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Demand Systems under Rationing :
An Introduction with Special Reference to
the Implications of Separability Assumptions

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Abstract

In this paper we describe the nature of income, substitution and spill-over effects which characterize demand behaviour when a consumer experiences constraints on one or several markets. We do so by relying on the virtual price concept popularized by Neary & Roberts (1980). In contrast to that paper, a primal approach is followed.

Furthermore, we trace out the implications of putting more structure on preferences. In particular, it is shown that the reduction of spill-over effects to budget effects requires an assumption which is stronger than Weak Separability, except in trivial cases.

1. Introduction.

The purpose of this note is to give a clear description of the income, substitution and spill-over effects which characterize systems of demand equations when a consumer experiences constraints on one or several markets. Unlike in a market clearing context, the consumer will be forced to consume less or more than desired so that his marginal willingness to pay no longer coincides with the ruling prices on these markets. A nonzero virtual price divergence is the result which will have strong consequences for the entire structure of consumer behaviour.

Demand behaviour under rationing has been carefully studied by e.g. Bronsard & Salvas-Bronsard (1980) and Neary & Roberts (1980). Their results are of a general nature and apply to any well-defined preference order. The specification of rationed demand systems and knowledge of its properties is important in several aspects, inter alia for :

- consistent estimation of demand equations when disequilibria or institutional constraints prevail (cf Ashenfelter, 1980, Deaton, 1981);
- addressing questions such as public sector pricing (cf Drèze, 1984), optimal taxation (cf Kakwani & Ray, 1989) and tax reform (cf Wibaut, 1989) for economies in disequilibrium regimes;
- modelling the effects of public goods provision on consumer behaviour;
- evaluating the use of quotas or rations as policy instruments in a Second-Best world (cf Bronsard & Wagneur, 1982, Guesnerie & Roberts, 1984).

Demand systems under rationing can be defined as a member of entire class of problems by assigning a particular status to the variables entering the consumer's optimization problem. Table 1 illustrates. The commodity bundle q is split into subbundles q_A

and q_B, with a similar partitioning of the price vector, p. Four sets of variables thus emerge, the status of which depends on whether a set is treated as parametric or not.

Table 1. A Typology of Demand Systems.

	p_A	p_B	q_A	q_B	Type of Demand System
1	X	X	N	N	Regular
2	X	X	N	X	Constrained Regular
3	N	X	X	N	Mixed
4	N	N	X	X	Inverse
5	N	X	X	X	Constrained Inverse

X = Exogeneous, N = Endogeneous.

From a theoretical point of view, systems of type 1 and types 2 and 3 are equally valid representations of the agent's preference order and the choice of working with either of them may be purely pragmatic (see e.g. Stern, 1986). For econometric purposes, however, these three types of systems are not equivalent. Parameters of interest together with weak exogeneity considerations may impose one of these types as the appropriate parameterization of the system for consistent estimation[1] (see e.g. Barten & Bettendorf, 1989, and Barten, 1989).

In systems of types 2 and 5, however, additional side constraints are introduced. Econometric considerations for using an Inverse Demand System to model the demand for inelastically supplied goods can be further augmented when the price of some of these goods is subjected to minimum levels (e.g. some agricultural product prices), leading to the choice of model 5. Similarly, quantity constraints will indicate a system of type 2 as the appropriate model to work with.

One could argue that the presence of rationing, quotas or other institutional constraints imposes some specific structure on demand behaviour. It is a structure which is imposed from the institutional economic framework in which the consumer transacts.

A much older tradition in the literature on demand theory, however, is concerned with the characterization of the structure of preferences, and tracing out its implications for the structure of demand behaviour in a market clearing context (i.e. models of type 1). It is a structure imposed by the preference order, in addition to the already prevailing Slutsky Structure, and is known under the heading of Separability Theory (see e.g. Barten, 1977, Section 3, Deaton & Muellbauer, 1980, Ch. 5, and the references therein). The implications of this kind of literature, inter alia for commodity aggregation, optimal taxation problems and efficient estimation, are very powerful. One might expect that their simplifying strength carries over to the structure of demand behaviour under rationing. And this is indeed the impression one gets from the recent literature on rationed demand systems where it is conjectured that 'if a rationed good is weakly separable from the other ones, the spill-over effect of a change in the ration level on the freely chosen goods reduces to a mere income effect'. In particular it is argued that rationing affects the demand for other goods only via the available budget (see e.g. Deaton, 1981, Neary, 1987, Blundell, 1988); a result which is frequently relied on in both theoretical and empirical applications (see e.g. Drèze & Stern, 1987, Section 2.4, and Kaiser, 1990).

