STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

SOME IMPLICATIONS OF A NEW DISTANCE FUNCTION FOR APPLIED WELFARE ECONOMICS

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A B S T R A C T

The purpose of this paper is to elaborate on the implications of a recently proposed distance function for duality theory and to show the usefulness of the new concept for applied welfare economics.

It is shown that the dual relation between the distance function and the expenditure function implies a simple procedure to derive the direct utility function from a given arbitrary but valid expenditure function. Moreover, the properties of the distance function are used to show that in some cases it is straightforward to derive the underlying direct utility function that corresponds to a given Marshallian demand function.

The distance function is further used to define and to evaluate Hicks' compensating and equivalent surplus. These welfare measures are particularly useful to analyze government policies that deal with public goods or operate in quantity constrained regimes. We propose a series of analytical and numerical procedures to determine exact surplus measures of a welfare change depending upon the information the researcher has available. An algorithm is presented that allows us to calculate the Hicksian surplus measures using only the system of ordinary demand functions.

0. Introduction

In a recent paper, W. Pauwels (1985) proposed a new distance function and indicated how this could be used to define Hicksian surplus measures in a straightforward way. The purpose of this paper is to elaborate on the implications of the distance function for duality theory and to show the usefulness of the new concept for applied welfare economics.

Organization of the paper is as follows. In order to make the paper self-contained we present the distance function in the first section and review its most important properties. It will become clear that it is similar to, but distinctly different from, the distance function described by Deaton (1979). We show that it can be considered to be dual to the expenditure function in a well-defined sense. This duality is explored in Section 2. Among other things, we show that it implies a procedure to derive the direct utility function from an arbitrary valid expenditure function, provided that a closed-form expression for the system of inverse compensated demand functions exists. Moreover, we indicate that in some cases it is straightforward to find the direct utility function that corresponds to a given inverse Marshallian demand function. These issues have not been carefully investigated in the literature. In Section 3 we use the distance function to define Hicks' equivalent and compensating surplus, which are important tools in applied welfare economics, as they are particularly suited for evaluating policies involving changes in imposed quantities. These policies may arise in the case of public goods and under a wide variety of government programs. Section 4 deals with analytical and numerical solution methods to evaluate the Hicksian surplus measures. Alternative procedures are developed depending upon the information the investigator has available. The distance function is used to generalize both Hausman's (1981) analytical and Vartia's (1983) numerical procedures for calculating exact welfare measures on the basis of market demand functions to quantity-constrained regimes. Finally, Section 5 summarizes the main findings of this paper.

1. A new distance function[1]

In this section we present the distance function proposed by Pauwels (1985) and review some useful properties. It should be noted that the distance function we consider is similar to the one suggested by Deaton (1979). However, by using a different normalization our function has a straightforward interpretation, which makes it a very convenient tool for both theoretical and applied economics.

Assume a consumer's preferences can be represented by a direct utility function $u(z,x)$, where x is a vector in R^n_+ and z is a numeraire commodity with unit price equal to one. For any vector $x \in R^n_+$ and any utility level v the distance function $F(x,v)$ is defined by the implicit equation

$$u\big(F(x,v),x\big) = v \tag{1.1}$$

The function $F(x,v)$ gives, for any vector of commodities x, the quantity of z that is required for the consumer in order to attain a given utility level v. It follows that

$$u(z,x) = v \quad \text{iff} \quad z = F(x,v) \tag{1.2}$$

Moreover, we have

$$\forall(x,z) \in R^{n+1}_+, \; F\big(x,u(z,x)\big) = z \tag{1.3}$$

Assuming positive marginal utilities for x and z it can easily be shown that $F(x,v)$ is decreasing in x and increasing in v (Pauwels (1985, p. 5)).

A crucial property of the distance function is that it can be obtained as the optimal value of the objective function of a simple maximization pro-

[1]Throughout this paper vectors are treated as column vectors. Row vectors are treated as transposed column vectors. The transpose of a matrix A will be denoted by A'.

blem. To see this, denote the expenditure function - giving the minimum expenditures required to attain utility level v when facing a vector of prices p - by e(p,v). Now consider any vector $x \in R^n_+$ and any utility level v. The difference [e(p,v) - p'x] is the quantity of z the consumer can buy after consuming x at prices p. For arbitrarily chosen p and x, the quantity [e(p,v)-p'x] of z will normally be insufficient for the consumer to attain utility level v through the bundle [e(p,v)-p'x,x]. Since F(x,v) is by definition the quantity of z required to attain utility level v, provided that x is consumed, we have, for arbitrary x and v,

$$\forall\ p \in R^n_+,\ F(x,v) \geq e(p,v) - p'x \tag{1.4}$$

Following Gorman (1976) and Deaton (1979) we say that vectors $\tilde{p}$ and $\tilde{x}$ are conjugate at v iff the consumption of $\tilde{x}$ combined with the quantity $[e(\tilde{p},v)-\tilde{p},\tilde{x}]$ of z allows the consumer to attain utility level v. Consequently,

$$F(\tilde{x},v) = e(\tilde{p},v) - \tilde{p}'\tilde{x} \tag{1.5}$$

Using (1.4) and (1.5) we obtain

$$F(x,v) = \underset{p \in R^n_+}{\text{Max}} [e(p,v) - p'x] \tag{1.6}$$

The distance function is illustrated in figures 1 and 2. In figure 1 we derive the function F(x,v) in (z,x) space, where x is now a scalar. Consider arbitrarily chosen price $\bar{p}$ and quantity x^0. It is obvious that the bundle $[e(\bar{p},v)-\bar{p}x^0,x^0]$ does not yield v. Given x^0 we obtain $F(x^0,v)$ by choosing price p*. The bundle $[e(p^*,v)-p^*x^0,x^0]$ does yield utility level v. That p* is the price which maximizes $[e(p,v)-px^0]$ is obvious both from figure 1 and figure 2. The latter illustrates the derivation of the distance function and the fundamental property (1.6) in expenditure-price space.

A simple argument can also be used to show that the expenditure function can be derived from the distance function by solving a minimization problem which is closely related to (1.6). Consider any vectors p and

