STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

SPECIFICATION AND ESTIMATION OF QUARTERLY MULTI-MARKET DISEQUILIBRIUM MODELS

An Application to the Belgian Manufacturing Sector

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Report 85/173

April 1985

*We acknowledge the National Science Foundation in Belgium for financial support through FKFO-contract 2.0018.79. The first author is also at the Department of Econometrics at the University of Tilburg, where this research is part of the V.F.-program : "Equilibrium and Disequilibrium in Supply and Demand". We cordially thank Mrs. Patricia De Bruyn for her patient and skilful typing.

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D/1985/1169/05

Abstract

Since the production elasticities of the (total) number of employed people and the (average) number of working hours per employed person are generally unequal, it is sensible to split the labour market into two sub-markets. In this paper, an analytical and empirical comparison will be performed between a 2-markets (goods and labour) and a 3-markets (goods and two types of labour) quantity rationing model (QRM).

A comprehensive maximum likelihood estimation procedure will be derived for each QRM under both circumstances that producers and consumers know and do not know beforehand the constraints perceived on the other market(s). After an explicit consideration of a rationing scheme for micro-markets, an econometric estimation will be carried out for "aggregate" and "disaggregate" 2- and 3-markets QRM's, applied on quarterly data of the Belgian manufacturing sector for the period 1963^{III}-1979^{IV}.

Contents

Introduction

Since prices and wages adjust too sluggishly to ensure permanent market clearing, agents at the "long side" of each market will be rationed and, as a direct consequence of their (assumed) optimizing behaviour, they will revise their demands and supplies on the other markets. These "spill-over" effects account for the distinction between notional and effective demands and supplies.

These concepts have been illustrated by Barro and Grossman (1971) for an aggregate two-markets model (goods and labour). One of the major characteristics of such quantity rationing models (QRM's) is that, in such a situation, the economy passes through (four) distinct regimes, in each of which different behavioural relationships are valid. Hence, traditional policy instruments may exert quite different impacts depending on the prevailing regime. In this respect, Meersman and Plasmans (1983) demonstrated that a price decrease (with constant wages) will in general have a reducing effect on unemployment in the Keynesian Unemployment regime where there is excess supply on both markets (as has often occurred during the last decade) and an increasing effect on unemployment in the Classical Unemployment regime with excess demand on the goods market. In other words, an income policy made possible by the exogenous manipulation of the real wage rate has even opposite effects on unemployment in these two regimes. Similarly, the impact of changes in the exogenous variables (prices, wages, autonomous consumption, budgets) on the location of regimes has been studied extensively.

Up to now, only a few attempts have occurred to specify and estimate an aggregate two-markets QRM. Under the assumption of exogenous and rigid prices and wages, Artus, Laroque and Michel (1982) for France, Kooiman and Kloek (1981) for the Netherlands and Sneessens (1981) for Belgium have made such an effort. Only the first-mentioned paper discusses a quarterly QRM, the other papers are based on yearly data, although the two-markets QRM has a short-term character.

It should be stressed, however, that although the Barro-Grossman model provides a very useful framework for the study of theoretical QRM's, its simple econometric transposition may not be very realistic for applied macro-economics because of the aggregate min(imum)-condition, which implies that the whole economy may switch from one regime to another, making suddenly ineffective the economic policy which was the most appropriate at the previous period. The simultaneous occurrence of unemployment and vacancies and/or of stocks and unfilled orders is impossible to account for in this type of aggregate QRM's. Therefore, an aggregate market should be considered as a total over a large number of micro-markets in disequilibrium. The "smoothing by aggregation" approach, proposed by Muellbauer (1978) and Malinvaud (1980) and applied by Kooiman (1982) and Lambert (1984), takes account of this aggregation principle and leads to an aggregate transaction function which is continuous and non-linear in the aggregate demand and supply quantities. A major advantage of this approach is that the effects of distinct economic policies evolve continuously over the business cycle, depending on the proportion of goods and labour markets being in various disequilibrium situations.

In this paper, both alternative approaches, i.e. the "aggregate" approach by means of a discontinuous min-condition for the aggregate transactions and the "disaggregate" approach leading to a continuous aggregate transaction function, will be mutually compared for as well a quarterly two-markets as a quarterly three-markets QRM. The three-markets QRM emerges because the labour market has been split into two submarkets : one market for the number of employed people and one market for the (average) number of working hours per employed person. The purpose of this splitting is, among others, to study the impact of a shorter working time and of a growing unemployment in a non-Walrasian economy and to take due account of the finite elasticity between working time and employment.

