

The above discussion does not depend on the choice of the length scale ℓ, as it should be. However, the electron field has an intrinsic length scale $\kappa^{-1} = \hbar/mc \simeq 4 \times 10^{-13}$m. It is therefore obvious to choose ℓ equal $1/\kappa$.

3.4 Symmetries

3.4.1 Representation independence

The so-called standard representation (3.16) of the gamma matrices is used explicitly in the present work. Other choices are found in the literature. The essential property of the gamma matrices is that they satisfy the anti-commutation relations

$$
\{\gamma^\mu, \gamma^\nu\}_+ = 2g^{\mu,\nu}. \qquad (3.47)
$$

Pauli's fundamental theorem (see for instance Section 3.2 and Appendix C of [11], or Appendix A2 of [7]) states that given two different sets of matrices, γ^μ respectively γ'^μ, there exists a non-singular matrix S such that $S\gamma^\mu = \gamma'^\mu S$ for all μ. This matrix is unique up to multiplication with a scalar. Conversely, given the standard representation of the γ^μ any non-singular matrix S defines a set of alternative gamma matrices γ'^μ by the transformation $\gamma'^\mu = S^{-1}\gamma^\mu S$.

The definition of the current operators $\hat{J}^\mu(x)$ does not depend on the choice of representation of the gamma matrices. This is most easily seen from the explicit expressions (3.38, 3.39). The argument is that when γ^μ transforms into γ'^μ also the polarization vectors $u^{(s)}$ and $v^{(s)}$ transform into $S^{-1}u^{(s)}$, respectively $S^{-1}v^{(s)}$. The argument works when $S^{-1} = S^\dagger$. In the present work formulas are presented in such a way that they only hold when γ^0 is hermitian and the γ^α are anti-hermitian.

3.4.2 Lorentz covariance of the Dirac equation

Consider a proper Lorentz transform $x \mapsto x'$. It is determined by 4-by-4 matrices $R_{\mu\nu}$ so that

$$x'^{\mu} = g^{\mu,\nu} R_{\nu,\lambda} x^{\lambda}.$$

These matrices must satisfy

$$R^{\mathrm{T}}_{\lambda,\mu} g^{\mu,\nu} R_{\nu,\tau} = g_{\lambda,\tau} \tag{3.48}$$

so as to conserve the pseudo-metric $k'_{\mu} x'^{\mu} = k_{\mu} x^{\mu}$. One has also the inverse transformation

$$x^{\mu} = g^{\mu,\lambda} R^{\mathrm{T}}_{\lambda,\nu} x'^{\nu}.$$

The Dirac field operators $\hat{\psi}_r(x)$ transform into operators $\hat{\psi}'_r(x)$ which must also satisfy Dirac's equation (3.17). One has

$$\begin{aligned} \kappa\hat{\psi}(x) &= i\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} \hat{\psi}(x) \\ &= i\gamma^{\mu} \frac{\partial x'^{\nu}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\nu}} \hat{\psi}(x) \\ &= i\gamma'^{\nu} \frac{\partial}{\partial x'^{\nu}} \hat{\psi}(x) \end{aligned} \tag{3.49}$$

with

$$\gamma'^{\nu} = \gamma^{\mu} \frac{\partial x'^{\nu}}{\partial x^{\mu}} = g^{\nu,\lambda} R_{\lambda,\mu} \gamma^{\mu}.$$

The matrices γ'^{ν} still satisfy the anti-commutation relations (3.47) because of (3.48). By the fundamental theorem of Pauli, discussed in the previous section, there exists a non-singular matrix S, unique up to scalar multiplication, for which $\gamma'^{\nu} = S\gamma^{\nu} S^{-1}$. Then (3.49) becomes

$$\kappa\hat{\psi}(x) = iS\gamma^{\nu} S^{-1} \frac{\partial}{\partial x'^{\nu}} \hat{\psi}(x).$$

This can be written as

$$\kappa S^{-1}\hat{\psi}(x) = i\gamma^{\nu} \frac{\partial}{\partial x'^{\nu}} S^{-1}\hat{\psi}(x)$$

and shows that $\psi'_{\mathbf{k}'}(x')$ defined by

$$\psi'_{\mathbf{k}'}(x') = S^{-1}\psi_{\mathbf{k}}(x)$$

satisfies Dirac's equation in the transformed Lorentz frame.

Let us now figure out the effect a Lorentz transformation has on the field operators $\phi^{(\pm)}_{s,\mathbf{k}}(x)$. From the inverse formula (3.21) follows for $s'=1,2$

$$\begin{aligned}
\phi^{(+)\prime}_{s',\mathbf{k}'}(x') &= \frac{k'_0}{\kappa}\frac{1}{\sqrt{2\ell k'_0}}\sum_r \overline{u^{(s')}_r(-\mathbf{k}')}\psi'_{r,\mathbf{k}'}(x') \\
&= \frac{k'_0}{\kappa}\frac{1}{\sqrt{2\ell k'_0}}\sum_{r,r'} \overline{u^{(s')}_r(-\mathbf{k}')}S^{-1}_{r,r'}\psi_{r',\mathbf{k}}(x) \\
&= \frac{\sqrt{k_0k'_0}}{\kappa}\sum_{r,r'} \overline{u^{(s')}_r(-\mathbf{k}')}S^{-1}_{r,r'} \\
&\quad\times\left[\sum_{s=1,2} u^{(s)}_{r'}(\mathbf{k})\phi^{(+)}_{s,\mathbf{k}}(x) + \sum_{t=3,4} v^{(t)}_{r'}(\mathbf{k})\phi^{(-)}_{t,\mathbf{k}}(x)\right] \\
&= \frac{\sqrt{k_0k'_0}}{\kappa}\left[\sum_{s=1,2}\langle u^{(s')}(-\mathbf{k}')|S^{-1}u^{(s)}(\mathbf{k})\rangle\phi^{(+)}_{s,\mathbf{k}}(x)\right. \\
&\qquad\left. + \sum_{t=3,4}\langle u^{(s')}(-\mathbf{k}')|S^{-1}v^{(t)}(\mathbf{k})\rangle\phi^{(-)}_{t,\mathbf{k}}(x)\right].
\end{aligned} \tag{3.50}$$

Similarly, from (3.22) follows for $t'=3,4$

$$\begin{aligned}
\phi^{(-)\prime}_{t',\mathbf{k}'}(x') &= \frac{\sqrt{k_0k'_0}}{\kappa}\left[\sum_{s=1,2}\langle v^{(t')}(-\mathbf{k}')|S^{-1}u^{(s)}(\mathbf{k})\rangle\phi^{(+)}_{s,\mathbf{k}}(x)\right. \\
&\qquad\left. + \sum_{t=3,4}\langle v^{(t')}(-\mathbf{k}')|S^{-1}v^{(t)}(\mathbf{k})\rangle\phi^{(-)}_{t,\mathbf{k}}(x)\right].
\end{aligned} \tag{3.51}$$

These equations show that under a Lorentz transformation each of the field operators $\phi^{(\pm)}_{s,\mathbf{k}}$ is replaced by a linear combination involving all of them.