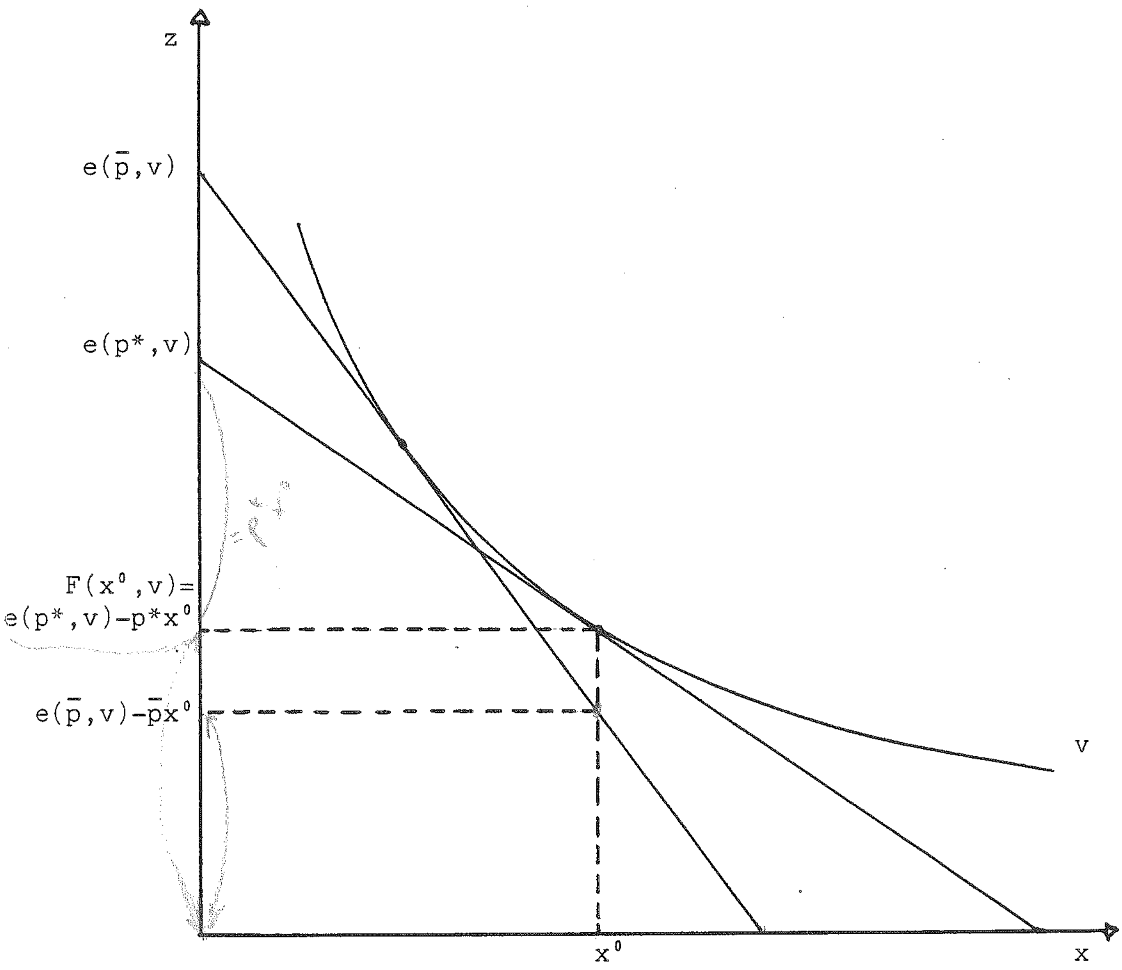

figure 1 : illustration of the distance function in (z,x) space

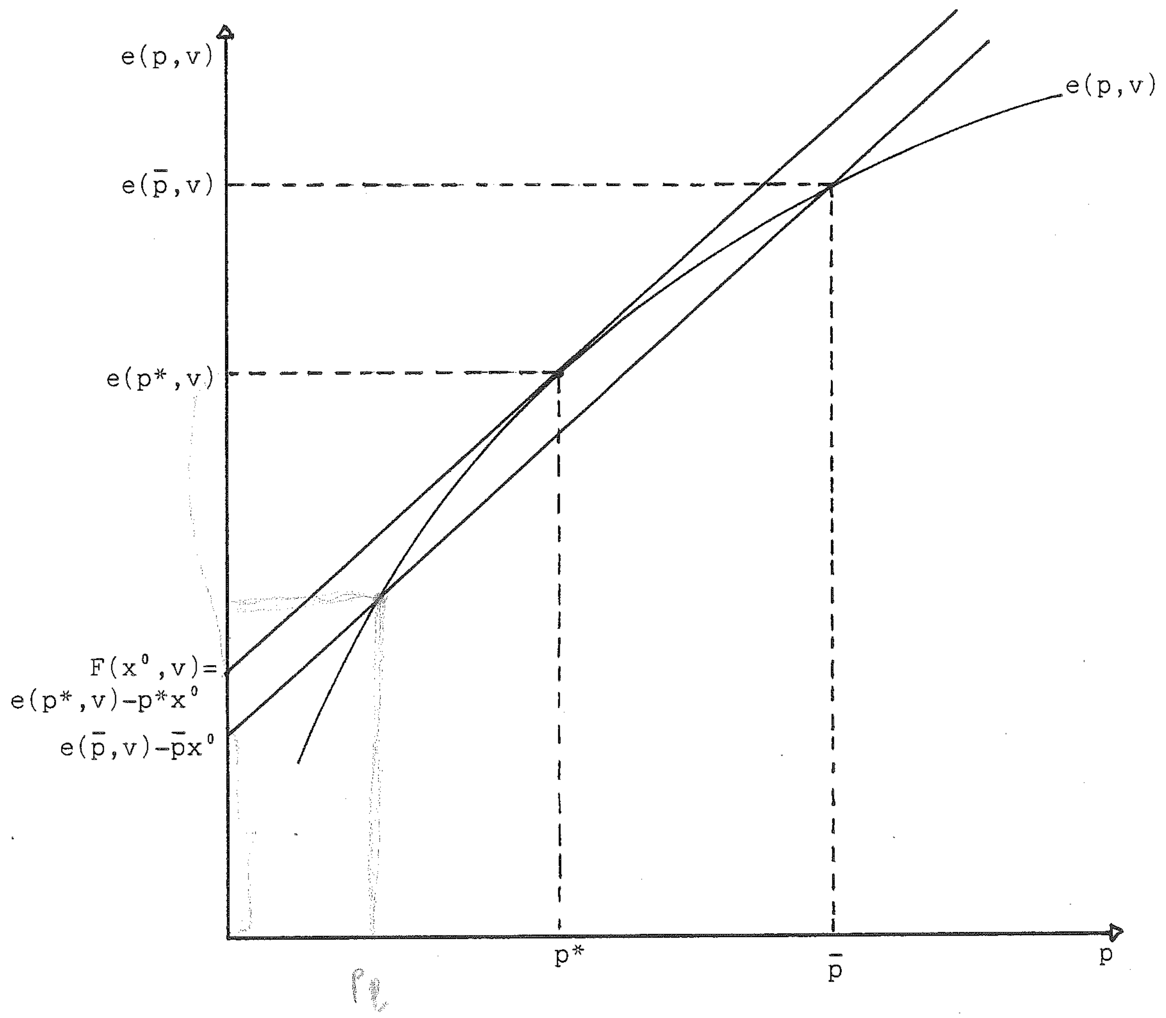

figure 2 : illustration of the distance function in expenditure-price space

$x \in R^n_+$ and any utility level v. The cost of bundle $[F(x,v),x]$ is $[F(x,v)+p'x]$. Given an arbitrarily chosen price this need not be the cheapest bundle to attain utility level v. Consequently, for any p and v, we have

$$\forall\ x \in R^n_+,\ e(p,v) \leq F(x,v) + p'x \tag{1.7}$$

However, if $\tilde{x}$ and $\tilde{p}$ are conjugate at v it follows from (1.5) that the cheapest bundle to attain v at prices $\tilde{p}$ is $[F(\tilde{x},v),\tilde{x}]$, viz.

$$e(\tilde{p},v) = F(\tilde{x},v) + \tilde{p}'\tilde{x} \tag{1.8}$$

Combining (1.7) and (1.8) we find

$$e(p,v) = \underset{x \in R^n_+}{\text{Min}} [F(x,v) + p'x] \tag{1.9}$$

The expenditure function e(p,v) and the distance function F(x,v) are dual to one another in the sense defined by equations (1.6) and (1.9).

Applying the envelope theorem to (1.9) yields the familiar result that the vector of compensated demand functions can be obtained by simple differentiation of the expenditure function with respect to the price vector. It is also obvious that the vector x solving the minimization problem at the right-hand side of (1.9) is just the vector of compensated demands evaluated at prices p and utility v.

Similarly, it can easily be shown that differentiation of the distance function with respect to x yields the vector of inverse compensated demand functions. Applying the envelope theorem to (1.6) implies

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$$\frac{\delta F(x,v)}{\delta x} = - p^c(x,v) \tag{1.10}$$

The price vector solving the optimization problem in (1.6) is obviously just the vector of inverse compensated demands evaluated at quantities x and utility level v.

A further useful property of $F(x,v)$ can be derived by applying the implicit function theorem to (1.1). We obtain

$$\frac{\delta F(x,v)}{\delta x} = - \frac{\frac{\delta u\left(F(x,v),x\right)}{\delta x}}{\frac{\delta u\left(F(x,v),x\right)}{\delta z}} \tag{1.11}$$

where, of course, $z = F(x,v)$. Combining (1.10) and (1.11) yields

$$\frac{\frac{\delta u\left(F(x,v),x\right)}{\delta x}}{\frac{\delta u\left(F(x,v),x\right)}{\delta z}} = p^c(x,v) \tag{1.12}$$

Equation (1.12) shows that, for a given vector x and utility level v, the i-the component $p_i^c(x,v)$ of the vector $p^c(x,v)$ is to be interpreted as the marginal willingness to pay for commodity i as it exists in the bundle $\left(F(x,v),x\right)$. Graphically, $p^c(x,v)$ can be interpreted in two-dimensional space as the slope of the indifference curve in the point $\left(x,F(x,v)\right)$.

Using the fact that $v = u(z,x)$ and $z = F(x,v)$ it also follows from (1.12) that

$$\frac{\frac{\delta u(z,x)}{\delta x}}{\frac{\delta u(z,x)}{\delta z}} = p^c\left(x,u(z,x)\right) = p^m(z,x) \tag{1.13}$$

The vector $p^m(z,x)$ is the vector of inverse Marshallian demand functions, expressing demand prices as a function of the quantities of z and x.[1]

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[1] The term 'inverse' Marshallian demand functions results from the fact that the (n+1)-equation system of ordinary Marshallian demand functions

$$\left.\begin{aligned} x &= x^m(p,y) \\ z &= y - p'x^m(p,y) \end{aligned}\right] \quad (A)$$

and the system

$$\left.\begin{aligned} p &= p^m(z,x) \\ y &= p^m(z,x)'x + z \end{aligned}\right] \quad (B)$$

are each other's inverse (W. Pauwels (1985, p. 18), also see Katzner (1970) and W. Barnett (1977, p. 1132 - 1133)). For a given vector (z,x) the solution of (A) is given by (B). Similarly, for a given vector (p,y) the solution of (B) is given by (A). The empirical relevance of system (B) has been emphasized by Barnett (1977). Also note that the vector $p^m(z,x)$ can be obtained by solving the system $x = x^m(p,y)$ for prices as a function of x and y and using the budget constraint $y = p'x + z$ to obtain price as a function of z and x.

Equation (1.13) shows that $p^m(z,x)$ should be interpreted as the marginal rate of substitution between z and x as it exists in the point (z,x). Graphically, $p^m(z,x)$ can be interpreted in two-dimensional space as the slope of the indifference curve through the bundle (z,x) in the point (x,z).

A final interesting relation to be derived in this section is a 'Slutsky equation' as applied to inverse demand systems[2]. Note that by definition

$$p^m\big(F(x,v),x\big) = p^c(x,v) \tag{1.14}$$

The desired equation can be obtained by differentiating both sides of (1.14) with respect to x and using (1.10). We find

$$\frac{\delta p^m(z,x)}{\delta x} = \frac{\delta p^c(x,v)}{\delta x} + \frac{\delta p^m(z,x)}{\delta z}\, p^m(z,x)' \tag{1.15}$$

where z = F(x,v).

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[2] For an alternative formulation of the 'Slutsky equation' see R. W. Anderson (1980).

2. Implications of the duality between the distance and the expenditure functions

The duality between the direct utility function $u(z,x)$ and the expenditure function $e(p,v)$ has been discussed at length in the literature. It leads to a set of important relations between $u(z,x)$, $e(p,v)$, the indirect utility function $v(p,y)$, the Marshallian demand functions $x^m(p,y)$ and the Hicksian demand functions $x^c(p,v)$. A carefully written overview is in Deaton and Muellbauer (1980, p. 37 - 42). In this section we summarize the relations between the functions $F(x,v)$, $u(z,x)$, $e(p,v)$, $v(p,y)$ and the Marshallian and Hicksian inverse demand functions $p^m(x,z)$ and $p^c(x,v)$ that follow directly from the properties of the distance function, described in the previous section. Moreover, we indicate the potential usefulness of the duality between $e(p,v)$ and $F(x,v)$.

It should be pointed out that the distance function that has been proposed in the literature (see, e.g., Gorman (1970, 1976), Diewert (1976), Blackorby, Lovell and Thursby (1976) and Deaton (1979)) leads to a series of quite similar relations.[1] These are briefly described in Deaton and Muellbauer

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[1] The distance function common in the literature is not defined in terms of the numeraire commodity z. Let the utility function be $u(x)$. Then the distance function proposed in the literature, $d(v,x)$, is defined as the amount by which x must be divided in order to yield a given utility level v. It is defined by the implicit equation $u(\frac{x}{d(v,x)}) = v$. Consequently $v = u(x)$ iff $d(v,x) = 1$. It can be shown that the following constrained optimization problems establish the duality between the distance function $d(v,x)$ and the expenditure function $e(p,v)$

$$d(v,x) = \underset{p}{\text{Min}}(p'x) \text{ subject to } e(p,v) = 1$$

$$e(p,v) = \underset{x}{\text{Min}}(x'p) \text{ subject to } d(v,x) = 1$$

For more details, see Deaton (1979, p. 392 - 396).

→ d.i. in feite de "numéraire" constraint

→ Dit is precies mijn voorstel: de hier ontwikkelde D-fct is enkel geldig als een numéraire met prijs ≠ 1 ingevoerd wordt (hetgeen m.i. geen bezwaar is omdat dit de praktische bruikbaarheid niet schaadt; zelfs verhoogt!)