In this paper we consider, both for the aggregate and the disaggregate version, an open economy without any endogenous treatment of the government. The decisions are taken by consumers and producers. The (representative) producer supplies the commodities and demands a number of workers and also an average number of hours of work per worker. The (representative) consumer demands commodities, decides whether or not he will enter the labour market and supplies a number of hours of work. The (representative) producer is supposed to maximize expected profits under a short run C.E.S.-production technology in both labour inputs. The (representative) consumer is supposed to maximize a strongly separable Johansen-type utility function in commodities, leisure and real cash balances subject to a budget constraint. The aggregate supply of the number of workers can be based on a labour force participation model.

It is supposed that producers and consumers can be rationed in the commodity market, in the market for the number of workers and in the market for the average number of hours of work per worker. Hence, eight disequilibrium or quantity rationing regimes can be derived.

In section 1 of this paper, notional and effective demands and supplies are derived for the above-mentioned three-markets QRM and the spill-over effects of the 8 quantity rationing regimes are evaluated explicitly.

Under the implicit assumption of this section that the agents know in theory the constraints perceived on the other markets (because of the assumed CES-production function and the Johansen-type utility function), the joint likelihood function of the three-markets QRM is evaluated by means of the spill-over effects in section 2.

To simplify this maximum likelihood estimation procedure as much as possible in the three-markets case, we follow the Malinvaud's (1977) reasoning that the regime of underconsumption, where the producer is rationed on both the commodity market as well as on the labour market(s) (situation of "labour hoarding") is not likely to occur in reality because of efficiency rules. Hence, we can exclude this regime and also the two related ones with changing opposite disequilibria in the labour markets. A statistically identified 5-regime QRM results, which is estimated by a full-information maximum likelihood (FIML)method.

In section 3, it is assumed that producers and consumers do not know a priori the (theoretical) constraints perceived on the other markets. It is proved that the resulting econometric three-markets QRM is not always statistically identified then and that the identification problem is solved by taking one of the dependent variables as exogenously determined (in our case : the labour force participation or the supply of workers). A FIML estimation procedure is applied on the resulting statistically identified 8-regime QRM.

In section 4, a comparison to a companion two-markets QRM is made, where labour is measured as the (total) number of working hours per period (quarter).

Finally, section 5 treats the transition from micro-to macro-markets and presents the econometric estimation results of the "aggregate" and "disaggregate" two- and three-markets QRM's, applied on quarterly data of the Belgian manufacturing sector for the period 1963^{III} - 1979^{IV}. The characteristics of the various empirical QRM's are mutually compared.

1. Walrasian and Effective Quantities

1.1. Introduction

Utilizing the following notation

- y the quantities on the commodity market
- x_1 the number of workers
- x_2 the number of hours per work,

with a superscript d* or s* denoting Walrasian demand or supply and a superscript d' or s' denoting effective demand or supply, and a superscript ¯ indicating the physical constraints determining the transactions (minimum-quantities)

the eight different quantity rationing regimes for our 3-market setting can be summarized as follows :

I-PCC	II-PCP	III-PPC	IV-PPP
$\bar{y} = y^{d'} < y^{s'}$ $\bar{x}_1 = x_1^{d'} < x_1^{s'}$ $\bar{x}_2 = x_2^{d'} < x_2^{s'}$	$\bar{y} = y^{d'} < y^{s'}$ $\bar{x}_1 = x_1^{d'} < x_1^{s'}$ $x_2^{d'} > x_2^{s'} = \bar{x}_2$	$\bar{y} = y^{d'} < y^{s'}$ $x_1^{d'} > x_1^{s'} = \bar{x}_1$ $\bar{x}_2 = x_2^{d'} < x_2^{s'}$	$\bar{y} = y^{d'} < y^{s'}$ $x_1^{d'} > x_1^{s'} = \bar{x}_1$ $x_2^{d'} > x_2^{s'} = \bar{x}_2$
V-CCC	**VI-CCP**	**VII-CPC**	**VIII-CPP**
$y^{d'} > y^{s'} = \bar{y}$ $\bar{x}_1 = x_1^{d'} < x_1^{s'}$ $\bar{x}_2 = x_2^{d'} < x_2^{s'}$	$y^{d'} > y^{s'} = \bar{y}$ $\bar{x}_1 = x_1^{d'} < x_1^{s'}$ $x_2^{d'} > x_2^{s'} = \bar{x}_2$	$y^{d'} > y^{s'} = \bar{y}$ $x_1^{d'} > x_1^{s'} = \bar{x}_1$ $\bar{x}_2 = x_2^{d'} < x_2^{s'}$	$y^{d'} > y^{s'} = \bar{y}$ $x_1^{d'} > x_1^{s'} = \bar{x}_1$ $x_2^{d'} > x_2^{s'} = \bar{x}_2$