In this note we will show that the above mentioned conjecture is correct when preferences are characterized by Strong Separability, or by an equivalent form of Weak Separability, but <u>not</u> in general under Weak Separability.

The paper is structured as follows. Section 2 reviews standard demand theory (Type 1). The impact of rationing (Type 2) is derived in Section 3 using the primal approach, while the concept of Weak Separability is briefly discussed in Section 4. These sections set the stage and introduce notation for analyzing in detail the comparative statics of the model under Weak and Strong Separability (Sections 5 and 6). A brief conclusion ends the paper.

2. Demand Systems under Market Clearing : a Reminder.

Consider a commodity bundle $q\epsilon\mathbb{R}^n_+$. The preference order of the agent is assumed to be representable by a strictly monotonic and strongly quasi-concave utility function u(.), whose gradient and Hessian are resp. denoted by u_q and U_{qq} [2]. Let the corresponding price vector be given by $p\epsilon\mathbb{R}^n_+$ and autonomous income by m. The first order conditions of the consumer's maximization problem $\max_q\{u(q),\ s.t.\ p'q=m\}$ are then given by $u_q=\lambda p$ and $p'q=m$. After total differentiation, these first order conditions can be written in matrix notation as

$$\begin{bmatrix} U_{qq} & p \\ p' & 0 \end{bmatrix}\begin{bmatrix} dq \\ (-d\lambda) \end{bmatrix} = \begin{bmatrix} 0 & \lambda I \\ 1 & -q' \end{bmatrix}\begin{bmatrix} dm \\ dp \end{bmatrix}, \qquad (2.1)$$

also known as the Fundamental Matrix Equation of Consumer Demand (cf Barten & Bohm, 1982, Section 12). In (2.1) λ is the Lagrange multiplier and I is the identity matrix of dimension n. Since the strongly quasi-concave nature of u(.) ensures nonsingularity of the bordered Hessian on the LHS (see Barten, Kloek & Lempers, 1967), this system of equations may be solved for dq and dλ, resulting in[3]

$$\begin{bmatrix} dq \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda}K & q_m \\ q_m' & -\lambda_m \end{bmatrix} \begin{bmatrix} 0 & \lambda I \\ 1 & -q' \end{bmatrix} \begin{bmatrix} dm \\ dp \end{bmatrix}, \tag{2.2}$$

where the first matrix on the RHS is the inverse of the bordered Hessian. Hence the differential expression for demand behaviour looks like

$$dq = q_m(dm - q'dp) + Kdp, \tag{2.3}$$

Here q_m denotes the vector of income effects, while K refers to the matrix of substitution effects; clearly, both are evaluated at the tuple (m,p).

This Local Slutsky Structure has the following well known properties :

$$p'q_m = 1,\ p'K = 0', \quad (adding\text{-}up); \tag{2.4a}$$

$$Kp = 0, \quad (homogeneity); \tag{2.4b}$$

$$K = K', \quad (symmetry); \tag{2.4c}$$

$$\zeta'K\zeta < 0,\ \forall\ \zeta \neq \alpha p,\ \alpha \text{ real scalar}, \quad (negativity). \tag{2.4d}$$

The structure of system (2.2) and its properties (2.4) will reappear in the sequel.

3. <u>Demand Sytems under Rationing : The General Case</u>.

In this section we derive the general differential form of a system of demand equations with rationing prevailing on some markets. In analogy with Section 2, we do so by setting up the Fundamental Matrix Equation. For a dual approach, see Neary & Roberts (1980).

Let the commodity bundle q be partitioned as $(q_1',q_2')'$ where q_1 and q_2 have resp. dimensionality n_1 and n_2. Then in the case where the consumer is forced to set q_2 equal to x, say, this will have consequences for the optimal choice w.r.t. the unrationed set of commodities q_1, which we relabel as the n_1 vector z. Let us partition the (n_1+n_2) price vector p conformly as $p'=(p_z',p_x')$. Then the agent's decision problem can be formulated as $\max_q \{u(q)\ s.t.\ p'q = m,\ q_2 = x\}$, where the quantity constraint has been written as an equality as we assume it will be binding at equilibrium. The first order conditions can be stated as