The two subcases of a spatial rotation and of a Lorentz boost are treated separately in the following two sections. They complete the discussion of Lorentz transformations because any proper Lorentz transformation can be decomposed into a succession of a spatial rotation, a Lorentz boost in the third direction, and a final spatial rotation.

3.4.3 Spatial rotations

In this section the Lorentz transformation R is assumed to be a spatial rotation. As before, let S be a matrix for which

$$\gamma^\mu g^{\nu,\lambda}R_{\lambda,\mu} = \gamma'^\nu = S\gamma^\nu S^{-1}.$$

Because R is a rotation matrix these equations reduce to

$$\gamma^0 = S\gamma^0 S^{-1} \quad \text{and} \quad -\sum_\alpha \gamma^\alpha R_{\beta,\alpha} = S\gamma^\beta S^{-1}.$$

This matrix S is not unique, even not if one requires that it is unitary. It is of the form

$$S = \begin{pmatrix} A & 0 \\ 0 & \pm A \end{pmatrix}, \tag{3.52}$$

where A is a solution of $A\sigma_\alpha A^{-1} = \pm\sum_\beta R_{\alpha,\beta}\sigma_\beta$. This equation establishes a relation between the rotation group SO(3) and the group SU(2) of unitary two-by-two matrices with determinant 1. For a given rotation matrix R it has two solutions $+A$ and $-A$, where A satisfies $A^\dagger A = AA^\dagger = \text{id}$ and $\det A = 1$.

Consider now the operator $T(\mathbf{k})$, defined before. From the explicit expression (3.20) and the observation that S commutes with γ^0 follows that

$$ST(\mathbf{k})S^{-1} = T(\mathbf{k}').$$

This allows to write (3.50) as

$$\begin{aligned}\phi^{(+)\prime}_{s',\mathbf{k}'}(x') &= \frac{\sqrt{k_0k_0'}}{\kappa}\Bigg[\sum_{s=1,2}\langle T(-\mathbf{k})S^{-1}u^{(s')}(0)|u^{(s)}(\mathbf{k})\rangle\phi^{(+)}_{s,\mathbf{k}}(x) \\ &\qquad + \sum_{t=3,4}\langle T(-\mathbf{k})S^{-1}u^{(s')}(0)|v^{(t)}(\mathbf{k})\rangle\phi^{(-)}_{t,\mathbf{k}}(x)\Bigg].\end{aligned} \tag{3.53}$$

Note that $S^{-1}u^{(s')}(0)$ is a linear combination of $u^{(1)}(0)$ and $u^{(2)}(0)$ so that $T(-\mathbf{k})S^{-1}u^{(s')}(0)$ is a linear combination of $u^{(1)}(-\mathbf{k})$ and $u^{(2)}(-\mathbf{k})$. From the orthogonality relations (3.18) then follows that the latter term of (3.53) vanishes. The result is

$$\phi^{(+)\prime}_{s',\mathbf{k}'}(x') = \frac{\sqrt{k_0k_0'}}{\kappa}\sum_{s=1,2}\langle T(-\mathbf{k})S^{-1}u^{(s')}(0)|u^{(s)}(\mathbf{k})\rangle\phi^{(+)}_{s,\mathbf{k}}(x).$$

It shows that each of the electron field operators $\hat\phi^{(\pm)}_s$, $s = 1,2$ transforms into a linear combination of these operators. In the same way one shows a similar property for the positron operators $\hat\phi^{(\pm)}_t$, $s = 3,4$. This confirms the well-known observation that spatial rotations do not mix up electrons and positrons.

3.4.4 Lorentz boost

In this section the Lorentz transformation matrix R is assumed to be a Lorentz boost in direction 3. The non-vanishing elements of R_μ^ν are

$$\begin{aligned} R_{0,0} &= R_{3,3} = \cosh(\chi), \\ R_{0,3} &= R_{3,0} = \sinh(\chi), \\ R_{1,1} &= R_{2,2} = 1, \end{aligned}$$

with $\chi \in \mathbb{R}$. As before, let S be a matrix for which $g^{\nu,\lambda} R_{\lambda,\mu}\gamma^\mu = \gamma'^\nu = S\gamma^\nu S^{-1}$. It is explicitly given by

$$S \quad = \quad \frac{1}{\sqrt{(1-\alpha^2}} \begin{pmatrix} \mathbb{I} & \alpha\sigma_3 \\ -\alpha\sigma_3 & -\mathbb{I} \end{pmatrix},$$

with

$$\alpha = \frac{\sinh(\chi)}{\cosh(\chi)+1} = \frac{\cosh(\chi)-1}{\sinh(\chi)} = \tanh\left(\frac{\chi}{2}\right).$$

The normalization is chosen such that $\det S = 1$. Note that $S^2 = \mathbb{I}$.

The transformed gamma matrices $\gamma'^\mu = S\gamma^\mu S^{-1}$ are $\gamma'^1 = \gamma^1$, $\gamma'^2 = \gamma^2$,

$$\begin{aligned} \gamma'^0 &= \frac{1}{1-\alpha^2} \begin{pmatrix} (1+\alpha^2)\mathbb{I} & 2\alpha\sigma_3 \\ -2\alpha\sigma_3 & -(1+\alpha^2)\mathbb{I} \end{pmatrix}, \\ \gamma'^3 &= \frac{1}{1-\alpha^2} \begin{pmatrix} -2\alpha\mathbb{I} & -(1+\alpha^2)\sigma_3 \\ (1+\alpha^2)\sigma_3 & 2\alpha\mathbb{I} \end{pmatrix}. \end{aligned}$$

Note that for $\alpha \neq 0$ the matrix γ'^0 is not anymore hermitian. The freedom of choice of representation of the gamma matrices is a gauge freedom of the Dirac equation. The present work relies heavily on choosing the standard representation in which γ^0 is a hermitian matrix. This particular choice is the reason why the Klein Gordon equation (3.27) splits into two equations.