where the first letter of regime classification indicates the (representative) agent, i.e., consumer or producer, who is rationed in the commodity market, the second letter that agent being rationed in the labour force market and the third letter the agent being rationed in the working hours market. Hence, the first quantity rationing regime PCC, denoting an excess supply in all markets, implies that the producer is only rationed in the commodity market and that the consumer is being rationed in both labour type markets. It is the situation of unemployment and starting stock formation, i.e., a Keynesian underutilization, which can be restored by extra expenditures.

If there is any rationing in a market, this will influence the demand and supply in the other markets. This influence is called the spill-over effect (see Patinkin (1949, 1956)). Hence, the concept of a "spill-over" refers to a situation where an economic agent is forced to revise his notional or desired level of transactions in one market, once he meets at least one constraint on the level of his transactions in another market.

An example : the producer's spill over effects from the commodity market to the labour markets amount to

$$\frac{\partial x_1^{d'}}{\partial \bar{y}} \text{ and } \frac{\partial x_2^{d'}}{\partial \bar{y}} \qquad \text{respectively.}$$

When the producer is confronted with a quantity constraint in the commodity market, this means that he cannot realize his Walrasian supply. As we have assumed absence of (lasting) stock formation, the producer will have to shift his production below the Walrasian output. Therefore, he will need less labour input. The effective demand for workers will be smaller than the Walrasian demand. The same holds for the effective demand for average hours of work per worker.

Now, a specific QRM will be elaborated and Walrasian and effective demands and supplies will be derived.

We get the notional or Walrasian demand and supply functions, when these functions depend only on the commodity prices and the wage cost for the labour markets, but where there is no quantity rationing from another market.

In order to derive the Walrasian quantities, we have to work out the consumer's and producer's optimizing behaviour.
First, we consider a (representative) consumer who maximizes

the utility function $U := \sum_{t=0}^{\infty} l_t U_t$, with the Johansen-type utility for each period t = 0, 1, 2, ... :

$$U_t = \frac{\beta_1}{\alpha_1}\left(\frac{y_t}{\beta_1}\right)^{\alpha_1} - \frac{\beta_2}{\alpha_2}\left(\frac{x_{2t}}{\beta_2}\right)^{\alpha_2} + \frac{\beta_3}{\alpha_3}\left(\frac{M_t/p_{c,t}}{\beta_3}\right)^{\alpha_3} \tag{1.1}$$

where y_t represents the quantities transacted on the commodity market at period t,
x_{2t} is the average number of hours of work for the individual during period t,
M_t is the nominal money stock at period t,
$p_{c,t}$ stands for the consumer price index at period t and
l_t a discount factor which attributes less importance to future utilities,

and where the parameters have to satisfy the following conditions :

$$\alpha_1, \alpha_2, \alpha_3 < 1$$
$$\beta_1, \beta_2, \beta_3 > 0$$

The budget restriction is given by :

$$p_{c,t}\, y_t + M_t = \left\{ w_t\, (1-q_t)\, x_{2t} + N_t \right\} (1-v_t) + M_{t-1} \tag{1.2}$$

with q_t the average ratio denoting the employee's share of the social security payroll taxes
w_t the nominal wage rate per hour of work
N_t the non-labour income
v_t the average personal income tax rate

Money is assumed to be a pure exchange object, which market is always in equilibrium; provisionally, it is assumed that the labour force participation is exogenous (or $x_{1t} := LFP_t$; see Meersman and Plasmans (1980, 1983)).

The Walrasian commodity demand and labour supply functions can be found by solving the maximization problem by use of the Lagrangean technique.

Under the assumption that the (Walrasian) money stock is known, the consumer is confronted with the following Walrasian commodity demand and labour supply :

$$\ln y_t^{d*} = \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_1} \ln\left(\frac{M_t}{p_{c,t}}\right)^*$$

$$\ln x_{2t}^{s*} = \ln \beta_2 - \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_3 - \frac{1}{1-\alpha_2} \ln\left\{\frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\right\}$$

$$+ \frac{1-\alpha_3}{1-\alpha_2} \ln\left(\frac{M_t}{p_{c,t}}\right)^* \qquad (1.3)$$

$$\ln x_{1t}^{s*} = \ln LFP_t \text{ (exogenous)},$$

from which the negative impact of real net wages on the Walrasian supply of hours of work becomes clear.