$$u_z = \lambda p_z, \qquad (3.1a)$$

$$u_x = \lambda (p_x+\xi), \qquad (3.1b)$$

$$p_z'z+p_x'x = m, \qquad (3.1c)$$

where, as in section 2, λ is the Lagrange multiplier w.r.t. the budget constraint and

$$\xi = \frac{u_x}{\lambda}-p_x = p_x^v-p_x$$

denotes the deviation of the virtual price vector ($p_x^v \equiv u_x/\lambda$) from the market price vector (p_x) w.r.t. the rationed commodities. The virtual price vector refers to <u>that</u> set of prices which would induce an unrationed consumer to choose in a market clearing setting exactly the subbundle q_2 equal to x. The use of virtual prices to model consumer behaviour under rationing goes back to Rothbarth (1940-1) and their unique existence was shown in Neary & Roberts (1980). Note that $\lambda\xi$ equals the vector of Lagrange multipliers associated with the quantity constraints q_2=x.

If one is only interested in utility compensated behaviour, virtual prices are the only tools one needs. However, within the primal analysis we follow, the use of virtual prices requires an adjustment of income so as to allow this consumer to transact at these virtual prices. Therefore, <u>virtual income</u> is defined as

$$m^v \equiv m+\xi'x,$$

so that the budget constraint (3.1c) can be restated as

$$p_z'z+(p_x+\xi)'x = m^v = m+\xi'x \; . \tag{3.2}$$

The advantage of working with virtual prices and income is that one can approach the comparative statics analysis as a two step procedure. First one traces out the effects of a changing environment on virtual prices and income. Next, standard demand theory can be invoked to check how these changing prices and income affect the allocation.

We are now in a position to express the first order conditions in differential matrix form:

$$\begin{bmatrix} U_{zz} & U_{zx} & p_z \\ U_{zx} & U_{xx} & p_x^v \\ p_z' & p_x^{v\prime} & 0 \end{bmatrix} \begin{bmatrix} dz \\ dx \\ (-d\lambda) \end{bmatrix} = \begin{bmatrix} 0 & \lambda I & O \\ 0 & O & \lambda I \\ 1 & -z' & -x' \end{bmatrix} \begin{bmatrix} dm \\ dp_z \\ dp_x \end{bmatrix} + \begin{bmatrix} O \\ \lambda I \\ 0' \end{bmatrix} d\xi + \begin{bmatrix} O \\ O \\ \xi' \end{bmatrix} dx. \tag{3.3}$$

Invoking the same regularity theorem referred to in the previous section, eq. (3.3) can be multiplied through by the inverse of the LHS bordered Hessian , viz.

$$\begin{bmatrix} \frac{1}{\lambda}K_{zz} & \frac{1}{\lambda}K_{zx} & z_m \\ \frac{1}{\lambda}K_{xz} & \frac{1}{\lambda}K_{xx} & x_m \\ z_m' & x_m' & -\lambda_m \end{bmatrix}.$$

The system can now be solved for the endogeneous differentials dz, $d\xi$ and $d\lambda$ in terms of the exogeneous differentials dm, dp_z, dp_x, and dx. Doing so first for the virtual price divergence, one obtains

$$d\xi = K_{xx}^{-1}[dx - x_m(dm - q'dp) - K_{xz}dp_z - x_m\xi'dx] - dp_x.$$

Inserting this into the equation for dz yields:

$$dz = z_m^R(dm - q'dp) + K_{zz}^R dp_z + Q_{zx}^R dx. \tag{3.4}$$

In this expression the superscript R refers to a derivative of the constrained demand system. In particular we have

$$z_m^R = z_m - K_{zx}K_{xx}^{-1}x_m\ , \tag{3.5a}$$

$$K_{zz}^R = K_{zz} - K_{zx}K_{xx}^{-1}K_{xz}\ , \tag{3.5b}$$

$$Q_{zx}^R = K_{zx}K_{xx}^{-1} + z_m^R\xi', \tag{3.5c}$$

where $q_m' = (z_m', x_m')$ is the usual vector of income effects and where K_{ij} stands for block (i,j) of K, the matrix of substitution effects. Note that both of this vector and matrix are evaluated at virtual prices and income, i.e. at (p_z, p_x^v, m^v). Consequently, one may obtain from the homogeneity property $K(p_z', p_x^{v\prime})'=0$ the following expression for the virtual price divergence :

$$\xi = -K_{xx}^{-1}K_{xz}p_z - p_x, \tag{3.6}$$

which is only of an analytical interest since the $n_2 \times n_1$ matrix $K_{xx}^{-1}K_{xz}$ itself needs to be evaluated at the unknown virtual prices. The often encountered problem in applied studies of quantifying shadow prices therefore remains.