Appendix A

Polarization of electromagnetic waves

Let us first make an explicit choice for the rotation matrix $\Xi(\mathbf{k})$. By definition it rotates the vector $\mathbf{k}$ into the positive third direction. Choose to do this by rotating around the first axis and then around the second axis. This gives a matrix of the form

$$\begin{aligned}\Xi(\mathbf{k}) &= \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta & -\sin\alpha\sin\beta & -\sin\beta\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \\ \sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix}.\end{aligned}$$

The first rotation eliminates component 2, the second eliminates component 1. From these requirements one deduces that

$$\cos\alpha = \frac{k_3}{\sqrt{k_2^2 + k_3^2}}, \qquad \sin\alpha = \frac{k_2}{\sqrt{k_2^2 + k_3^2}},$$

and

$$\cos\beta = \frac{1}{|\mathbf{k}|}\sqrt{k_2^2 + k_3^3}, \qquad \sin\beta = \frac{k_1}{|\mathbf{k}|}.$$

It is now straightforward to verify that $\Xi(\mathbf{k})\mathbf{k} = |\mathbf{k}|e_3$.

Next let R be an arbitrary rotation matrix and Λ the corresponding 4-dimensional Lorentz transformation. We want to calculate

$$M_\Lambda(\mathbf{k}) = \Xi(R\mathbf{k})R\Xi(\mathbf{k})^\dagger.$$

We know that $M_\Lambda(\mathbf{k})$ is of the form

$$M_\Lambda(\mathbf{k}) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Use the notation $\mathbf{k}' = R\mathbf{k}$, and α', β' for the value of α respectively β when evaluated at $\mathbf{k}'$ instead of $\mathbf{k}$. Then the first row of the matrix $\Xi(\mathbf{k}')^\dagger M_\Lambda(\mathbf{k})$ evaluates to

$$\left[\cos\beta'\cos\gamma, \quad -\cos\beta'\sin\gamma, \quad \sin\beta'\right].$$

The first row of the matrix $R\Xi(\mathbf{k})^\dagger$ evaluates to

$$\left[R_{1,1}\cos\beta - (R_{1,2}\sin\alpha + R_{1,3}\cos\alpha)\sin\beta, \quad R_{1,2}\cos\alpha - R_{1,3}\sin\alpha, \right. \\ \left. R_{1,1}\sin\beta + (R_{1,2}\sin\alpha + R_{1,3}\cos\alpha)\cos\beta\right].$$

Equating both rows yields

$$\begin{aligned} \cos\beta'\cos\gamma &= R_{1,1}\cos\beta - (R_{1,2}\sin\alpha + R_{1,3}\cos\alpha)\sin\beta, && \text{(A.1)} \\ -\cos\beta'\sin\gamma &= R_{1,2}\cos\alpha - R_{1,3}\sin\alpha, && \text{(A.2)} \\ \sin\beta' &= R_{1,1}\sin\beta + (R_{1,2}\sin\alpha + R_{1,3}\cos\alpha)\cos\beta. \end{aligned}$$

The latter implies

$$\begin{aligned} \cos\gamma &= \frac{R_{1,1} - \sin\beta\sin\beta'}{\cos\beta\cos\beta'}, && \text{(A.3)} \\ \sin\gamma &= -\frac{R_{1,2}\cos\alpha - R_{1,3}\sin\alpha}{\cos\beta'}. \end{aligned}$$

Appendix B

Polarization of electron waves

Let us first consider solutions of the equation

$$\gamma^\mu k_\mu u^{(s)}(\mathbf{k}) = \kappa u^{(s)}(\mathbf{k}). \tag{B.1}$$

When $\mathbf{k} = 0$ the equation reduces to

$$\gamma^0 u_0^{(s)} = u_0^{(s)}.$$

This has indeed two independent solutions $u_0^{(1)}$ and $u_0^{(1)}$. Assuming the standard representation of the gamma matrices one can choose $(1,0,0,0)^{\mathrm{T}}$ and $(0,1,0,0)^{\mathrm{T}}$.

Next, choose $u_{r,\mathbf{k}}^{(s)}$ of the form

$$u_r^{(s)}(\mathbf{k}) = \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{k_0+\kappa}} \left[\sum_{r'} k_\nu \gamma^\nu_{r,r'} u^{(s)}_{r',0} + \kappa u^{(s)}_{r,0} \right]. \tag{B.2}$$

Note that $u^{(s)}(0) = u_0^{(s)}$. Then one finds, using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu,\nu}$,

$$\begin{aligned}
k_\mu \gamma^\mu u^{(s)}(\mathbf{k}) &= \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{k_0+\kappa}} \left[k_\mu k_\nu \gamma^\mu \gamma^\nu u_0^{(s)} + \kappa k_\mu \gamma^\mu u_0^{(s)} \right] \\
&= \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{k_0+\kappa}} \left[k_\mu k^\mu u_0^{(s)} + \kappa k_\mu \gamma^\mu u_0^{(s)} \right] \\
&= \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{k_0+\kappa}} \left[\kappa^2 u_0^{(s)} + \kappa k_\mu \gamma^\mu u_0^{(s)} \right] \\
&= \kappa u^{(s)}(\mathbf{k}).
\end{aligned}$$

One concludes that $u_r^{(s)}(\mathbf{k})$, defined by (B.2), solves the equation (B.1).

Next one verifies that

$$\sum_r \overline{u_r^{(s)}}(\mathbf{k}) u_r^{(t)}(\mathbf{k}) = \frac{1}{2k_0} \frac{1}{k_0+\kappa} \left\langle u_0^{(s)} \middle| (k_\nu (\gamma^\nu)^\dagger + \kappa)(k_\sigma \gamma^\sigma + \kappa) u_0^{(t)} \right\rangle$$

$$
\begin{aligned}
&= \frac{1}{2k_0}\frac{1}{k_0+\kappa}\Big[\left\langle u_0^{(s)}\middle|k_\nu(\gamma^\nu)^\dagger k_\sigma\gamma^\sigma u_0^{(t)}\right\rangle \\
&\quad +\kappa\left\langle u_0^{(s)}\middle|k_\sigma\gamma^\sigma u_0^{(t)}\right\rangle+\kappa\left\langle u_0^{(s)}\middle|k_\nu(\gamma^\nu)^\dagger u_0^{(t)}\right\rangle+\kappa^2\delta_{s,t}\Big].
\end{aligned}
$$

Now use that $(\gamma^\nu)^\dagger\gamma^0=\gamma^0\gamma^\nu$ and that $\gamma^0 u_0^{(s)}=u_0^{(s)}$ to obtain

$$
\begin{aligned}
k_\nu(\gamma^\nu)^\dagger k_\sigma\gamma^\sigma u_0^{(s)} &= \gamma^0 k_\nu\gamma^\nu\gamma^0 k_\sigma\gamma^\sigma u_0^{(s)} \\
&= \left(k_0-\sum_\alpha k_\alpha\gamma^\alpha\right)\left(k_0+\sum_\beta k_\beta\gamma^\beta\right)u_0^{(s)} \\
&= (k_0^2+|\mathbf{k}|^2)u_0^{(s)}. \qquad \text{(B.3)}
\end{aligned}
$$