(1980, p. 54 - 56). However, the definition of the distance function $F(x,v)$ in terms of a numeraire commodity z is more straightforward and definitely more convenient for ~~our purposes~~. The simple duality structure described by equations (1.6) and (1.9) yields the functions $F(x,v)$ and $e(p,v)$ as the optimal values of the objective function of <u>unconstrained</u> optimization problems. This property has some useful applications, to be discussed below.

Important relations are summarized in figure 3. The left branch shows how the solution of problem (1.6) yields the vector of inverse compensated demands $p^c(x,v)$ and, by substitution, the distance function $F(x,v)$. Inversion of $F(x,v)$ leads to the direct utility function $u(z,x)$. Application of (1.13) finally yields the inverse Marshallian demand functions $p^m(z,x)$. The right branch starts from problem (1.9) to obtain $x^c(p,v)$ and the expenditure function $e(p,v)$. Then the well-known inversion to the indirect utility function $v(p,y)$ and application of Roy's theorem to find Marshallian demands follow.

Figure 3 suggests a straightforward way to recover the direct representation of preferences $u(z,x)$, provided that it can be written as an explicit function of z and x, from a given arbitrary but valid indirect utility or expenditure function[1]. The procedure is to solve the problem $\underset{p}{\mathrm{Max}}[e(p,v)-p'x]$ to find $F(x,v)$ and to obtain $u(z,x)$ by inversion, using $F\big(x,u(z,x)\big) = z$. This simple method will yield $u(z,x)$ provided that the inverse compensated demand functions that solve the maximization problem can be written as explicit functions of x and v.[2]

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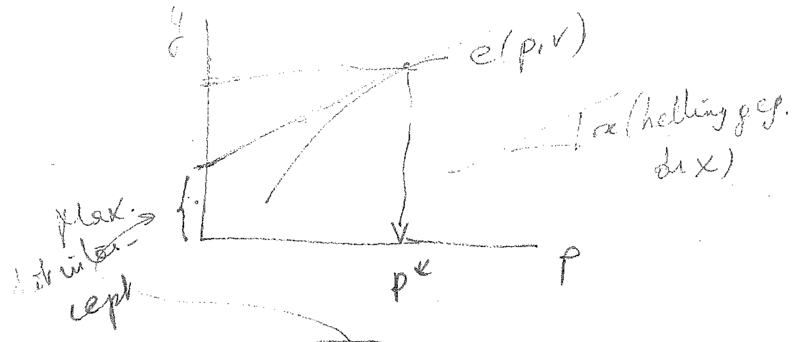
[1] Obviously, the direct utility function can only be found up to a monotonic transformation.

[2] A similar problem exists in the case of deriving $v(p,y)$ from $u(z,x)$. It is only possible to write the indirect utility function $v(p,y)$ analytically if the Marshallian demand functions that solve the constrained utility maximization problem can be written as explicit functions of prices and income.



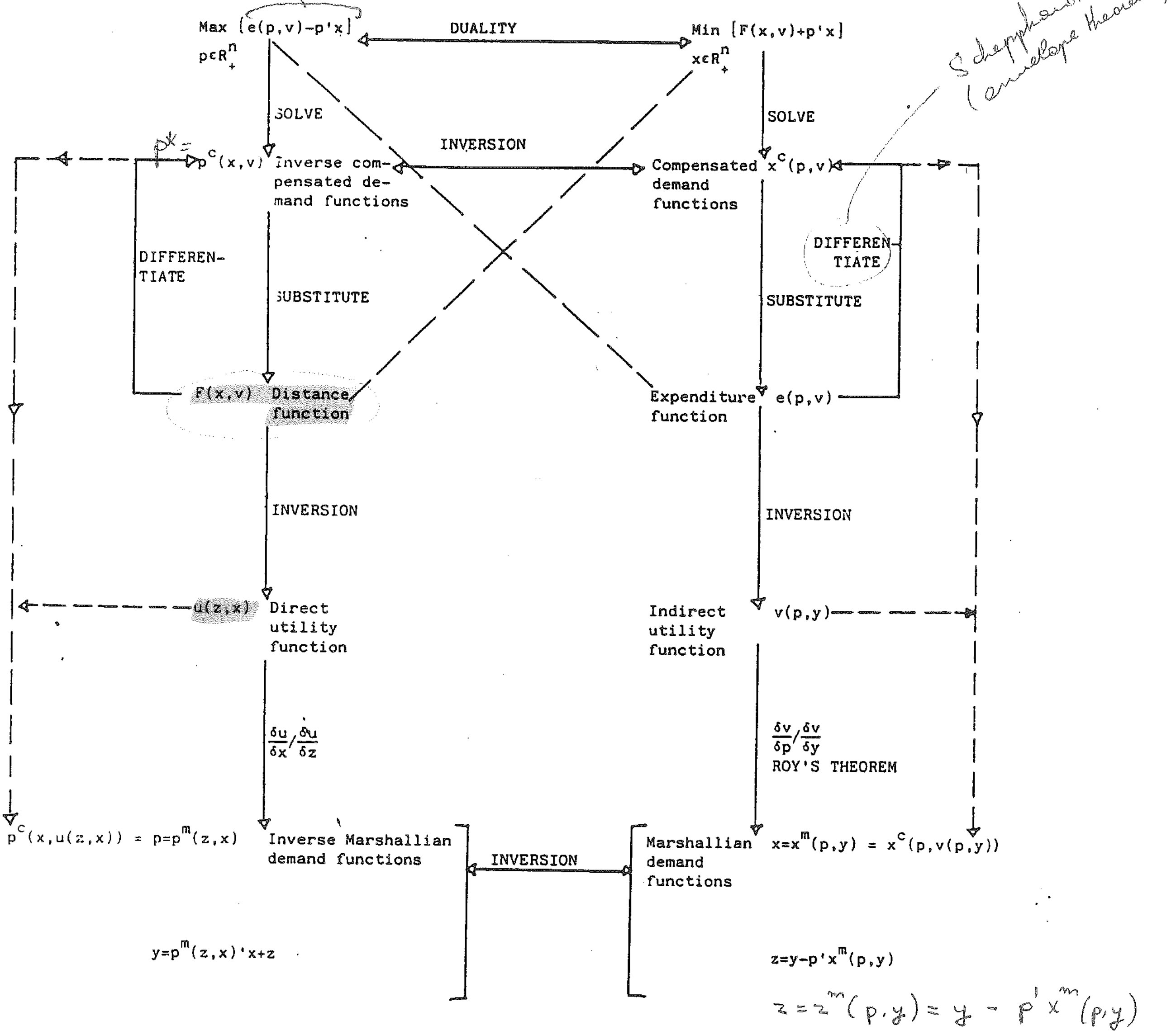


figure 3 : Duality between expenditure and distance functions

The described procedure is interesting from a theoretical viewpoint because it provides a simple way to go from an indirect to a direct representation of preferences[1]. It is also interesting for applied research. Although many empirical studies start from a specified indirect utility function for practical reasons – the demand functions to be estimated can be directly obtained via Roy's theorem –, researchers may be interested in the direct representation of preferences and its economic properties such as elasticities of substitution[2]. Moreover, in some applications in the area of welfare measurement the knowledge of the direct representation of preferences is essential, as will become obvious in Section 3.

Note that the described procedure starts from an expenditure function defined on a vector of n prices and utility and that the resulting direct utility function is defined on (n+1) goods, that is, n goods contained in the vector x and the numeraire commodity z, obtained through the distance function. As a consequence, it is instructive to consider the expenditure function as being defined on (n+1) prices, the price of the numéraire commodity normalized to equal unity.

To illustrate the procedure we present several examples. Let n = 2 and let the expenditure function be

$$e(p_1,p_2,v) = -\frac{1}{v}(\sqrt{p_1}+\sqrt{p_2}+1)^2 \tag{2.1}$$

The reader can verify that this is a valid expenditure function, provided that the utility index v is negative[3]. Solving problem (1.6), viz.

$$\underset{p_1,p_2}{\text{Max}} \quad [-\frac{1}{v}(\sqrt{p_1}+\sqrt{p_2}+1)^2 - p_1x_1 - p_2x_2]$$

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[1] It should be emphasized that the distance function previously proposed in the literature could be used to recover the direct representation of preferences in a related way. Although they do not elaborate on this issue, the possibility is implicit in the discussion in Deaton and Muellbauer (1980, p. 55). This would be a more difficult exercise in practice, however, because it would require the solution of a constrained rather than an unconstrained optimization problem. Moreover, our distance function F(x,v) has important applications in welfare economics, see Section 3.

[2] Alternatively, researchers may wonder whether an analytical direct utility function exists that is consistent with the specified indirect representation of preferences.

[3] This turns out to be the case, see equation 2.5 .

we find $$p_1 = p_1^c(x_1,x_2,v) = \left(\frac{x_2}{vx_1x_2+x_1+x_2}\right)^2 \qquad (2.2)$$

$$p_2 = p_2^c(x_1,x_2,v) = \left(\frac{x_1}{vx_1x_2+x_1+x_2}\right)^2 \qquad (2.3)$$

Substituting these expressions in the objective function yields the distance function

$$F(x_1,x_2,v) = \frac{-x_1x_2}{vx_1x_2+x_1+x_2} \qquad (2.4)$$

Equating $z = F(x_1,x_2,v)$ and solving for v we finally obtain the direct utility function

$$u(z,x_1,x_2) = v = -\frac{1}{z} - \frac{1}{x_1} - \frac{1}{x_2} \qquad (2.5)$$

The interested reader can check that the utility function (2.5) does indeed lead to the expenditure function (2.1) we started from.[1]

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[1] Generalization to more than 3 goods is straightforward. One shows that the expenditure function

$[-\frac{1}{v}(\sum_{i=1}^{n}\sqrt{p_i} + 1)^2]$ leads to the direct utility function

$[-\frac{1}{z} - \sum_{i=1}^{n}\frac{1}{x_i}]$.

men je moet de Deel. Hulls. feit niet hebben: $\left[-\frac{1}{v}\left(\sum_{i=1}^{n+1}\sqrt{p_i}\right)^2\right]$

en $\left[-\sum_{i=1}^{n+1}\frac{1}{x_i}\right]$

als meer algemene oplossing?