It is worth pointing out that also this rationed demand system is endowned with a Local Slutsky Structure:

Proposition 1 : The Constrained Demand System (3.4) is endowned with a Local Slutsky Structure, viz.

$$p_z' z_m^R = 1,\ p_z' K_{zz}^R = 0',\ p_z' Q_{zx}^R = -p_x' \quad (\textit{adding-up}), \tag{3.7a}$$

$$K_{zz}^R p_z = 0 \quad (\textit{homogeneity}), \tag{3.7b}$$

$$K_{zz}^R = K_{zz}^{R'} \quad (\textit{symmetry}), \tag{3.7c}$$

$$\zeta' K_{zz}^R \zeta < 0,\ \forall\ \zeta \neq \alpha p_z,\ \alpha \textit{ real scalar} \quad (\textit{negativity}). \tag{3.7d}$$

<u>Proof</u> : See Appendix.

Letting $z^R(p,m,x)$ be the constrained demand function for q_1, then the indirect utility function can be defined as $v(p,m,x)=u(z^R(p,m,x),x)$. Relying on the Slutsky properties it can then be easily checked that

$$\frac{\partial v}{\partial p} = -\lambda q,\ \frac{\partial v}{\partial m} = \lambda,\ \textit{and} \tag{3.8a}$$

$$\frac{\partial v}{\partial x} = \lambda\xi. \tag{3.8b}$$

Roy's identities then follow immediately :

$$\frac{\partial v}{\partial p}+\frac{\partial v}{\partial m}q = 0 \quad and \quad \frac{\partial v}{\partial x}-\frac{\partial v}{\partial m}\xi = 0. \tag{3.9}$$

Two conclusions may now be drawn. First, it is clear that the desirability of a change in the economic environment (du/λ) equals the adjustment of real income evaluated at virtual prices, viz. $p_z'dz+p_x^{v}{}'dx$ [4]. Second, eq. (3.8b) reveals that the $z_m^R\xi'$ part of the spill-over effect (3.5c) can be associated with a utility or income effect. Whence

$$Q_{zx}^R = K_{zx}^R+z_m^R\xi',$$

where $K_{zx}^R = K_{zx}K_{xx}^{-1}$ is the substitution effect of a changing quota.

Notice the duality between the Roy/Slutsky identities w.r.t. quantities and the more familiar ones w.r.t. prices.

Remarks:

(i) Consider the situation considered by Tobin & Houthakker (1950-1) where the quantity constraints initially 'just bite', i.e. the consumer's free choice w.r.t. the subvector q_2 exactly coincides with the exogeneous quota x. Then it is obvious that the virtual price vector p_x^v required to induce the choice of x, does not diverge from the market price vector p_x, whence $\xi=0$. In this case, the comparative statics expressions (3.5) remain valid, although the spill-over effects now reduce to the utility-compensated part K_{zx}^R;

(ii) The result in (i) constitutes one of the arguments in the paper by Guesnerie & Roberts (1984). A Planner in a Second-Best environment can impose "small" quota's or rations to an initially unconstrained agent. To a second order, this does not affect this agent's utility, cf eq. (3.8b). However, the net proceedings of this operation are non-zero since the Planner evaluates the bundle dx at social values which differ from effective market prices (for we are in a Second-Best world). Subject to the conditions of their Theorem 1, these authors show that a Pareto improvement is feasible when redistributing the net proceedings over all consumers;

(iii) Conditioning on the signs of $(K^R)_{i,j\epsilon q}$, it is possible to develop an alternative typology of Complements and Substitutes. See Madden (1990).

To end this Section, we would like to draw attention to the LeChatelier Principle that applies to the utility compensated demand functions for the free goods z (cf Samuelson, 1947, pp. 168-9). This principle refers to the result that the (compensated) effect of a shift in a parameter upon the solution of a decision problem is smaller under the presence of additional side constraints than under their absence. It follows from the Generalized Enveloppe Theorem (cf Beavis & Dobbs, 1990, p. 113-7).

Proposition 2: $K^R_{zz} - K_{zz}$+ *a positive definite matrix.*

Proof : From (3.5b) and the negative definite character of K^{-1}_{xx}.

Hence own substitution effects under rationing are of a smaller magnitude than the corresponding ones in a market clearing context (with the consumer facing the same but this time parametric virtual prices and income), viz.