Because also

$$\left\langle u_0^{(s)}\middle|k_\sigma\gamma^\sigma u_0^{(t)}\right\rangle = k_0\delta_{s,t}$$

there follows

$$\sum_r \overline{u_r^{(s)}}(\mathbf{k})u_r^{(t)}(\mathbf{k}) = \delta_{s,t}.$$

Next consider the equation

$$\gamma^\mu k_\mu v^{(s)}(\mathbf{k}) = -\kappa v^{(s)}(\mathbf{k}). \qquad \text{(B.4)}$$

At $\mathbf{k}=0$ it becomes $\gamma^0 v_0^{(s)}=-v_0^{(s)}$. The matrix γ^0 has two eigenvectors with eigenvalue -1. We choose

$$v_0^{(3)}=(0,0,1,0)^{\mathrm{T}} \quad\text{and}\quad v_0^{(4)}=(0,0,0,1)^{\mathrm{T}}.$$

Next one verifies that

$$v_r^{(s)}(\mathbf{k}) = \frac{1}{\sqrt{2k_0}}\frac{1}{\sqrt{k_0+\kappa}}\left[-\sum_{r'}k_\nu\gamma^\nu_{r,r'}v^{(s)}_{r',0}+\kappa v^{(s)}_{r,0}\right] \qquad \text{(B.5)}$$

is a solution of (B.4). The normalization

$$\sum_r \overline{v_r^{(s)}}(\mathbf{k})v_r^{(t)}(\mathbf{k}) = \delta_{s,t}$$

is proved in the same way as for the u-vectors.

Finally, let us calculate

$$\sum_r \overline{u_r^{(s)}}(\mathbf{k})v_r^{(t)}(-\mathbf{k}) = \frac{1}{2k_0}\frac{1}{k_0+\kappa}\left\langle u_0^{(s)}\middle|(k_0\gamma^0-\sum_\alpha k_\alpha\gamma^\alpha+\kappa)\right.$$

$$
\begin{aligned}
& \times(-k_0\gamma^0 + \sum_\beta k_\beta\gamma^\beta + \kappa)v_0^{(t)}\Big\rangle \\
= & -\frac{1}{2k_0}\frac{1}{k_0+\kappa}\left\langle u_0^{(s)}\middle|(k_0+\kappa)^2 - \left(\sum_\alpha k_\alpha\gamma^\alpha\right)^2 v_0^{(t)}\right\rangle \\
= & -\frac{1}{2k_0}\frac{1}{k_0+\kappa}\left((k_0+\kappa)^2 + |\mathbf{k}|^2\right)\langle u_0^{(s)} v_0^{(t)}\rangle \\
= & \ 0.
\end{aligned}
$$

This ends the verification of the orthogonality relations.

The inverse relations (3.21, 3.22) follow easily. Using the orthogonality relations one obtains

$$
\sum_r \overline{u_r^{(s)}(-\mathbf{k})}\psi_{r,\mathbf{k}}(x) = \sqrt{2\ell k_0}\sum_{t=1,2}\langle u^{(s)}(-\mathbf{k})|u^{(t)}(\mathbf{k})\rangle\phi_{t,\mathbf{k}}^{(+)}(x). \quad \text{(B.6)}
$$

Now use the definitions to evaluate

$$
\begin{aligned}
& \langle u^{(s)}(-\mathbf{k})|u^{(t)}(\mathbf{k})\rangle \\
= & \ \frac{1}{2k_0(k_0+\kappa)}\langle(k_0+\kappa-\sum_\alpha k_\alpha\gamma^\alpha)u_0^{(s)}|(k_0+\kappa+\sum_\alpha k_\alpha\gamma^\alpha)u_0^{(t)}\rangle \\
= & \ \frac{1}{2k_0(k_0+\kappa)}\langle u_0^{(s)}|(k_0+\kappa+\sum_\alpha k_\alpha\gamma^\alpha)^2 u_0^{(t)}\rangle \\
= & \ \frac{1}{2k_0(k_0+\kappa)}\langle u_0^{(s)}|(k_0+\kappa)^2+\sum_\alpha k_\alpha^2(\gamma^\alpha)^2 u_0^{(t)}\rangle \\
= & \ \frac{(k_0+\kappa)^2-|\mathbf{k}|^2}{2k_0(k_0+\kappa)}\delta_{s,t} \\
= & \ \frac{\kappa}{k_0}\delta_{s,t}.
\end{aligned}
$$

Hence (B.6) implies (3.21). The derivation of (3.22) is similar.

Appendix C

Transformation

Let $T(\mathbf{k})$ be the matrix defined by (3.19). It maps the orthonormal bazis vectors $u^{(s)}(\mathbf{k}=0) = u_0^{(s)}$ and $v^{(t)}(\mathbf{k}=0) = v_0^{(t)}$, defined at wave vector $\mathbf{k}=0$, onto their values at arbitrary k

$$\begin{aligned} T(\mathbf{k})u_0^{(s)} &= u^{(s)}(\mathbf{k}), \quad s = 1, 2, \\ T(\mathbf{k})v_0^{(t)} &= v^{(t)}(\mathbf{k}), \quad t = 3, 4. \end{aligned}$$

The matrix elements are

$$\begin{aligned} \langle u_0^{(s')}|T(\mathbf{k})u_0^{(s)}\rangle &= \sqrt{\frac{k_0+\kappa}{2k_0}}\delta_{s,s'}, \\ \langle v_0^{(t')}|T(\mathbf{k})v_0^{(t)}\rangle &= \sqrt{\frac{k_0+\kappa}{2k_0}}\delta_{t,t'}, \\ \langle v_0^{(t')}|T(\mathbf{k})u_0^{(s)}\rangle &= \frac{1}{\sqrt{2k_0}\sqrt{k_0+\kappa}}\langle v_0^{(t')}|\sum_\alpha k_\alpha\gamma^\alpha u_0^{(s)}\rangle, \\ \langle u_0^{(s')}|T(\mathbf{k})v_0^{(t)}\rangle &= \frac{-1}{\sqrt{2k_0}\sqrt{k_0+\kappa}}\langle u_0^{(s')}|\sum_\alpha k_\alpha\gamma^\alpha v_0^{(t)}\rangle. \end{aligned}$$

Use $(\gamma^\alpha)^\dagger = -\gamma^\alpha$ to verify that the matrix $T(\mathbf{k})$ is hermitian.