As a second example, consider

$$e(p_1,p_2,v) = v^{\frac{1}{\rho}}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} + 1\right)^{\frac{\rho-1}{\rho}} \tag{2.6}$$

where $0 < \rho < 1$. Going through the same procedures we consecutively find the inverse compensated demand functions

$$p_1 = p_1^c(x_1,x_2,v) = \left(\frac{x_1^\rho}{v-x_1^\rho-x_2^\rho}\right)^{\frac{\rho-1}{\rho}} \tag{2.7}$$

$$p_2 = p_2^c(x_1,x_2,v) = \left(\frac{x_2^\rho}{v-x_1^\rho-x_2^\rho}\right)^{\frac{\rho-1}{\rho}} \tag{2.8}$$,

the distance function

$$F(x_1,x_2,v) = (v-x_1^\rho-x_2^\rho)^{\frac{1}{\rho}} , \tag{2.9}$$

and, finally, the direct utility function

$$u(z,x_1,x_2) = z^\rho+x_1^\rho+x_2^\rho , \tag{2.10}$$

which is obviously a CES specification.[1]

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[1]Again, generalization is easy. One shows that the expenditure function

$$[v^{\frac{1}{\rho}}(1+\sum_{i=1}^{n} p_i^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}}]$$

is consistent with the direct utility function

$$[z^\rho + \sum_{i=1}^{n} x_i^\rho].$$

As a final example, let n = 1. Consider the expenditure function that corresponds to the linear demand equation

$$x = \alpha p + \delta y \tag{2.11}$$

where both price and income are deflated by the price of the numeraire good. Hausman (1981, p. 668) has shown that this expenditure function is given by

$$e(p,v) = e^{\delta p}v - \frac{1}{\delta}[\alpha p + \frac{\alpha}{\delta}] \tag{2.12}$$

We assume that $\alpha < 0$, $\delta > 0$ and that e(p,v) is strictly concave in p, so that (2.12) is a valid expenditure function arising from utility maximization. Strict concavity of e(p,v) implies

$$\frac{\delta^2 e(p,v)}{\delta p^2} = \frac{\delta x^c(p,v)}{\delta p} < 0.$$

Using the Slutsky equation it is easy to show that the latter condition can be written as[1]

$$\alpha + \delta x < 0.$$

Many authors in the past have estimated equations of the form presented in (2.11). Using the procedures previously developed it is now straightforward

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[1] Note that $\frac{\delta^2 e(p,v)}{\delta p^2} = \delta^2 e^{\delta p} v$. For this expression to be negative the utility index v itself should be negative. This will be the case provided that $\alpha + \delta x < 0$, see equation (2.15) below.

to find the direct utility function, up to a monotonic transformation, that generates the demand function linear in both price and income. The utility function will have x and the numeraire good z, which may be interpreted as a composite 'all other goods', as arguments.

The solution to

$$\max_{p} \left[e^{\delta p} v - \frac{1}{\delta}\left(\alpha p + \frac{\alpha}{\delta}\right) - px \right]$$

yields the following inverse compensated demand function

$$p^{c}(x,v) = \frac{1}{\delta}\left[\ln\left(\frac{\alpha+\delta x}{\delta^{2} v}\right)\right] \quad . \tag{2.13}$$

Substitution in the objective function yields the distance function

$$F(x,v) = \frac{x}{\delta} - \left[\ln\left(\frac{\alpha+\delta x}{\delta^{2} v}\right)\right] \left(\frac{\delta x+\alpha}{\delta^{2}}\right) \quad , \tag{2.14}$$

from which we derive the direct utility function

$$u(z,x) = \left(\frac{\delta x+\alpha}{\delta^{2}}\right) e^{\frac{\delta^{2}}{\delta x+\alpha}\left(z - \frac{x}{\delta}\right)} \quad . \tag{2.15}$$

The reader can check that, provided that the Slutsky condition holds, that is, $\delta x + \alpha < 0$, this utility function is quasi-concave in (z,x). Moreover it is easily verified that constrained maximization of (2.15) subject to $px + z = y$ will indeed result in the demand function (2.11).

Two final remarks with respect to this example are in order. First, unlike the demand for x, the demand function for good z is not linear in both price and income. Using the budget constraint it is easy to show that the demand for z is given by $z = y - \alpha p^{2} - \delta py$. Second, in order for the utility function (2.15) to generate positive marginal utilities it should be the case

Dus, als we dit veralgemenen wil dit zeggen dat een lineaire vraagfct. voor 1 goed in feite een ingewikkelde fct. voor een ander meebrengt that

$$\frac{\delta u(z,x)}{\delta x} = \frac{1}{\delta}\, e^{\frac{\delta^2}{\delta x+\alpha}(z-\frac{x}{\delta})} + (\frac{\delta x+\alpha}{\delta^2})\, e^{\frac{\delta^2}{\delta x+\alpha}(z-\frac{x}{\delta})} [-\frac{(\alpha\delta+\delta^3 z)}{(\delta x+\alpha)^2}] > 0$$

$$\frac{\delta u(z,x)}{\delta z} = e^{\frac{\delta^2}{\delta x+\alpha}(z-\frac{x}{\delta})} > 0.$$

The second condition will always be satisfied. The first one can be shown to be equivalent with $\delta z - x > 0$, which places an additional constraint on the parameters of the demand function.

To conclude this section we want to point out that in some cases it is possible to recover the underlying utility function from given ordinary demand functions, using the inverse Marshallian demand functions and the following equation, implied by (1.13)

$$\frac{\frac{\delta u(z,x)}{\delta x}}{\frac{\delta u(z,x)}{\delta z}} = p^m(z,x) \tag{2.16}$$

The procedure will be especially useful in a two-good world because in that case it only requires the solution of a first-order, ordinary differential equation (see below). Obviously, it will only yield an analytical expression for the direct utility function if the inverse Marshallian demand function $p^m(z,x)$ can be written in explicit form and, in addition, an analytical solution for the differential equation exists.[1]

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[1]Note that several frequently used demand functions $x^m(p,y)$ do not yield an analytical expression for $p^m(x,z)$. A simple example is the loglinear demand function $\ln x = \gamma_1 \ln p + \gamma_2 \ln y$. Substituting $(px + z)$ for y the resulting equation cannot be solved explicitly for p as a function of x and z.

The solution method we propose is analogous to that of Hausman (1981). He suggested a way of finding a closed-form expression for the indirect utility function on the basis of a given Marshallian demand function by solving Roy's identity

$$\frac{\frac{\delta v(p,y)}{\delta p}}{\frac{\delta v(p,y)}{\delta y}} = - x^{m}(p,y)$$

for v(p,y). We use a similar procedure to solve (2.16) for u(z,x).

As an example, consider again the linear demand function (2.11)

$$x = \alpha p + \delta y \ .$$

This can be rewritten, using $y = px + z$, as

$$p = p^{m}(z,x) = \frac{x-\delta z}{\alpha+\delta x} \ . \qquad (2.17)$$

This is the inverse Marshallian demand function.[1]

Throughout the calculations we remain on the same utility level, say v^0. Consequently

$$du(z,x) = \frac{\delta u(z,x)}{\delta x} \, dx + \frac{\delta u(z,x)}{\delta z} \, dz = 0$$

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[1] In order to generate nonnegative prices and given the Slutsky condition $\alpha + \delta x < 0$ it should be the case that $x - \delta z < 0$. This condition was previously shown to be necessary for the marginal utility of x corresponding to the utility function (2.15) to be positive.

It follows that

$$\frac{dz}{dx} = - \frac{\frac{\delta u(z,x)}{\delta x}}{\frac{\delta u(z,x)}{\delta z}} \tag{2.18}$$

Combining (2.16), (2.17) and (2.18) we have

$$\frac{dz}{dx} = \frac{\delta z - x}{\alpha + \delta x} \tag{2.19}$$

or, alternatively

$$(\alpha + \delta x)dz + (x - \delta z)dx = 0 \tag{2.20}$$

Expressions (2.19) and (2.20) are alternative representations of a first-order differential equation that can be solved for z as a function x. The solution is[1]

$$z = -(\alpha+\delta x)c - \frac{\alpha}{\delta^2} - \frac{1}{\delta^2}(\alpha+\delta x)\ [\ln(-\alpha-\delta x)] \tag{2.21}$$

where c is an arbitrary constant of integration, which depends on the initial utility level v^0. We simply choose the constant c equal to the initial utility index v^0. Noting that equation (2.21) should hold for arbitrary c, i.e. for arbitrary initial value of the utility index, we can derive the utility function by solving (2.21). We find[2]

$$u(z,x) = \frac{-z}{\alpha+\delta x} - \frac{1}{\delta^2}\left[\frac{\alpha}{\alpha+\delta x} + \ln(-\alpha-\delta x)\right] \tag{2.22}$$

---

[1] The fact that $\delta x + \alpha < 0$ should be taken into account when deriving and interpreting this solution. The solution is unique for a given initial condition.

[2] Any monotonic transformation of this equation will also satisfy the differential equation, the only change being in the constant of integration c. This reflects the fact that ordinal utility is only determined up to a monotonic transformation.

Again the reader can verify that this utility function is quasi-concave in (z,x) and that it generates the linear demand function (2.11).

Note that we have shown that both (2.15) and (2.22) generate the linear demand function $x = \alpha p + \delta y$. This is not surprising because the utility functions defined in (2.15) and (2.22) are monotonic transformations of one another, representing the same preferences. Denoting the right hand sides of (2.15) and (2.22) by u_1 and u_2, respectively, it takes some algebra to show that $u_2 = -\frac{1}{\delta^2}\,[\ln(-u_1) + 1 + \ln \delta^2]$, which is indeed a monotonic transformation.

As a final example, consider the semi-log specification

$$\ln x = \gamma p + \varepsilon y \tag{2.23}$$

Using the Slutsky equation it is easy to show that the Slutsky negativity condition requires $\gamma + \varepsilon x < 0$. The inverse Marshallian demand function can be written as[1]

$$p^m(z,x) = \frac{\ln x - \varepsilon z}{\gamma + \varepsilon x} \tag{2.24}$$

Applying the same reasoning as before the following differential equation has to be solved for z as a function of x :

$$\frac{dz}{dx} = \frac{\varepsilon z - \ln x}{\gamma + \varepsilon x} \tag{2.25}$$

or, alternatively,

$$(\gamma + \varepsilon x)dz + (\ln x - \varepsilon z)dx = 0.$$

---

[1]Given the Slutsky condition $(\gamma + \varepsilon x) < 0$ the expression $(\ln x - \varepsilon z)$ has to be negative in order to obtain positive prices, see equation (2.24). It will be shown later that this is also the condition that has to hold for the utility function to have positive marginal utility for good x.