$|(K_{zz})_{ii}| > |(K^R_{zz})_{ii}|, \forall i \epsilon z.$

4. The Case of Weak Separability.

In this Section we briefly review the results of demand theory under Weak Separability.Consider the case where the commodity bundle q can be partitioned into N subsets $(q_A', q_B', \ldots, q_N')'$ and where the utility index u(q) can be written (up to any monotone increasing transformation) as

$$u(q) = u^*[u_A(q_A), u_B(q_B), \ldots, u_N(q_N)].$$

Preferences which can be represented in such a way are said to be Weakly Seperable (cf Katzner, 1970).

Such a preference structure allows for a two stage budgetting process in the following sense. At an upper stage the allocation of total income m over the different commodity groups is determined, i.e.

$$m_F \equiv (p_F' q_F) = m_F(p, m), \tag{4.1}$$

where $p' = (p_A', p_B', \ldots, p_N')$. Here we may define the marginal propensity to spend on commodities of subset F as $\varepsilon_F \equiv \partial m_F / \partial m$,

satisfying the property $\Sigma_F \, \varepsilon_F = 1$.

The lower stage budgetting process consists of the allocation of m_F over the outlays on the different commodities of group F; the resulting demand system for group F can therefore be written as

$$q_F = q_F(p_F, m_F). \tag{4.2}$$

It is the solution to the optimization of the sub-problem $\max_{q_F}\{u_F(q_F),\ s.t.\ p_F' q_F = m_F\}$, and hence must be endowned with a Local Slutsky Structure. In analogy with the results in section 2, one may therefore write

$$dq_F = q_{eF}(dm_F - q_F' dp_F) + K^F dp_F\ , \tag{4.3}$$

where q_{eF} and K^F denote income and substitution effects referring to the sub-problem; they satisfy the properties (2.3) of Section 2.

On the other hand, there is no harm in interpreting q_F as part of the solution to the overall optimization problem $\max_q\{u(q),\ s.t.\ p'q = m\}$, so that an alternative expression for dq_F is

$$dq_F = q_{mF}(dm - q'dp) + \Sigma_G\, K_{FG} dp_G\ . \tag{4.4}$$

In this expression q_{mF} is block F from q_m, the usual vector of income effects; similarly, K_{FG} refers to block (F,G) of the overall matrix of substitution effects, K.

The relation between both types of Slutsky structures is of particular interest. It looks as follows :

$$q_{mF} = \varepsilon_F\, q_{eF}\ , \tag{4.5}$$

$$\begin{aligned} K_{FG} &= K^F + \psi_{FF}\, q_{eF} q_{eF}',\ \ for\ F=G, \\ &= \qquad \psi_{FG}\, q_{eF} q_{eG}',\ \ for\ F \neq G, \end{aligned} \tag{4.6}$$

where the (N×N) matrix Ψ with typical element $\psi_{FG}=\psi_{FG}(p,m)$ shares the following properties:

$$\Psi\iota = 0, \quad (\textit{homogeneity}), \tag{4.7a}$$

$$\Psi = \Psi', \quad (\textit{symmetry}), \textit{ and,} \tag{4.7b}$$

$$\mu'\Psi\mu < 0, \ \forall\, \mu\neq\alpha\iota,\ \alpha \textit{ real scalar}, \quad (\textit{negativity}), \tag{4.7c}$$

ι denoting the (N×1) vector of units. For a proof we refer again to the excellent survey of Barten & Böhm (1982, pp. 418-421).

Note that the analysis of this section remains true in a rationing context provided the Local Structures are evaluated at virtual prices and income.

5. Implications of Separability for Rationing.

Suppose now that the consumer is constrained w.r.t. the choice of the entire subbundle N, i.e. $q_N=x$ (at the end of this Section, something will be said on partial rationing of the bundle q_N). In the notation of Section 3, we now have $z'=(q_A', q_B', \ldots, q_{N-1}')$. Making use of results (3.4)-(3.5) we may write

$$dq_F = q_{mF}^R(dm-q'dp) + \Sigma_{G\neq N}\, K_{FG}^R dp_G + Q_{FN}^R dq_N\ , \ \forall\ F\neq N, \tag{5.1}$$

where

$$q_{mF}^R = q_{mF} - K_{FN}K_{NN}^{-1}q_{mN}\ , \tag{5.2a}$$

$$K^{R}_{FG} = K_{FG} - K_{FN}K^{-1}_{NN}K_{NG}, \quad G \neq N, \tag{5.2b}$$

$$Q^{R}_{FN} = K_{FN}K^{-1}_{NN} + q^{R}_{mP}\xi'_{N} \; . \tag{5.2c}$$

The implications of weakly separable preferences on the Local Slutsky Structure in the presence of rationing of a particular subset can then be seen by applying the properties (4.5)-(4.7). From (5.2) and (4.6) it is clear this will involve the inverse of the $(n_N \times n_N)$ matrix $K^{N} + \psi_{NN}\, q_{eN} q'_{eN}$. Since the matrix K^N has rank $(n_N - 1)$, one cannot apply the Bartlett Inverse Formula. Hence we present the following