The determinant of the matrix $T(\mathbf{k})$ equals

$$\begin{aligned} \det T &= \left(\frac{k_0+\kappa}{2k_0}\right)^2 + \frac{k_0+\kappa}{2k_0}\left(|T_{13}|^2 + |T_{14}|^2 - |T_{23}|^2 - |T_{24}|^2\right) \\ &\quad -|T_{13}T_{24} - T_{14}T_{23}|^2, \end{aligned}$$

with

$$T_{1,3} = \langle u_0^{(1)}|T(\mathbf{k})v_0^{(3)}\rangle = \frac{k_3}{\sqrt{2k_0}\sqrt{k_0+\kappa}}$$

$$
\begin{aligned}
T_{1,4} &= \langle u_0^{(1)} | T(\mathbf{k}) v_0^{(4)} \rangle = \frac{k_1 - ik_2}{\sqrt{2k_0}\sqrt{k_0+\kappa}} \\
T_{2,3} &= \langle u_0^{(2)} | T(\mathbf{k}) v_0^{(3)} \rangle = \frac{k_1 + ik_2}{\sqrt{2k_0}\sqrt{k_0+\kappa}} \\
T_{2,4} &= \langle u_0^{(2)} | T(\mathbf{k}) v_0^{(4)} \rangle = \frac{-k_3}{\sqrt{2k_0}\sqrt{k_0+\kappa}} = -T_{13}.
\end{aligned}
$$

The result simplifies to

$$
\det T(\mathbf{k}) = \frac{(k_0+\kappa)^4 - |\mathbf{k}|^4}{4k_0^2(k_0+\kappa)^2)} = \frac{\kappa}{k_0}.
$$

Appendix D

Useful relations

A useful relation is

$$
\begin{aligned}
\gamma^0 u^{(s)}(\mathbf{k}) &= \gamma^0 \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{k_0+\kappa}} \left[k_0\gamma^0 + \sum_{\alpha=1}^{3} k_\alpha \gamma^\alpha + \kappa \right] u_0^{(s)} \\
&= \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{k_0+\kappa}} \left[k_0 - \sum_{\alpha=1}^{3} k_\alpha \gamma^\alpha + \kappa \right] u_0^{(s)} \\
&= u^{(s)}(-\mathbf{k}). \qquad \text{(D.1)}
\end{aligned}
$$

Similarly is $\gamma^0 v^{(s)}(\mathbf{k}) = -v^{(s)}(-\mathbf{k})$. Both relations are used in

$$
\begin{aligned}
\langle v^{(4)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(4)}(\mathbf{k}')\rangle &= \langle C\overline{u^{(1)}(-\mathbf{k})}|\gamma^0\gamma^\mu C\overline{u^{(1)}(-\mathbf{k}')}\rangle \\
&= \langle\overline{u^{(1)}(-\mathbf{k})}|C^\dagger\gamma^0\gamma^\mu C\overline{u^{(1)}(-\mathbf{k}')}\rangle \\
&= \langle\overline{u^{(1)}(-\mathbf{k})}|(\gamma^\mu\gamma^0)^{\mathrm{T}}\overline{u^{(1)}(-\mathbf{k}')}\rangle \\
&= \langle u^{(1)}(-\mathbf{k}')|\gamma^\mu\gamma^0 u^{(1)}(-\mathbf{k})\rangle \\
&= \langle u^{(1)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(1)}(\mathbf{k})\rangle. \qquad \text{(D.2)}
\end{aligned}
$$

Similarly is

$$
\begin{aligned}
\langle v^{(4)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(3)}(\mathbf{k}')\rangle &= \langle u^{(2)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(1)}(\mathbf{k})\rangle; \qquad \text{(D.3)} \\
\langle v^{(3)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(3)}(\mathbf{k}')\rangle &= \langle u^{(2)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(2)}(\mathbf{k})\rangle. \qquad \text{(D.4)}
\end{aligned}
$$

One has also

$$
\begin{aligned}
\langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(4)}(\mathbf{k}')\rangle &= \langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\mu C\overline{u^{(1)}(-\mathbf{k}')}\rangle \\
&= \langle u^{(1)}(\mathbf{k})|C(\gamma^\mu\gamma^0)^{\mathrm{T}}\overline{u^{(1)}(-\mathbf{k}')}\rangle \\
&= -\langle u^{(1)}(-\mathbf{k}')|\gamma^\mu\gamma^0 C\overline{u^{(1)}(\mathbf{k})}\rangle \\
&= -\langle u^{(1)}(-\mathbf{k}')|\gamma^\mu\gamma^0 v^{(4)}(-\mathbf{k})\rangle \\
&= \langle u^{(1)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(4)}(\mathbf{k})\rangle. \qquad \text{(D.5)}
\end{aligned}
$$

Similarly is

$$\begin{aligned}
\langle u^{(2)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(4)}(\mathbf{k}')\rangle &= \langle u^{(1)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(3)}(\mathbf{k})\rangle; && \text{(D.6)}\\
\langle u^{(2)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(3)}(\mathbf{k}')\rangle &= \langle u^{(2)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(3)}(\mathbf{k})\rangle. && \text{(D.7)}
\end{aligned}$$

Next we calculate

$$\begin{aligned}
& \langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle \\
= & \frac{1}{2k_0(k_0+\kappa)}\langle (k_\nu\gamma^\nu+\kappa)u_0^{(s)}|\gamma^0\gamma^\mu(k_\tau\gamma^\tau+\kappa)u_0^{(t)}\rangle \\
= & \frac{1}{2k_0(k_0+\kappa)}\langle u_0^{(s)}|(k_\nu\gamma^\nu+\kappa)\gamma^\mu(k_\tau\gamma^\tau+\kappa)u_0^{(t)}\rangle \\
= & \frac{1}{2k_0(k_0+\kappa)}\langle u_0^{(s)}|[\gamma^\mu(-k_\nu\gamma^\nu+\kappa)+2k^\mu](k_\tau\gamma^\tau+\kappa)u_0^{(t)}\rangle \\
= & \frac{k^\mu}{k_0(k_0+\kappa)}\langle u_0^{(s)}|(k_\tau\gamma^\tau+\kappa)u_0^{(t)}\rangle + \frac{1}{2k_0(k_0+\kappa)}\langle u_0^{(s)}|\gamma^\mu[-k_\nu\gamma^\nu k_\tau\gamma^\tau+\kappa^2]u_0^{(t)}\rangle \\
= & \frac{k^\mu}{k_0}\delta_{s,t}. && \text{(D.8)}
\end{aligned}$$