Taking into account $(\gamma + \varepsilon x) < 0$ we derive the solution

$$z = -(\gamma + \varepsilon x)\ [c + \frac{1}{\gamma\varepsilon} \ln(\frac{x}{-\gamma-\varepsilon x}) - \frac{\ln x}{\varepsilon(\gamma+\varepsilon x)}] \tag{2.26}$$

where c is an arbitrary constant of integration. Choosing c equal to the initial utility level and solving equation (2.26) we find

$$u(z,x) = -\frac{z}{\gamma+\varepsilon x} - \frac{1}{\gamma\varepsilon} \ln [\frac{x}{-(\gamma+\varepsilon x)}] + \frac{\ln x}{\varepsilon(\gamma+\varepsilon x)} \tag{2.27}$$

Given the fact that $\gamma + \varepsilon x < 0$ this direct utility function will be quasi-concave in (z,x). The reader can verify that it leads to the semi-log specification (2.23). In order for (2.27) to yield positive marginal utilities we should have

$$\frac{\delta u(z,x)}{\delta x} = \frac{\varepsilon z}{(\gamma+\varepsilon x)^2} - \frac{1}{\gamma\varepsilon x} + \frac{1}{\gamma\varepsilon}\,\frac{\varepsilon}{\gamma+\varepsilon x} + \frac{\frac{\varepsilon}{x}(\gamma+\varepsilon x) - \varepsilon^2 \ln x}{\varepsilon^2(\gamma+\varepsilon x)^2} > 0$$

$$\frac{\delta u(z,x)}{\delta z} = -\frac{1}{\gamma+\varepsilon x} > 0$$

The latter condition is satisfied due to the Slutsky negativity condition. The former restriction implies $\varepsilon z - \ln x > 0$.

3. The definition of measures of welfare change in quantity constrained regimes

It is suggested in the literature to use Hicks' equivalent and compensating variations to evaluate the welfare implications of price changes (see, e.g., Varian (1978, p. 209 - 213). In the empirical literature the equivalent variation has also been used to measure the change in welfare due to the transition from an unconstrained regime to a regime in which a price subsidy for some commodities is combined with a restriction on the quantity consumed (Clarkson (1976), Olsen and Barton (1983), De Borger (1985)).

In many situations it would be preferable to use measures of welfare change that are independent of prices, such as Hicks' equivalent and compensating surplus. This will be the case whenever market prices do not exist, as in the case of public goods, or whenever certain commodities are rationed so that prices do not reflect consumers' willingness to pay. In particular, the Hicksian surplus measures are convenient tools to analyze the welfare effects of policies involving changes in imposed quantities.

In this section we show how the distance function can be used to define the Hicksian surplus measures.[1] This results in very simple expressions that stand in sharp contrast with alternative expressions proposed in the literature. We consider a consumer in each of two states. The initial situation, state 0, is characterized by a price vector p^0, income y^0 and a consumption bundle (z^0, x^0), yielding utility level $v^0 = u(z^0, x^0)$. We do not require that (z^0, x^0) is the optimal bundle given prices p^0 and income y^0. Due to restrictions on consumption, e.g., it may be the case that the quantities consumed are not on the Marshallian demand curves, that is

$$x^0 \neq x^m(p^0, y^0)$$
$$z^0 \neq y^0 - p^{0\prime} x^m(p^0, y^0)$$

[1] The remainder of this section closely follows W. Pauwels (1985, p. 20 - 24).

We do require the budget constraint $p^{0\prime} x^0 + z^0 = y^0$ to hold. Similarly, state 1 is characterized by prices p^1, income y^1 and a commodity bundle (z^1, x^1), yielding utility level $v^1 = u(z^1, x^1)$. Again, the bundle (z^1, x^1) need not be optimal at prices p^1 and income y^1 but the budget constraint $p^{1\prime} x^1 + z^1 = y^1$ is assumed to hold.

Consider a consumer in state 1. The compensating surplus is defined as the quantity of z that can be taken away from the consumer and still yield the initial utility level $v^0 = u(z^0, x^0)$, assuming the quantities of x remain fixed at x^1. It is implicitly defined by the equation

$$u(z^1 - CS, x^1) = v^0 \qquad (3.1)$$

Using the definition $u(F(x^1, v^0), x^1) = v^0$ the solution can be shown to be

$$CS = z^1 - F(x^1, v^0). \qquad (3.2)$$

This can be written as

$$CS = z^1 - z^0 + F(x^0, v^0) - F(x^1, v^0) \qquad (3.3)$$

$$= z^1 - z^0 + \int_{x^1}^{x^0} \frac{\delta F(x, v^0)'}{\delta x} dx . \qquad (3.4)$$

Using equation (1.10) we finally derive the following expression

$$CS = z^1 - z^0 + \int_{x^0}^{x^1} p^c(x, v^0)' \, dx . \qquad (3.5)$$

Next consider a consumer in state 0. The equivalent surplus ES is defined as the quantity of z that should be given to the consumer in order to make him as well off as in state 1, assuming that x remains fixed at x^0. It is defined by

$$u(z^0 + ES, x^0) = v^1 \qquad (3.6)$$

Noting that $u(F(x^0,v^1),x^0) = v^1$ it can be written as

$$ES = F(x^0, v^1) - z^0 \tag{3.7}$$

$$ES = z^1 - z^0 + F(x^0, v^1) - F(x^1, v^1) \tag{3.8}$$

$$ES = z^1 - z^0 + \int_{x^1}^{x^0} \frac{\delta F(x,v^1)'}{\delta x}\, dx \tag{3.9}$$

Using (1.10) we finally have

$$ES = z^1 - z^0 + \int_{x^0}^{x^1} p^C(x, v^1)'\, dx \tag{3.10}$$

Expressions (3.2) and (3.7) show the extremely simple definitions of the surplus measures in terms of the distance function $F(x,v)$. They stand in contrast with alternative expressions that have been proposed in the literature. Recently, Lankford (1983) considered the calculation of the surplus measures for changes in imposed quantities. Although his definition is different from the one we used, his expressions can be shown to equal those presented in (3.3) and (3.8). He defined ES as the payment which, if made to be individual in state 0 rather than the change in quantity, would mean the person was just able to obtain the post-quantity-change utility level v^1. Similarly, CS was defined as the payment which, if made by the consumer in state 1, would allow the pre-quantity-change utility level v^0 to be attained even though the post-quantity-change amount was consumed. Lankford uses the concept of constrained expenditure function and the important relation between constrained and unconstrained expenditure functions (due to Neary and Roberts (1980)) to show that CS and ES can be written as follows :[1]

---

[1] Lankford assumed that the change in imposed quantities was not accompanied by a change in prices or income. We present his expressions in the more general case that does allow for changes in price and income.

$$CS = y^1 - y^0 + e(p^C(x^0,v^0),v^0) - e(p^C(x^1,v^0),v^0)$$

$$+ (p^0 - p^C(x^0,v^0))' x^0 - (p^1 - p^C(x^1,v^0))' x^1 \qquad (3.11)$$

$$ES = y^1 - y^0 + e(p^C(x^0,v^1),v^1) - e(p^C(x^1,v^1),v^1)$$

$$+ (p^0 - p^C(x^0,v^1))' x^0 - (p^1 - p^C(x^1,v^1))' x^1 \qquad (3.12)$$

One easily shows that (3.11) and (3.12) reduce to equations (3.3) and (3.8) respectively. Remember from Section 1 that the price vector solving (1.6) is the vector of inverse compensated demand functions $p^C(x,v)$. Consequently, (1.6) implies

$$F(x,v) = e(p^C(x,v),v) - p^C(x,v)' x$$

Moreover, it is obvious that

$$p^{0\prime} x^0 - p^{1\prime} x^1 = y^0 - y^1 + z^1 - z^0$$

Appropriately substituting these expressions in (3.11) and (3.12) one immediately shows the equivalence of (3.11) - (3.12) and (3.3) - (3.8). Note, however, that the definition of ES and CS in terms of the distance function is a lot more intuitive.

4. Analytical and numerical procedures to evaluate Hicksian surplus measures

Although most empirical studies start from a specified direct utility function, it has been shown in the literature that exact welfare measures of a price change can be calculated (Hausman (1981)) or numerically approximated up to any desired degree of accuracy (Vartia (1983)) on the basis of an estimated system of Marshallian demand functions. The purpose of this section is to use the distance function to develop algebraic and numerical procedures to evaluate the Hicksian surplus measures ES and CS. This may have important applications in the areas of health, education and housing, where goverment programs often introduce quantity-constrained regimes. The methods will be particularly useful to evaluate the welfare implications of changes in imposed quantities.

The appropriate solution procedure for the surplus measures depends upon the information a researcher has available, or alternatively, the assumptions the investigator is willing to make. We can distinguish three cases. First, it is possible that the investigator is willing to specify a direct utility function and empirically estimates its parameters using the derived demand equations. This case is dealt with in subsection 4.1. Algebraic calculation of ES and CS is straightforward. Second, in subsection 4.2 we consider the case where the researcher prefers not to specify a direct utility function but starts out with a valid indirect utility or expenditure function and empirically estimates the Marshallian demand functions, derived using Roy's theorem. In this case the properties of the distance function can be used to determine the surplus measures ES and CS algebraically or numerically. Third, it is possible that the investigator arbitrarily estimates a large number of different specifications for the Marshallian demand functions and chooses an appropriate specification on the basis of statistical considerations, without specifying a direct or indirect representation of preferences. Sometimes economists just estimate a large number of different functional forms and choose the demand system that yields the best fit to the data. In subsection 4.3 we show that in some cases it is still possible to determine ES and CS algebraically, using the procedures presented

Mostly

in section 2. Normally, however, numerical methods will have to be used. We provide an algorithm, based on the properties of the distance function, which is similar to Vartia's procedure to approximate the compensating income in the case of a pure price change. The algorithm is easy to understand and computer programs can easily be written for practical applications.

4.1. The direct utility function is known

Whenever the investigator is willing to specify a direct utility function u(z,x), calculation of the surplus measures is obvious. The distance function can be obtained by inversion and the result can be used in (3.2) and (3.7) to yield CS and ES, respectively. To give an example, let the utility function be

$$u(z,x) = \sum_{i=1}^{n} \alpha_i x_i^{\delta_i} + \beta z^{\gamma}$$

By inversion,

$$F(x,v) = \left(\frac{\sum_{i=1}^{n} \alpha_i x_i^{\delta_i} - v}{\beta} \right)^{\frac{1}{\gamma}}$$

Substituting v = u(z,x) and applying (3.2) and (3.7) we immediately find

$$ES = \left[\frac{\sum_{i=1}^{n} \alpha_i (x_i^1)^{\delta_i} - \sum_{i=1}^{n} \alpha_i (x_i^0)^{\delta_i} + \beta (z^1)^{\gamma}}{\beta} \right]^{\frac{1}{\gamma}} - z^0$$

$$CS = z^1 - \left[\frac{\sum_{i=1}^{n} \alpha_i (x_i^0)^{\delta_i} - \sum_{i=1}^{n} \alpha_i (x_i^1)^{\delta_i} + \beta (z^0)^{\gamma}}{\beta} \right]^{\frac{1}{\gamma}}$$

4.2. The expenditure function is known

Suppose the investigator specifies a valid indirect utility or expenditure function and estimates the system of Marshallian demands that corresponds to the chosen function. The properties of the distance function can be employed to determine ES and CS. In particular, whenever the vector of inverse compensated demand functions that solve the maximization problem on the right-hand side of (1.6) can be written as a closed form expression, the distance function is obtained by substitution of the solution in the objective function. Knowledge of the distance function suffices to determine the surplus measures using (3.2) and (3.7).