Proposition 3 : Consider the (n×n) symmetric matrix $S = K + \psi x_e x'_e$ with the following properties :

(i) K has rank (n-1) and can be partitioned as $\begin{bmatrix} \tilde{K} & k \\ k' & \kappa \end{bmatrix}$, $\tilde{K}$ an (n-1)×(n-1) matrix of full rank ;

(ii) $\exists\, p \in \mathbb{R}^n_+$: (iia) $Kp = 0$ and (iib) $p'x_e = 1$;

(iii) p and x_e can be partitioned accordingly as $p' = (\tilde{p}', \pi)$, $x'_e = (\tilde{x}'_e, \chi_e)$, where $\tilde{p} \in \mathbb{R}^{n-1}_+$, and $\tilde{x}_e \in \mathbb{R}^{n-1}$.

Then

(iv) $S = M' \begin{bmatrix} \tilde{K} & 0 \\ 0' & \psi \end{bmatrix} M$, where $M = \begin{bmatrix} I & -\frac{1}{\pi}\tilde{p} \\ \tilde{x}'_e & \chi_e \end{bmatrix}$;

(v) $\det(M) = 1/\pi$, $M^{-1} = \begin{bmatrix} I - \tilde{p}\tilde{x}'_e & \tilde{p} \\ -\pi\chi_e & \pi \end{bmatrix}$, and

(vi) $x'_e M^{-1} = (0' \;\; 1)$.

<u>Proof</u> : See Appendix.

With this result and properties (4.5)-(4.7) in mind, it is easy to establish that expressions (5.2a)-(5.2b) reduce to

$$q^R_{mF} = (\varepsilon_F - \varepsilon_N \frac{\psi_{FN}}{\psi_{NN}}) q_{eF} , \tag{5.3a}$$

$$K^R_{FG} = (\psi_{FG} - \frac{\psi_{FN}\psi_{NG}}{\psi_{NN}}) q_{eF} q'_{eG}, \text{ for } F \neq G,\ G \neq N, \tag{5.3b}$$

$$K^R_{FF} = K^F + (\psi_{FF} - \frac{\psi^2_{FN}}{\psi_{NN}}) q_{eF} q'_{eF} . \tag{5.3c}$$

The spillover effects (5.2c) specialize to

$$Q^R_{FN} = \psi_{FN}\, q_{eF} p^{v\prime}_N + (\varepsilon_F - \varepsilon_N \frac{\psi_{FN}}{\psi_{NN}}) q_{eF} (p^v_N - p_N)' \tag{5.3d}$$

where the RHS terms denote resp. the substitution and income effects.

Another way to write the spill-over effect is

$$Q^R_{FN} = (q^R_{mF} + \frac{\psi_{FN}}{\psi_{NN}} q_{eF}) p^{v\prime}_N - q^R_{mF} p'_N , \tag{5.3}$$

or

$$Q^R_{FN} = q^R_{mF} (\beta_{FN}\, p^v_N - p_N)' dq_N, \quad \text{where } \beta_{FN} = 1 + \frac{\psi_{FN}}{\psi_{NN}\varepsilon_F - \psi_{FN}\varepsilon_N} . \tag{5.4}$$

From this last expression, it becomes clear that <u>as long as β_{FN} is non-zero, the impact of rationing is not only via the budget available</u>, i.e. $(m - p_N' q_N)$. The next section indicates when this will be the case.