On the producer's side, we consider a (representative) producer who maximizes the expected present value of net after-tax profits subject to a stochastic short run C.E.S.-production function in both labour inputs (the number of workers and the average number of working hours) with Hicks-neutral technological progress, or the producer has to solve the problem :

$$\max.\ E\left\{\sum_{t=o}^{\infty} \kappa_t(1-u_t)\left[p_t y_t - w_t(1+s_t)x_{1t}x_{2t}-c_t\right]\right\}$$

$$\text{subject to } y_t = Ae^{\lambda t}\left[\delta x_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\right]^{-\frac{\mu}{\rho}} + \varepsilon_t \qquad (1.4)$$

with

κ_t the discount factor at period t defined as :

$$\kappa_t := 1 \text{ for } t = o$$

$$\kappa_t := \prod_{\theta=o}^{t}\left(\frac{1}{1+r_\theta}\right) \text{ for } t > o;$$

u_t the average corporate income tax at period t;

p_t the wholesale price index for output at period t;

w_t the average wage rate per hour at period t;

s_t the average share of the employer's contributions to the social security at period t;

c_t other costs as capital maintenance costs, net depreciation costs, quasi-fixed non-wage labour costs (as, e.g., vocational training costs, certain social allowances, clothes, canteen, payments for days not worked, etc...)(*);

A a positive scaling parameter, λ a nonnegative Hicks-neutral technological progress parameter, δ a distribution parameter ($o<\delta<1$), ρ a substitution parameter ($\rho \geq -1$, or the elasticity

(*) Hart (1984,p.19) estimates the ratio between quasi-fixed non-wage labour costs and total variable labour costs

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of substitution satisfies $\sigma = \frac{1}{1+\rho} \geqq o$), μ a positive returns to scale parameter and

$\underset{\sim}{\varepsilon}_t$ an independently and identically distributed (iid) error term, denoting various stochastic impacts on the production of output, caused by weather components, deficiencies, break-downs, etc... with mathematical expectation zero and constant variance σ_ε^2, or $\underset{\sim}{\varepsilon}_t \overset{iid}{\sim} (o,\sigma_\varepsilon^2)$ (t=o,1,2,...).

Under general regularity conditions (as, e.g. summability in quadratic mean), the order of the expectation and summation operators E and Σ may be mutually interchanged in the objective functional of (1.4), so that the producer has to maximize :

$$\sum_{t=o}^{\infty} \kappa_t (1-u_t) \left[p_t \, E(\underset{\sim}{y}_t) - w_t(1+s_t)x_{1t} \, x_{2t} - c_t\right] \qquad (1.5)$$

with respect to $E(\underset{\sim}{y}_t) = y_t$, x_{1t} and x_{2t} to obtain the Walrasian output supply and the Walrasian labour demands for the number of working people and for the (average) number of hours of work in natural logarithms :

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(i.e., per man hour costs) at about 20 % in the U.S.A. and U.K. during the late 70's, beginning 80's. Due to lack of precise aggregate (annual or quarterly) data, these costs will be included in the (quasi) fixed cost term c_t in this paper. Note, however, that certain quasi-fixed labour costs could also be included into the wage costs, so that they become variable (e.g., training costs as a lump sum of wage costs, related to the total number of hours worked in a firm, canteen costs, clothes, etc...). In this case, these costs are involved in w_t and we shall continue on this line of thought.

$$\ln y_t^{s*} = \frac{2}{2-\mu}\ln A + \frac{\mu}{2-\mu}\ln\mu - \frac{\mu}{\rho(2-\mu)}\ln\delta - \frac{\mu}{\rho(2-\mu)}\ln(1-\delta) - \frac{\mu(\rho+2)}{\rho(2-\mu)}\ln 2 + \frac{2\lambda}{2-\mu}t - \frac{\mu}{2-\mu}\ln w_t + \frac{\mu}{2-\mu}\ln p_t - \frac{\mu}{2-\mu}\ln(1+s_t) \quad (1.6)$$