However, in many situations no analytical solution to (1.6) exists. This is the case for the most general specifications such as the translog indirect utility function (Christensen, Jorgenson and Lau (1975)) or the expenditure function corresponding to the Almost Ideal Demand System of Deaton and Muellbauer (1980). When no analytical expression for $p^c(x,v)$ can be obtained, the following ~~simple~~ numerical exercise will yield the Hicksian surplus measures.

First consider the equivalent surplus. Using (1.6) in (3.7) we have

$$ES = \{\underset{p}{\text{Max}}\ [e(p,v^1) - p'x^0]\} - z^0 \tag{4.1}$$

If the utility level $v^1 = u(z^1,x^1)$, attained in state 1, were known we could substitute v^1 into $e(p,v^1)$ and use a numerical optimization routine to determine ES[1]. Unfortunately, this is not the case[2]. Therefore, we first have to evaluate v^1 as follows. Note that by definition

$$v^1 = v[p^*,\ e(p^*,v^1)] \tag{4.2}$$

$$x^1 = x^m[p^*,\ e(p^*,v^1)] \tag{4.3}$$

---

[1]Available routines include the Newton-Raphson method as well as procedures proposed by Powell (1964) and Goldfeld, Quandt and Trotter (1966).

[2]Note that $v^1 \neq v(p^1,y^1)$ due to imposed constraints on quantity.

where $p^* = p^c(x^1,v^1)$. Moreover, since income y^1 corresponds to a constrained regime we can use the relation between constrained and unconstrained expenditure functions (Neary and Roberts (1980)) to find

$$e(p^*,v^1) = y^1 - (p^1 - p^*)' x^1 . \tag{4.4}$$

Substituting this result in (4.3) yields

$$x^1 = x^m[p^*,y^1 - (p^1 - p^*)' x^1] \tag{4.5}$$

This implicit equation can numerically be solved for p^* using a standard algorithm (see Conte and De Boor (1972) for some useful methods). Substituting the value of p^* in (4.4) and both p^* and the result into (4.2) leads to the value of the utility level v^1. The final step is to use this value in (4.1) and maximize the expression $\left(e(p,v^1) - p'x^0\right)$ over p . This yields ES.[1]

A similar procedure can be employed to determine CS. To evaluate $v^0 = u(z^0,x^0)$ note that

$$v^0 = v[\bar{p}, e(\bar{p},v^0)] \tag{4.6}$$

$$x^0 = x^m[\bar{p}, e(\bar{p},v^0)] \tag{4.7}$$

where $\bar{p} = p^c(x^0,v^0)$. Using the relation

$$e(\bar{p},v^0) = y^0 - (p^0 - \bar{p})' x^0 \tag{4.8}$$

in (4.7) we find an implicit equation in $\bar{p}$

$$x^0 = x^m[\bar{p}, y^0 - (p^0-\bar{p})' x^0] \tag{4.9}$$

---

[1]An obvious alternative is to solve in the final step the first-order conditions $x^c(p,v^1) = x^0$ for p.

Numerically solving for $\bar{p}$, and substituting the result back in (4.8) and (4.6) we find the value of v^0. This can be used to maximize $[e(p,v^0) - p'x^1]$ over p and to use the result in the definition of CS (see equations (1.6) and (3.2))

$$CS = z^1 - \{\underset{p}{\text{Max}}\ [e(p,v^0) - p'x^1]\}$$

The numerical solution procedures described in this subsection may seem cumbersome. However, computer routines to solve the numerical problems are readily available in many software packages. A program to determine ES and CS that incorporates these routines is easily written and can be executed at reasonable cost. This will certainly be the case when the number of goods dealt with in the analysis is small, a condition that will hold for most empirical applications[1].

4.3. Only the Marshallian demand functions are known

In this subsection we assume that the only information available to the investigator is the system of Marshallian demand functions, estimated on the basis of a sample of individual housheolds.[2] It may be the case that the specification of the demand system has been chosen on purely statistical grounds, a procedure that has been frequently used in empirical work.

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[1]Most studies dealing with the evaluation of government programs limit the empirical analysis to two goods, the subsidized good and a composite commodity 'all other goods'. See, e.g., Sonstelie (1982) and Olsen and Barton (1983).

[2]Note that the demand system will have to be estimated using a sample of families that is not subject to the quantity restrictions. Using these estimates to evaluate the effect of a government program for a sample of participating families introduces the possibility of selection bias in the welfare measures finally obtained. The problem of selection bias can be taken into account in the estimation procedure (Heckman (1979)).

It should be noted at the outset that in some special cases the Hicksian surplus measures can be determined using the results developed earlier in this paper. First, it is possible that the investigator is able to find the expenditure function that corresponds to the demand system, using the methodology developed by Hausman (1981). In that case the procedures in subsection 4.2 can be applied to find ES and CS. Second, we presented several examples in Section 3 that showed that in a two good world it is in some cases possible to write the inverse Marshallian demand function $p^m(z,x)$ in explicit form, to find a closed-form solution to the differential equation

$\frac{dz}{dx} = - p^m(z,x)$, and to use the result to determine the direct utility and distance functions. This information is sufficient to calculate ES and CS.

The purpose of this subsection is to provide a numerical method that generalizes the latter procedure to a many-good world and to arbitrary systems of demand functions, provided that they are consistent with an underlying utility function. The proposed method is close in spirit to Vartia's (1983) algorithm to solve for the compensated income of a price change on the basis of a system of market demand functions. Whereas he uses the properties of the expenditure function, we will employ the properties of the distance function to develop our algorithm. It yields a procedure to approximate the value of the distance function, which can appropriately be inserted into the relevant formulas for ES and CS.

Our method can be used even in the case where the demand system $x^m(p,y)$ does not lead to an explicit system of inverse Marshallian demands $p^m(z,x)$, although the computational cost will be substantially smaller when a closed form expression for $p^m(z,x)$ does exist. When discussing the algorithm we will initially assume that $p^m(z,x)$ exists in closed-form. It will be indicated later how the procedure should be changed in case the demand system cannot be inverted.[1]

---

[1]Under some assumptions it is possible to estimate $p^m(z,x)$ directly rather than via the ordinary demand system. For empirical applications of inverse demand functions, see Barnett (1977) and Christensen and Manser (1977).

Suppose the investigator has estimated a system of Marshallian demand functions for x and z. It is assumed that these demand functions are continuously differentiable, that they satisfy the adding-up restriction and that the corresponding Slutsky matrix is symmetric and negative semi-definite.[1] If in addition a regularity condition holds (Hurwics and Uzawa (1971)) the demand system is consistent with an underlying direct utility function (Vartia (1983)). All these conditions can in principle be checked for an empirically determined demand system or they can be imposed in the estimation procedure.[2]

To fix ideas suppose the investigator is considering the calculation of the compensating surplus CS of a transition from state 0 to state 1. The two states are characterized by consumption bundles (z^0,x^0) and (z^1,x^1) which are known to the investigator. Calculation of CS requires the numerical evaluation of $F(x^1,v^0)$, see equation (3.2). The procedure we propose runs in two steps: first, derive an ordinary first-order differential equation using the properties of the distance function and an auxiliary variable. Second, numerically evaluate the solution of the equation, starting from the initial value $F(x^0,v^0) = z^0$ and moving in steps towards $F(x^1,v^0)$, thereby remaining on the indifference surface determined by v^0.

To facilitate the exposition we introduce an auxiliary variable $s \in [0,1]$. Let $x(s)$ be an arbitrary differentiable path connecting x^0 and x^1 such that

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[1] Since the price of the numeraire good z is normalized at one all prices in the demand system are relative prices. Therefore, the condition that the system should be homogenous in prices and income does not apply.

[2] The conditions can also be checked on the system $p^m(z,x)$. The inverse demand functions should be differentiable, satisfy the adding-up constraint, and the Antonelli matrix of derivatives of the inverse compensated demand functions should be symmetric and negative semi-definite. This matrix can be determined using (1.15).

$x(0) = x^0$ and $x(1) = x^1$. The function $x(s)$ is to be specified by the researcher. We have

$$F(x,v^0) = F\big(x(s),v^0\big) = \phi(s) \tag{4.10}$$

which implies that, as long as we remain at utility level v^0, the value of the distance function is a function of s only. It also follows

$$v^0 = u\big(F(x,v^0),x\big) = u\big(\phi(s),x(s)\big) \tag{4.11}$$

Totally differentiating $u\big(\phi(s),x(s)\big)$ we find that

$$\frac{du\big(\phi(s),x(s)\big)}{ds} = \sum_i \frac{\delta u\big(\phi(s),x(s)\big)}{\delta x_i(s)} \frac{dx_i(s)}{ds} + \frac{\delta u\big(\phi(s),x(s)\big)}{\delta\phi(s)} \frac{d\phi(s)}{ds} \tag{4.12}$$

Throughout the calculations we remain on v^0. Letting $du\big(\phi(s),x(s)\big) = 0$, applying equation (1.12) at utility level v^0 and using (4.11) we derive after some straightforward algebra

$$\frac{d\phi(s)}{ds} = - \sum_i p_i^m\big(\phi(s),x(s)\big) \frac{dx_i(s)}{ds} \tag{4.13}$$

This first-order nonlinear differential equation in $\phi(s)$ is a generalization of the equation $dz = - p^m(z,x)dx$ we used in Section 2 in a two-commodity world. It is important to note that we assume at this stage of the analysis that the functions $p_i^m(.)$ and $x(s)$ are known to the investigator. Also observe that if we change s we remain at utility level v^0 since by definition $\phi(s) = F\big(x(s),v^0\big)$ and $p_i^c\big(x(s),v^0\big) = p_i^m\big(F(x(s),v^0),x(s)\big) = p_i^m(\phi(s),x(s))$.