Two remarks conclude the present Section :

(i) The negative conclusion above becomes quite intuitive when we rule out all relative price changes within each group (i.e. $dp_G \propto p_G$ s.t. $K^G dp_G=0$). In that case group-outlays can be considered as Hicksian aggregate commodities, whose Local Slutsky Structure is given by the terms in round brackets in expressions (5.3). One then appreciates the analogy with these terms and the ones in (3.5) where no separability assumptions where introduced;

(ii) For simplicity and ease of presentation, we only considered the case where the consumer is rationed w.r.t. his consumption of the entire subbundle q_N. Nevertheless the interest in the more general case with partial rationing of a subbundle may be more than purely academic: think of dual labour market situations with excess supply of only high quality labour services. In this case it can be shown that, apart from the appearance of a correction factor in the ratio terms, the structure of expressions (5.3) remain valid. One has of course the additional 'links' between the free and rationed parts of subbundle q_N; these are then, up to the same correction factor, similar to the expressions (3.5).

6. Strongly Separable Preferences.

Preferences are said to be strongly separable when the utility index can be written as

$$u(q) = v_A(q_A)+v_B(q_B)+\ldots+v_N(q_N) \ , \qquad (6.1)$$

or as any monotone increasing transformation of (6.1). It can be shown that under this preference structure, the coefficients ψ_{FG} take the form

$$\begin{aligned} \psi_{FF} &= -\tau \ \varepsilon_F(1-\varepsilon_F) \ , \\ \psi_{FG} &= \ \tau \ \varepsilon_F \varepsilon_G \quad , F \neq G, \end{aligned} \qquad (6.2)$$

where $\tau=\tau(p,m)$ is a factor which is not group specific (a proof can be found in Barten & Böhm, 1982, pp. 422).

Both terms of the spill-over effect (5.3d) now further specialize to

$$Q_{FN} = -\frac{\varepsilon_F}{1-\varepsilon_N} q_{eF} p_N^{v\prime} + \frac{\varepsilon_F}{1-\varepsilon_N} q_{eF} (p_N^v - p_N)'.$$

Consequently, Strong Separability implies that the substitution effect exactly cancels with the virtual price term of the income effect [or in terms of eq. (5.4): $\beta_{FN}=0$]; only a budget effect remains.

This result applies as well in the Weak Separability case when the set of commodities can only be partitioned into two subsets, F and N, say; for then $\varepsilon_F+\varepsilon_N=1$ and $\psi_{FF}+\psi_{FN}=0$. This is not surprising since Weak and Strong Separability amount to the same thing with only two subsets of commodities. Indeed any utility index of the form $u(q)=u^*[u_F(q_F),u_N(q_N)]$ can be transformed into $v(q)=F[u(q)]=v_F(q_F)+v_N(q_N)$ using the transformation described by Samuelson (1947, p. 178). Inference based on the textbook model (where q_F and q_N are scalars) is therefore already very restrictive as no degrees of freedom are left under rationing.

Finally, note that also under Strong Separability, one cannot get around the unobservable virtual prices to calculate the lump sum transfers necessary to compensate a consumer affected by a tightening of the rationing constraints, viz.

$$dm = -\xi'_N dq_N.$$

5. Conclusion.

In this note we have attempted to give a precise characterization of income, price, and spill-over effects for rationed demand systems under alternative assumptions w.r.t. the preference order. In this way, insight has been gained in the way structures of preferences further shape those imposed by the market (in casu rationing). Starting from the general model we specified the comparative statics results for both Weakly and Strongly Separable preferences.

The main conclusion of the analysis is that the strength of the Weak Separability hypothesis has often been overstated in the literature on consumer behaviour under rationing. Under either preference structure the rationing effect can be neutralized by an appropriate lump sum tax/transfer. However, of both hypotheses, only Strong Separability provides a sufficient condition for a tightening or loosening of the ration constraint to affect the allocation over the freely choosen commodities only via a change in the available budget.

In view of the severe reservations that are often made w.r.t. Strong or Additive Separability assumptions in both the literature on labour supply and intertemporal decisions, this means one should be more careful in dismissing rationing effects as budget effects.

Notes.

[1] We have in mind estimation on aggregate data. But clearly when (some of) the supply curves are not perfectly elastic or inelastic fitting either of these three types of models will not result in consistent estimation. One then should have recourse to Limited Information methods.

[2] Matrices will be denoted by capitals, vectors by lower case letters. Vectors are always column vectors and a prime indicates transposition.