Similarly is

$$\begin{aligned}
& \langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle \\
= & \frac{1}{2k_0(k_0+\kappa)}\langle (-k_\nu\gamma^\nu+\kappa)v_0^{(s)}|\gamma^0\gamma^\mu(-k_\tau\gamma^\tau+\kappa)v_0^{(t)}\rangle \\
= & \frac{1}{2k_0(k_0+\kappa)}\langle v_0^{(s)}|(-k_\nu\gamma^\nu+\kappa)^\dagger\gamma^0\gamma^\mu(-k_\tau\gamma^\tau+\kappa)v_0^{(t)}\rangle \\
= & -\frac{1}{2k_0(k_0+\kappa)}\langle v_0^{(s)}|(-k_\nu\gamma^\nu+\kappa)\gamma^\mu(-k_\tau\gamma^\tau+\kappa)v_0^{(t)}\rangle \\
= & -\frac{1}{2k_0(k_0+\kappa)}\langle v_0^{(s)}|[\gamma^\mu(k_\nu\gamma^\nu+\kappa)-2k^\mu](-k_\tau\gamma^\tau+\kappa)v_0^{(t)}\rangle \\
= & \frac{k^\mu}{k_0(k_0+\kappa)}\langle v_0^{(s)}|(-k_\tau\gamma^\tau+\kappa)v_0^{(t)}\rangle - \frac{1}{2k_0(k_0+\kappa)}\langle v_0^{(s)}|\gamma^\mu[-k_\nu\gamma^\nu k_\tau\gamma^\tau+\kappa^2]v_0^{(t)}\rangle \\
= & \frac{k^\mu}{k_0}\delta_{s,t}. && \text{(D.9)}
\end{aligned}$$

This finishes the proof of (3.23, 3.24).

Appendix E

Charge conjugation

Let $C_{\mathrm{c}}|\emptyset\rangle = i|\emptyset\rangle$ and require

$$\begin{aligned} C_{\mathrm{c}}\sigma_1^{(-)}C_{\mathrm{c}}^{-1} &= -\sigma_4^{(-)}, \\ C_{\mathrm{c}}\sigma_2^{(-)}C_{\mathrm{c}}^{-1} &= -\sigma_3^{(-)}, \\ C_{\mathrm{c}}\sigma_3^{(-)}C_{\mathrm{c}}^{-1} &= -\sigma_2^{(-)}, \\ C_{\mathrm{c}}\sigma_4^{(-)}C_{\mathrm{c}}^{-1} &= -\sigma_1^{(-)}. \end{aligned}$$

This implies in particular that

$$\begin{aligned} C_{\mathrm{c}}|\{1\}\rangle &= i|\{4\}\rangle, \\ C_{\mathrm{c}}|\{2\}\rangle &= i|\{3\}\rangle, \\ C_{\mathrm{c}}|\{3\}\rangle &= i|\{2\}\rangle, \\ C_{\mathrm{c}}|\{4\}\rangle &= i|\{1\}\rangle. \end{aligned}$$

Extend the definition to arbitrary basis vectors $|\Lambda\rangle$, $\Lambda \subset \{1,2,3,4\}$, and by linearity to all of $\mathcal{H}_{16}$. One verifies that $C_{\mathrm{c}}^2 = -\mathrm{id}$ and $C_{\mathrm{c}}^\dagger = C_{\mathrm{c}}^{-1} = -C_{\mathrm{c}}$. This implies that also

$$C_{\mathrm{c}}\sigma_s^{(+)}C_{\mathrm{c}}^{-1} = -\sigma_{5-s}^{(+)},$$

and

$$\begin{aligned} C_{\mathrm{c}}\phi_{s,\mathbf{k}}^{(+)}(x)C_{\mathrm{c}}^{-1} &= \frac{1}{N_\kappa(\mathbf{k})}e^{-ik_\mu x^\mu}C_{\mathrm{c}}\sigma_s^{(+)}C_{\mathrm{c}}^{-1} \\ &= -\phi_{5-s,\mathbf{k}}^{(+)}(x). \end{aligned}$$

From the definition (3.16) now follows

$$C_{\mathrm{c}}\psi_{r,\mathbf{k}}(x)C_{\mathrm{c}}^{-1}$$

$$
\begin{aligned}
&= \sqrt{2\ell k_0}\left[\sum_{s=1,2} u_r^{(s)}(\mathbf{k}) C_{\mathrm{c}} \phi_{s,\mathbf{k}}^{(+)}(x) C_{\mathrm{c}}^{-1} + \sum_{s=3,4} v_r^{(s)}(\mathbf{k}) C_{\mathrm{c}} \phi_{s,\mathbf{k}}^{(-)}(x) C_{\mathrm{c}}^{-1}\right] \\
&= -\sqrt{2\ell k_0}\left[\sum_{s=1,2} u_r^{(s)}(\mathbf{k}) \phi_{5-s,\mathbf{k}}^{(+)}(x) + \sum_{s=3,4} v_r^{(s)}(\mathbf{k}) \phi_{5-s,\mathbf{k}}^{(-)}(x)\right] \\
&= -\sqrt{2\ell k_0}\left[\sum_{s=3,4} u_r^{(5-s)}(\mathbf{k}) \phi_{s,\mathbf{k}}^{(+)}(x) + \sum_{s=1,2} v_r^{(5-s)}(\mathbf{k}) \phi_{s,\mathbf{k}}^{(-)}(x)\right] \\
&= -\sqrt{2\ell k_0}\sum_{r'} C_{r,r'}\left[-\sum_{s=3,4} \overline{v_{r'}^{(s)}(-\mathbf{k})} \phi_{s,\mathbf{k}}^{(+)}(x) + \sum_{s=1,2} \overline{u_{r'}^{(s)}(-\mathbf{k})} \phi_{s,\mathbf{k}}^{(-)}(x)\right] \\
&= -\sqrt{2\ell k_0}\sum_{r'} C_{r,r'}\left[-\sum_{s=3,4} v_{r'}^{(s)}(-\mathbf{k}) \phi_{s,\mathbf{k}}^{(-)}(x) + \sum_{s=1,2} u_{r'}^{(s)}(-\mathbf{k}) \phi_{s,\mathbf{k}}^{(+)}(x)\right]^{\dagger} \\
&= -\sqrt{2\ell k_0}\sum_{r'} C_{r,r'}\gamma^0_{r',r'}\left[\sum_{s=3,4} v_{r'}^{(s)}(\mathbf{k}) \phi_{s,\mathbf{k}}^{(-)}(x) + \sum_{s=1,2} u_{r'}^{(s)}(\mathbf{k}) \phi_{s,\mathbf{k}}^{(+)}(x)\right]^{\dagger} \\
&= -\sum_{r'} C_{r,r'}\psi^{\mathrm{a}}_{r',\mathbf{k}}(x).
\end{aligned}
$$

This proves (3.29).