The uniqueness property of the solution of first-order differential equations implies that $\phi(s) = F\big(x(s),v^0\big)$ is the unique solution of (4.13) for initial value $\phi(0) = F(x^0,v^0) = z^0$ for arbitrary function $x(s)$. The arbitrariness of $x(s)$ reflects the fact that the solution to (4.13) does not depend on the path of integration.[1]

To solve numerically for $F(x^1,v^0)$ we define a series of s_k values

$$s_0 = 0,\ s_1,\ s_2,\ \ldots,\ s_{N-1},\ s_N = 1$$

which implies a series of intermediate quantities between x^0 and x^1,

$$x^0 = x(s_0),\ x(s_1),\ x(s_2),\ \ldots,\ x(s_{N-1}),\ x(s_N) = x^1.$$

Then note that, by integration, the differential equation (4.13) is equivalent to the following integral equation

$$\phi(s_k) - \phi(s_{k-1}) = -\sum_i \int_{s_{k-1}}^{s_k} p_i^m\big(\phi(s),x(s)\big)\frac{dx_i(s)}{ds}\,ds \qquad (4.14)$$

We propose to move in N small discrete steps from $\phi(s_0) = z^0$ to $\phi(s_N) = F(x^1,v^0)$, using (4.14). Obviously the integral on the right-hand side of the equation cannot be solved explicitly because the distance function is unknown. Therefore, approximation techniques have to be used at each of the N steps. To facilitate notation, let

$$p_{i,k} = p_i^m\big(\phi(s_k),\ x(s_k)\big) \qquad (4.15)$$

---

[1]This is consistent with the observation that the Hicksian surplus measures are independent of the path of integration. Note, e.g., that CS can be written as (see equation 3.5)

$$CS = z^1 - z^0 + \int_{x^0}^{x^1} p^c(x,v^0)'dx$$

This expression is independent of the path x(s) that is used to integrate from x^0 to x^1 because of the symmetry property of the matrix of inverse compensated demand functions. The symmetry property itself follows from (1.10).

$$x_{i,k} = x_i(s_k) \tag{4.16}$$

$$\phi_k = \phi(s_k) \tag{4.17}$$

Many ways to approximate the integral exist in the literature (see, e.g., Collatz (1960), Bronson (1973)). By far the simplest procedure is to replace $p_i^m(\phi(s),x(s))$ by its value at the lower integration limit, $p_i^m(\phi(s_{k-1}),x(s_{k-1}))$. Using (4.15), (4.16) and (4.17), this implies that (4.14) can be rewritten as

$$\phi_k - \phi_{k-1} \cong -\sum_i \int_{s_{k-1}}^{s_k} p_{i,k-1}\, dx_i(s) \tag{4.18}$$

$$\phi_k - \phi_{k-1} \cong -\sum_i p_{i,k-1}(x_{i,k}-x_{i,k-1}) \tag{4.19}$$

However, Vartia (1983) shows in a different context that a much faster converging method is to replace $p_i^m(\phi(s),x(s))$ by the mean of its values at the lower and upper integration bounds, that is,[1]

$$p_i^m(\phi(s),x(s)) \cong \frac{p_{i,k} + p_{i,k-1}}{2} \tag{4.20}$$

Equation (4.14) then becomes

$$\phi_k - \phi_{k-1} \cong -\sum_i \int_{s_{k-1}}^{s_k} \left(\frac{p_{i,k} + p_{i,k-1}}{2}\right) dx_i(s) \tag{4.21}$$

$$\cong -\sum_i \left(\frac{p_{i,k} + p_{i,k-1}}{2}\right)(x_{i,k} - x_{i,k-1}) \tag{4.22}$$

[1] Also see Collatz (1960).

Rewriting (4.22) in vector notation and slightly rearranging, we obtain

$$\phi_k \stackrel{\sim}{=} \phi_{k-1} - \left(\frac{p_k + p_{k-1}}{2}\right)' (x_k - x_{k-1}) \tag{4.23}$$

where p_k and x_k are the vectors having the $p_{i,k}$ and $x_{i,k}$ as elements, respectively.

Starting from the initial value $\phi_0 = \phi(s_0) = \phi(0) = z^0$ we can use relation (4.23) to calculate approximate values for $\phi_1, \phi_2, \ldots \phi_{N-1}$ until we finally obtain an approximation for $\phi_N = \phi(s_N) = \phi(1) = F(x^1, v^0)$. However, a final problem remains to be solved. Indeed, unlike the values of p_{k-1}, x_k and x_{k-1} the value of p_k depends on ϕ_k because by definition

$$p_k = p^m(\phi_k, x_k).$$

Clearly, then, equation (4.23) is an implicit equation in ϕ_k. The proposal is to solve the equation iteratively using

$$\phi_k^{(m)} = \phi_{k-1} - \frac{1}{2}(p^m(\phi_k^{(m-1)}, x_k) + p_{k-1})'(x_k - x_{k-1}) \tag{4.24}$$

where one uses ϕ_{k-1} as starting value for the iteration, that is,

$$\phi_k^{(0)} = \phi_{k-1}$$

It seems desirable at this moment to summarize the practical steps to be taken in the following algorithm, in which a simple linear path x(s) is chosen[1].

ALGORITHM :

1. Let the path x(s) be $x(s) = x^0 + s(x^1 - x^0)$, $0 \leq s \leq 1$.
2. Choose the number of steps N.
3. Let $s_k = k/N$, $x_k = x(s_k)$

[1]Remember that the path x(s) is arbitrary.

4. Generate a sequence $\phi_1, \phi_2, \ldots \phi_N$ using the equation

$$\phi_k = \phi_{k-1} - \frac{1}{2}(p_k + p_{k-1})'(x_k - x_{k-1}) \qquad (*)$$

where $p_k = p^m(\phi_k, x_k)$. Use as initial values for ϕ, p and x the values corresponding to the consumption bundle (z^0, x^0) provided in state 0, viz.

$$\phi_0 = F(x^0, v^0) = z^0$$

$$p_0 = p^m(x^0, x^0)$$

$$x_0 = x^0$$

The implicit equation (*) can in each of the N steps be solved iteratively using

$$\phi_k^{(m)} = \phi_{k-1} - \frac{1}{2}\left(p^m(\phi_k^{(m-1)}, x_k) + p_{k-1}\right)'(x_k - x_{k-1})$$

where $\phi_k^{(0)} = \phi_{k-1}$.

Since $x(s_N) = x^1$ and $\phi_N = \phi(s_N) = F\left(x(s_N), v^0\right)$ we know by construction of the algorithm that, if the procedure converges, it yields the desired value of the distance function $F(x^1, v^0)$. It remains to be shown that the algorithm converges. Recently, Vartia (1983) applies the same general technique, in which the distance and inverse Marshallian demand functions are replaced by the expenditure function and ordinary Marshallian demand functions, to approximate the compensated income of a price change. Using results derived from the literature dealing with the numerical solution of differential equations he shows the convergence of the procedure. It can be shown by analogy that for large N the quantity ϕ_N converges to $F(x^1, v^0)$. Moreover, since $p_N = p^m(\phi_N, x_N)$ it follows that p_N converges to $p^m\left(F(x^1, v^0), x^1\right) = p^c(x^1, v^0)$.

A few remarks are in order. First, the described procedure to move in small steps along an indifference curve is generally applicable and not restricted to a two-good world. Second, it should be noted that the algorithm requires a double iteration. Within each of the N steps an iterative technique is used to approximate ϕ_k. However, it is clear that the computations to be performed are extremely simple and computationally cheap. In small problems a hand calculator would be sufficient to do the calculations. Third, an analogous method can be used to start from the bundle (z^1,x^1) and to move along the indifference curve corresponding to v^1 to the bundle $(F(x^0,v^1),x^0)$. This would yield a numerical approximation to $F(x^0,v^1)$ which can be used to calculate the equivalent surplus ES.

To illustrate the use of the algorithm in practice we consider a simple example. Consider the inverse Marshallian demand function

$$p^m(z,x) = \left(\frac{x}{z}\right)^{-0.5} \tag{4.25}$$

The reader can verify that this function corresponds to the following direct utility and distance functions, respectively

$$u(z,x) = x^{0.5} + z^{0.5} \tag{4.26}$$

$$F(x,v) = (v - x^{0.5})^2 \tag{4.27}$$

Let the initial situation be represented by the bundle $(z^0,x^0) = (1,2)$. Assume a policy change results in a final consumption bundle $(z^1,x^1) = (2,3)$. Obviously, knowledge of the distance function could be used to find $F(x^1,v^0)$ using (4.26) and (4.27) :

$$F(x^1,v^0) = F(3,u(1,2)) = (2^{0.5} + 1^{0.5} - 3^{0.5})^2 = 0.465346.$$

Consequently, the compensating surplus CS of the policy change is

$$CS = z^1 - F(x^1,v^0) = 2\text{-}0.465346 = 1.534653$$

However, to double-check the proposed numerical solution procedure we will calculate the value of $F(x^1, v^0)$ without using the distance function. We will only use the inverse Marshallian demand function (4.25) and apply our algorithm. If the algorithm is at all reliable we should find a numerical value for $F(x^1, v^0)$ which closely approximates the correct value 0.465346.

We propose to use a linear path x(s) and to use as few as 4 steps. We have

$$x(s) = x^0 + s(x^1 - x^0) = 2 + s$$

Values for s and x(s) at each of the 4 steps are easily determined :

$$s_0 = 0 \qquad x_0 = x(s_0) = 2$$

$$s_1 = 0.25 \qquad x_1 = x(s_1) = 2.25$$

$$s_2 = 0.5 \qquad x_2 = x(s_2) = 2.5$$

$$s_3 = 0.75 \qquad x_3 = x(s_3) = 2.75$$

$$s_4 = 1 \qquad x_4 = x(s_4) = 3$$

Starting values for the algorithm are $(x_0, p_0, \phi_0) = (x^0, p^m(z^0, x^0), z^0) =$ (2,0.7071068,1). Using these values we determine ϕ_1, ϕ_2, ϕ_3 and ϕ_4 iteratively on the basis of

$$\phi_k^{(m)} = \phi_{k-1} - 0.125 \, [p^m(\phi_k^{(m-1)}, x_k) + p_{k-1}]$$

For example, the first iteration is

$$\phi_1^{(1)} = 1 - 0.125 \, [p^m(1, 2.25) + 0.7071068] = 0.828278.$$

An overview of the calculations is in table 1. At each step k it takes 4 or 5 iterations to achieve convergence for ϕ_k. We finally obtain

$$\phi_N = \phi_4 = 0.464537$$

$$p_N = p_4 = 0.393504$$

k	m	p_k	$\phi_k^{(m)}$	ϕ_k
0		0.707168	1	1
1	1	0.609351	0.828278	
	2		0.835770	
	3		0.835428	
	4		0.835444	0.835444
2	1	0.526666	0.687015	
	2		0.693747	
	3		0.693427	
	4		0.693442	0.693442
3	1	0.455538	0.543992	
	2		0.572013	
	3		0.570599	
	4		0.570670	
	5		0.570666	0.570666
4	1	0.393504	0.459206	
	2		0.464819	
	3		0.464521	
	4		0.464537	0.464537

Table 1 : numerical illustration of the algorithm

Taking into account that we only used 4 steps these values are very close to the correct values $F(x^1,v^0)$ and $p^c(x^1,v^0)$ that can be calculated using the 'unknown' distance function (4.27) :

$$F(x^1,v^0) = 0.465346$$

$$p^c(x^1,v^0) = 0.393847$$

The reader who is dissatisfied with the approximation of $F(x^1,v^0)$ and $p^c(x^1,v^0)$ by ϕ_4 and p_4, respectively, can always obtain a higher degree of accuracy by increasing the number of steps N.