[3] A shortcut is taken here. Let the inverse of the bordered Hessian be the (n+1)×(n+1) matrix

$$\begin{bmatrix} V & v \\ v' & \nu \end{bmatrix}.$$

Let the Jacobian of the system of (n+1) equations q(p,m) and -λ(p,m) be given by the (n+1)×(n+1) matrix

$$\begin{bmatrix} q_m & Q_p \\ -\lambda_m & -\lambda'_p \end{bmatrix}.$$

Equation (2.1) then implies the following identity :

$$\begin{bmatrix} q_m & Q_p \\ -\lambda_m & -\lambda'_p \end{bmatrix} = \begin{bmatrix} V & v \\ v' & \nu \end{bmatrix} \begin{bmatrix} 0 & \lambda I \\ 1 & -q' \end{bmatrix}.$$

Therefore, the vector of income effects and the matrix of price effects satisfy the following identities :

$$q_m = v, \quad and \quad Q_p = \lambda V - q_m q'.$$

The second matrix equation is the Slutsky decomposition of the price effect into the substitution and income effect. Defining the substitution term as K then motivates the writing of expression (2.2).

[4] From the equations (3.8) dv is given by $\lambda(dm-q'dp+\xi dx)$. But as $dm-q'dp=p'dq$, $dv=\lambda(p_z'dz+p_x^{v}{}'dx)$ follows.

Appendix.

Proof of Proposition 1 :

(3.7a), (3.7b) and (3.7c) follow from the fact that, when evaluated at the virtual prices and income the Local Slutsky Structure satisfies the properties (2.3). E.g.

$$p_z' z_m + p_x^{v'} x_m = 1, \text{ and } p_z' K_{zx} + p_x^{v'} K_{xx} = 0',$$

imply $p_z' z_m^R = 1$, etc.

To prove negativity, note that from (2.4d) we have

$$[\zeta_z', \zeta_x'] \begin{bmatrix} K_{zz} & K_{zx} \\ K_{xz} & K_{xx} \end{bmatrix} \begin{bmatrix} \zeta_z \\ \zeta_x \end{bmatrix} < 0, \ \forall \begin{bmatrix} \zeta_z \\ \zeta_x \end{bmatrix} \neq \alpha \begin{bmatrix} p_z \\ p_x^v \end{bmatrix}.$$

Choosing $\zeta_x = -K_{xx}^{-1} K_{xz} \zeta_z$ results in $\zeta_z' K_{zz}^R \zeta_z < 0$.

However, for $\zeta_z = \alpha p_z \Rightarrow \zeta_x = -\alpha K_{xx}^{-1} K_{xz} p_z = \alpha p_x^v \Rightarrow \zeta' = \alpha [p_z', p_x^{v'}]$,

which is not allowed. Therefore $\zeta_z \neq \alpha p_z$ ■

Proof of Proposition 3 :

(iv) Making use of property (iia), the matrix S can be rewritten as

$$S = \begin{bmatrix} \tilde{K} & -\frac{1}{\pi}\tilde{K}\tilde{p} \\ -\frac{1}{\pi}\tilde{p}'\tilde{K} & \frac{1}{\pi^2}\tilde{p}'\tilde{K}\tilde{p} \end{bmatrix} + \psi \begin{bmatrix} \tilde{x}_e\tilde{x}_e' & \chi_e\tilde{x}_e \\ \chi_e\tilde{x}_e' & \chi_e^2 \end{bmatrix},$$

which can be decomposed as

$$\begin{bmatrix} I & \tilde{x}_e \\ -\frac{1}{\pi}\tilde{p}' & \chi_e \end{bmatrix} \begin{bmatrix} \tilde{K} & 0 \\ 0' & \psi \end{bmatrix} \begin{bmatrix} I & -\frac{1}{\pi}\tilde{p} \\ \tilde{x}_e' & \chi_e \end{bmatrix} = M'\grave{K}M,$$

where $\grave{K}$ is non-singular since K has rank n-1.

(v) Det(M)= $|\chi_e+\frac{1}{\pi}\tilde{x}_e'\tilde{p}|\cdot|I| = \frac{1}{\pi}$ because of (iib). Making use of the

formula for partitionned matrix inversion and exploiting (iib) yields

$$M^{-1} = \begin{bmatrix} (I+\frac{1}{\pi\chi_e}\tilde{p}\tilde{x}_e')^{-1} & \tilde{p} \\ -\pi\chi_e & \pi \end{bmatrix}.$$

Applying the Bartlett Inverse Formula to the NW-block, one obtains

$$M^{-1} = \begin{bmatrix} I-\tilde{p}\tilde{x}_e' & \tilde{p} \\ -\pi\tilde{x}_e' & \pi \end{bmatrix}.$$

(vi) Using once more the adding-up property (iib), it is easy to check that

$x_e'M^{-1} = (0' \ 1)$ ■

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