Appendix F

The current operators

Here explicit expressions for the current operators $J^{\mu}_{\mathbf{k},\mathbf{k}'}(x)$ are calculated.

From the definition follows

$$
\begin{aligned}
J^{\mu}_{\mathbf{k},\mathbf{k}'}(x) &= \frac{1}{2}qc\left(R^{\mu}_{\mathbf{k},\mathbf{k}'}(x) - C_{\mathrm{c}}R^{\mu}_{\mathbf{k},\mathbf{k}'}(x)C_{\mathrm{c}}^{-1}\right) \\
&= \frac{1}{2}qc\sum_{r,r'}\gamma^{\mu}_{r,r'}\left(\psi^{\mathrm{a}}_{r,\mathbf{k}}(x)\psi_{r',\mathbf{k}'}(x) - C_{\mathrm{c}}\psi^{\mathrm{a}}_{r,\mathbf{k}}(x)\psi_{r',\mathbf{k}'}(x)C_{\mathrm{c}}^{-1}\right) \\
&= \frac{1}{2}qc\Bigg(\sum_{r,r'}\gamma^{\mu}_{r,r'}\psi^{\mathrm{a}}_{r,\mathbf{k}}(x)\psi_{r',\mathbf{k}'}(x) \\
&\quad + \sum_{r,r'}\gamma^{\mu}_{r,r'}\sum_{r'',r'''}C_{r,r''}\psi_{r'',\mathbf{k}}(x)C_{r',r'''}\psi^{\mathrm{a}}_{r''',\mathbf{k}'}(x)\Bigg) \\
&= \frac{1}{2}qc\left(\sum_{r,r'}\gamma^{\mu}_{r,r'}\psi^{\mathrm{a}}_{r,\mathbf{k}}(x)\psi_{r',\mathbf{k}'}(x) + \sum_{r,r'}(C^{\mathrm{T}}\gamma^{\mu}C)_{r,r'}\psi_{r,\mathbf{k}}(x)\psi^{\mathrm{a}}_{r',\mathbf{k}'}(x)\right)
\end{aligned}
$$

Use that $C^{\mathrm{T}}\gamma^{\mu}C = -(\gamma^{\mu})^{\mathrm{T}}$ to obtain

$$
J^{\mu}_{\mathbf{k},\mathbf{k}'}(x) = \frac{1}{2}qc\sum_{r,r'}\gamma^{\mu}_{r,r'}\psi^{\mathrm{a}}_{r,\mathbf{k}}(x)\psi_{r',\mathbf{k}'}(x) - \frac{1}{2}qc\sum_{r,r'}\gamma^{\mu}_{r',r}\psi_{r,\mathbf{k}}(x)\psi^{\mathrm{a}}_{r',\mathbf{k}'}(x).
$$

This is (3.37).

Use the definition of the field operators to obtain

$$
\begin{aligned}
J^{\mu}_{\mathbf{k},\mathbf{k}'}(x) &= \frac{1}{2}qc\sum_{r,r'}(\gamma^0\gamma^{\mu})_{r,r'}\psi^{\dagger}_{r,\mathbf{k}}(x)\psi_{r',\mathbf{k}'}(x) \\
&\quad - \frac{1}{2}qc\sum_{r,r'}(\gamma^0\gamma^{\mu})_{r,r'}\psi_{r',\mathbf{k}}(x)\psi^{\dagger}_{r,\mathbf{k}'}(x) \\
&= qc\ell\sqrt{k_0k_0'}\sum_{r,r'}(\gamma^0\gamma^{\mu})_{r,r'}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\sum_{s=1,2}\overline{u_r^{(s)}(\mathbf{k})}\phi_{s,\mathbf{k}}^{(-)}(x)+\sum_{s=3,4}\overline{v_r^{(s)}(\mathbf{k})}\phi_{s,\mathbf{k}}^{(+)}(x)\right]\\
&\times\left[\sum_{s'=1,2}u_{r'}^{(s')}(\mathbf{k}')\phi_{s',\mathbf{k}'}^{(+)}(x)+\sum_{s'=3,4}v_{r'}^{(s')}(\mathbf{k}')\phi_{s',\mathbf{k}'}^{(-)}(x)\right]\\
-qc\ell\sqrt{k_0k_0'}&\sum_{r,r'}(\gamma^0\gamma^\mu)_{r,r'}\\
&\times\left[\sum_{s=1,2}u_{r'}^{(s)}(\mathbf{k})\phi_{s,\mathbf{k}}^{(+)}(x)+\sum_{s=3,4}v_{r'}^{(s)}(\mathbf{k})\phi_{s,\mathbf{k}}^{(-)}\right]\\
&\times\left[\sum_{s'=1,2}\overline{u_r^{(s')}(\mathbf{k}')}\phi_{s',\mathbf{k}'}^{(-)}(x)+\sum_{s'=3,4}\overline{v_r^{(s')}(\mathbf{k}')}\phi_{s',\mathbf{k}'}^{(+)}(x)\right].
\end{aligned}
\tag{F.1}
$$

The two terms of (F.1) are evaluated separately, with omission of the prefactor. One has

$$
\begin{aligned}
\text{First term} &= \sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\,\phi_{s,\mathbf{k}}^{(-)}(x)\phi_{t,\mathbf{k}'}^{(+)}(x)\\
&\quad+\sum_{s=1,2}\sum_{t=3,4}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle\,\phi_{s,\mathbf{k}}^{(-)}(x)\phi_{t,\mathbf{k}'}^{(-)}(x)\\
&\quad+\sum_{s=3,4}\sum_{t=1,2}\langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\,\phi_{s,\mathbf{k}}^{(+)}(x)\phi_{t,\mathbf{k}'}^{(+)}(x)\\
&\quad+\sum_{s,t=3,4}\langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle\,\phi_{s,\mathbf{k}}^{(+)}(x)\phi_{t,\mathbf{k}'}^{(-)}(x)\\
&= \sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\,\phi_{s,\mathbf{k}}^{(-)}(x)\phi_{t,\mathbf{k}'}^{(+)}(x)\\
&\quad+\sum_{s=1,2}\sum_{t=3,4}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle\,\phi_{s,\mathbf{k}}^{(-)}(x)\phi_{t,\mathbf{k}'}^{(-)}(x)\\
&\quad+\sum_{s=3,4}\sum_{t=1,2}\langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\,\phi_{s,\mathbf{k}}^{(+)}(x)\phi_{t,\mathbf{k}'}^{(+)}(x)\\
&\quad-\sum_{s,t=3,4}\langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle\,\phi_{t,\mathbf{k}'}^{(-)}(x)\phi_{s,\mathbf{k}}^{(+)}(x)\\
&\quad+\sum_{s=3,4}\langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(s)}(\mathbf{k}')\rangle\,e^{i(k'_\nu-k_\nu)x^\nu},
\end{aligned}
$$