In the preceding analysis, we implicitly assumed that the investigator has available the system of inverse Marshallian demand functions $p^m(z,x)$. However, it may be the case that the researcher estimated a system of ordinary demand functions and that it is impossible or extremely difficult to invert the system and to write $p^m(z,x)$ in closed-form. In that case the proposed algorithm should be adjusted so as to evaluate the price function $p^m(.)$ at each iteration. These additional computations may for large-scale problems substantially increase the computational costs.

The problem is that we need the value of $p^m(\phi_k^{(m-1)},x_k)$ at each iteration of the algorithm. In case the functional form of the price function is unknown the required value should be obtained numerically. The procedure is as follows. Noting that in general x_j is the vector of quantities demanded when prices are $p^m(z_j,x_j)$ and income equals $[p^m(z_j,x_j)'x_j + z_j]$, the following equation holds by definition

$$x_k = x^m\ [p^m(\phi_k^{(m-1)},x_k),\ p^m(\phi_k^{(m-1)},\ x_k)'x_k + \phi_k^{(m-1)}] \tag{4.28}$$

To simplify notation, this can be rewritten more conveniently as

$$x_k = x^m \left[\bar{p},\ \bar{p}'\ x_k + \phi_k^{(m-1)}\right] \tag{4.29}$$

where $\bar{p} = p^m(\phi_k^{(m-1)},\ x_k)$. Equation (4.29) should be solved numerically for $\bar{p}$ at each iteration in order to find the value of $\phi_k^{(m)}$. It is clear that these additional computations are the price one has to pay for not choosing a demand system that yields a closed-form expression for the inverse Marshallian demand functions.

4.4. Evaluating Hicksian surplus measures : some concluding remarks

On figure 4 we summarize alternative methods to determine the Hicksian surplus measures of a welfare change, as discussed in this section. Depending upon the information the investigator has available alternative solution procedures were proposed. A few concluding remarks concerning the material presented in this section are in order, however.

First, it is obvious from the preceding discussion that the investigator has to trade off the restrictions on behavior implied by the specification of a direct utility function against computational convenience. If he prefers not to specify u(z,x) the material presented in this section shows that solving for the welfare measures CS and ES is more complicated but, more importantly, it shows that it can be done.

Second, the discussion in subsection (4.3) suggests that, in case the investigator starts from a Marshallian demand system, the numerical calculations are easier and computationally cheaper if the inverse demand functions $p^m(z,x)$ can be written in closed-form. As many Marshallian demand systems do not yield a closed-form expression for the inverse demand functions, this seems to suggest that a case can be made for the direct estimation of the inverse demand system. Unfortunately, from a theoretical viewpoint estimation of inverse demand functions only makes sense under some very specific assumptions that are not very realistic in most applications. The approach implies that prices are determined by quantities, that is, prices are endogenous and quantities exogenous. This assumption may not be unreasonable when using aggregate time-series data[1], but it can

[1] In an interesting empirical application W. Barnett (1977) uses time-series data to estimate a system of inverse demand functions and to predict commodity prices. He assumes supply is a function of lagged prices which implies that in any given period supply can be treated as a predetermined variable. This in turn implies that total demand, which equals supply, can be assumed to determine prices, which are the endogenous variables in the model.

hardly be defended when using cross-sectional or panel data on individual households.[1] Therefore, direct estimation of $p^m(z,x)$ will not be a very useful alternative in case the researcher is interested in evaluating the Hicksian surplus measures of welfare change. However, computational convenience may lead the investigator to choose a system of demand functions for x and z that does yield a closed-form expression for $p^m(z,x)$.

[1] In fact, when a single cross-sectional sample is drawn in one particular area (city, region), it may be difficult or even impossible to observe any price variation at all.

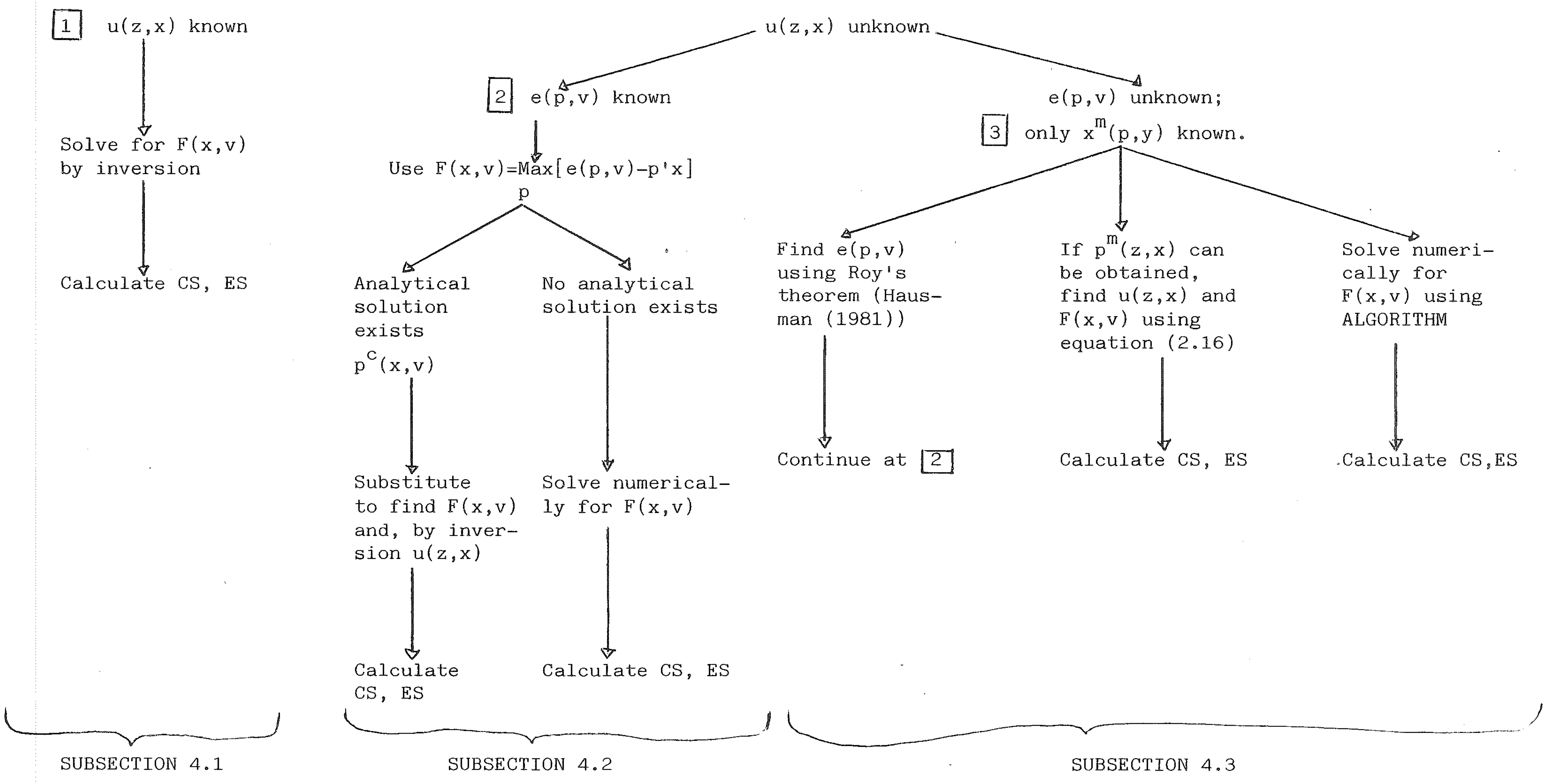

Figure 4 : Overview of procedures to determine the surplus measures ES and CS

5. Summary and Conclusion

In this paper we discussed the potential usefulness of a new distance function for applied research. The important dual relationship between this new distance function and the expenditure function is fully explored and several implications are studied in detail. Moreover, the properties of the distance function are used to define and evaluate Hicks' equivalent and compensating surplus, which are important tools for measuring the change in welfare in quantity-constrained regimes.

We show that the distance function and its duality with the expenditure function implies a simple procedure to derive the direct utility function, up to a monotonic transformation, from an arbitrary valid expenditure function. The procedure only requires the solution of an unconstrained optimization problem and will yield the desired direct utility function provided that a closed-form solution exists. We further use the properties of the distance function to show that under some conditions it is possible to solve for the direct utility function when only the (inverse) Marshallian demand function is known.

In the second part of the paper the distance function is employed to define the Hicksian surplus measures and to propose analytical and numerical evaluation procedures. We propose alternative methods to determine exact surplus measures of welfare change depending upon the information a researcher has available. We distinguish three cases that cover all possibilities that may be important in empirical research. Analytical and numerical procedures are developed to calculate Hicks' equivalent and compensating surplus starting from a direct utility function, a valid expenditure or indirect utility function and, finally, the system of Marshallian demand functions. To evaluate the surpluses using only the information provided by the market demand functions we propose an algorithm that involves the numerical solution of nonlinear first order differential equations. The algorithm is similar to Vartia's procedure to find the compensated income of a price change but it is based on the economic properties of the distance function rather than the expenditure function.

The results of this paper are particularly relevant for evaluating the welfare implications of government policies involving changes in imposed quantities. Both the theoretical analysis and the proposed numerical techniques will be useful in empirical research dealing with quantity constrained regimes and public goods. They provide a series of tools to evaluate variations in public good provision and changes in existing government programs in the area of health, education and housing.

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