and

$$
\begin{aligned}
\text{Second term} &= \sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle\,\phi_{t,\mathbf{k}}^{(+)}(x)\phi_{s,\mathbf{k}'}^{(-)}(x)\\
&\quad+\sum_{s=1,2}\sum_{t=3,4}\langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle\,\phi_{t,\mathbf{k}}^{(-)}(x)\phi_{s,\mathbf{k}'}^{(-)}(x)
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{s=3,4}\sum_{t=1,2} \langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle\, \phi^{(+)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x) \\
& + \sum_{s,t=3,4} \langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle\, \phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x) \\
= & - \sum_{s,t=1,2} \langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle\, \phi^{(-)}_{s,\mathbf{k}'}(x)\phi^{(+)}_{t,\mathbf{k}}(x) \\
& + \sum_{s=1,2} \langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(s)}(\mathbf{k})\rangle\, e^{i(k'_\nu - k_\nu)x^\nu} \\
& + \sum_{s=1,2}\sum_{t=3,4} \langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle\, \phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(-)}_{s,\mathbf{k}'}(x) \\
& + \sum_{s=3,4}\sum_{t=1,2} \langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle\, \phi^{(+)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x) \\
& + \sum_{s,t=3,4} \langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle\, \phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x).
\end{aligned}
$$

Next subtract the two contributions. It is shown in the Appendix D that

$$
\sum_{t=3,4} \langle v^{(t)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle \;=\; \sum_{s=1,2} \langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(s)}(\mathbf{k}')\rangle. \tag{F.2}
$$

Hence the two scalar terms cancel and one obtains

$$
\begin{aligned}
\text{First} - \text{second} = & \sum_{s,t=1,2} \langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\, \phi^{(-)}_{s,\mathbf{k}}(x)\phi^{(+)}_{t,\mathbf{k}'}(x) \\
& + \sum_{s,t=1,2} \langle u^{(s)}(\mathbf{k}'|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle\, \phi^{(-)}_{s,\mathbf{k}'}(x)\phi^{(+)}_{t,\mathbf{k}}(x) \\
& - \sum_{s,t=3,4} \langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle\, \phi^{(-)}_{t,\mathbf{k}'}(x)\phi^{(+)}_{s,\mathbf{k}}(x) \\
& - \sum_{s,t=3,4} \langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle\, \phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x) \\
& + \sum_{s=1,2}\sum_{t=3,4} \langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle\, \phi^{(-)}_{s,\mathbf{k}}(x)\phi^{(-)}_{t,\mathbf{k}'}(x) \\
& - \sum_{s=1,2}\sum_{t=3,4} \langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle\, \phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(-)}_{s,\mathbf{k}'}(x) \\
& + \sum_{s=3,4}\sum_{t=1,2} \langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\, \phi^{(+)}_{s,\mathbf{k}}(x)\phi^{(+)}_{t,\mathbf{k}'}(x) \\
& - \sum_{s=3,4}\sum_{t=1,2} \langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle\, \phi^{(+)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x).
\end{aligned}
$$

This result can be split into two pieces (3.38) and (3.39).

Appendix G

The continuity equation

Here follows the proof that $\hat{J}^{\mathrm{diag},\mu}(x)$ satisfies the continuity equation (3.40) and the symmetry property (3.41). The similar proofs for $\hat{J}^{\mathrm{off},\mu}(x)$ are omitted.

Use the explicit expression (3.38) to obtain

$$\begin{aligned}
i\partial_\mu J^{\mathrm{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x) \quad = \quad & -\frac{1}{2}\sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k}')\rangle(k_\mu - k'_\mu)\phi^{(-)}_{s,\mathbf{k}}(x)\phi^{(+)}_{t,\mathbf{k}'}(x) \\
& +\frac{1}{2}\sum_{s,t=3,4}\langle v^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k})\rangle(k_\mu - k'_\mu)\phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x) \\
& +\frac{1}{2}\sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k}')|\gamma^0\gamma^\mu u^{(t)}(\mathbf{k})\rangle(k_\mu - k'_\mu)\phi^{(-)}_{s,\mathbf{k}'}(x)\phi^{(+)}_{t,\mathbf{k}}(x) \\
& -\frac{1}{2}\sum_{s,t=3,4}\langle v^{(s)}(\mathbf{k})|\gamma^0\gamma^\mu v^{(t)}(\mathbf{k}')\rangle(k_\mu - k'_\mu)\phi^{(-)}_{t,\mathbf{k}'}(x)\phi^{(+)}_{s,\mathbf{k}}(x).
\end{aligned}$$

Each of the 4 contributions can be shown to vanish by use of the definition of the polarization vectors $u^{(s)}$ and $v^{(t)}$ as solutions of the equations (B.1, B.4).

Use the same relations to calculate

$$\begin{aligned}
k_\mu J^{\mathrm{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x) \quad = \quad & \frac{1}{2}\sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k})|\gamma^0 k_\mu\gamma^\mu u^{(t)}(\mathbf{k}')\rangle\phi^{(-)}_{s,\mathbf{k}}(x)\phi^{(+)}_{t,\mathbf{k}'}(x) \\
& -\frac{1}{2}\sum_{s,t=3,4}\langle v^{(s)}(\mathbf{k}')|\gamma^0 k_\mu\gamma^\mu v^{(t)}(\mathbf{k})\rangle\phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x) \\
& +(\mathbf{k}\leftrightarrow\mathbf{k}') \\
= \quad & \frac{1}{2}\kappa\sum_{s,t=1,2}\langle u^{(s)}(\mathbf{k})|\gamma^0 u^{(t)}(\mathbf{k}')\rangle\phi^{(-)}_{s,\mathbf{k}}(x)\phi^{(+)}_{t,\mathbf{k}'}(x) \\
& -\frac{1}{2}\kappa\sum_{s,t=3,4}\langle v^{(s)}(\mathbf{k}')|\gamma^0 v^{(t)}(\mathbf{k})\rangle\phi^{(-)}_{t,\mathbf{k}}(x)\phi^{(+)}_{s,\mathbf{k}'}(x)
\end{aligned}$$

$$+(\mathbf{k} \leftrightarrow \mathbf{k}').$$

For the part with $\mathbf{k}$ and $\mathbf{k}'$ exchanged one can use that the matrix $\gamma^0 k'_\mu \gamma^\mu$ is hermitian. The same expression is obtained when one calculates $k'_\mu J^{\mathrm{diag},\mu}_{\mathbf{k},\mathbf{k}'}(x)$. One concludes that (3.41) holds.