$$\ln x_{1t}^{d*} = \frac{1}{2-\mu}\ln A + \frac{1}{2-\mu}\ln\mu + \frac{1-\mu}{\rho(2-\mu)}\ln\delta - \frac{1}{\rho(2-\mu)}\ln(1-\delta) - \frac{\mu+\rho}{\rho(2-\mu)}\ln 2 + \frac{\lambda}{2-\mu}t - \frac{1}{2-\mu}\ln w_t + \frac{1}{2-\mu}\ln p_t - \frac{1}{2-\mu}\ln(1+s_t) \quad (1.7)$$

$$\ln x_{2t}^{d*} = \frac{1}{2-\mu}\ln A + \frac{1}{2-\mu}\ln\mu + \frac{1-\mu}{\rho(2-\mu)}\ln(1-\delta) - \frac{1}{\rho(2-\mu)}\ln\delta - \frac{\mu+\rho}{\rho(2-\mu)}\ln 2 + \frac{\lambda}{2-\mu}t - \frac{1}{2-\mu}\ln w_t + \frac{1}{2-\mu}\ln p_t - \frac{1}{2-\mu}\ln(1+s_t) \quad (1.8)$$

from which it follows that the Walrasian commodity supply is (generally) an increasing function of the commodity price and a decreasing function of variable nominal labour costs and the Walrasian demand for workers increases when these labour costs decrease. Finally, the Walrasian demand for working hours will decrease when nominal wages increase and wholesale prices decrease.

1.3. The effective demand and supply functions

The effective quantities are obtained when the functions do not only depend on the price or wage component but also on a quantity rationing component from another market. The Clower effective demand and supply functions result from the maximization of the trader's preferences taking account of all quantity constraints except those prevailing on that market. The Drèze effective quantities are calculated by taking account of all constraints.
Throughout this paper we employ the Clower effective functions.

To obtain these (Clower) effective demand and supply functions, we have to consider the various consumer's and producer's rationing schemes.

1.3.1. The consumer is rationed in the commodity market

Then, the effective commodity demand is equal to the Walrasian commodity demand in (1.3) and assuming that the rationing in the commodity market is reflected in the money stock by supposing that :

$$\ln\left(\frac{M_t}{p_{c,t}}\right)' = \ln\left(\frac{M_t}{p_{c,t}}\right)^* + \gamma_1(\ln y_t^{d*} - \ln \bar{y}_t)$$

$$= \ln\left(\frac{M_t}{p_{c,t}}\right)^* + \gamma_1 \ln \beta_1 - \gamma_1 \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3$$

$$+ \gamma_1 \frac{1-\alpha_3}{1-\alpha_1} \ln\left(\frac{M_t}{p_{c,t}}\right)^* - \gamma_1 \ln \bar{y}_t \qquad (1.9)$$

The effective consumer's supply of the average number of hours of work can be found by redefining the Walrasian labour supply in (1.3) as the effective labour supply, taking account of (1.9) :

$$\ln x^{s'}_{2t} = \ln \beta_2 + \gamma_1 \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_1 - \{\frac{1-\alpha_3}{1-\alpha_2} + \gamma_1 \frac{(1-\alpha_3)^2}{(1-\alpha_1)(1-\alpha_2)}\} \ln \beta_3$$

$$- \frac{1}{1-\alpha_2} \ln \{\frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\} + \frac{1-\alpha_3}{1-\alpha_2} \{1 + \gamma_1 \frac{1-\alpha_3}{1-\alpha_1}\} \ln (\frac{M_t}{p_{c,t}})^*$$

$$- \gamma_1 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{y}_t, \quad (*) \qquad (1.10)$$

where the spill-over elasticity is $-\gamma_1 \frac{1-\alpha_3}{1-\alpha_2}$.

Also

$$\ln x^{s'}_{1t} = \ln x^{s*}_{1t} = \ln x^{s}_{1t} = \ln LFP_t \text{ (exogenous)}.$$

(*) Assuming that the real Walrasian money demand could be approximated by an adaptive expectations model, we could write :

$$(\frac{M_t}{p_{c,t}})^* \cong A \prod_{i=0}^{n} (\frac{M_{t-i}}{p_{c,t-i}})^{w_i}, \quad w_i \geqq 0, \quad \sum_{i=0}^{n} w_i = 1 \qquad (1.11)$$

1.3.2. The consumer is rationed in the commodity market and on the number of workers

Following the assumption :

$$\ln\left(\frac{M_t}{p_{c,t}}\right)' = \ln\left(\frac{M_t}{p_{c,t}}\right)^* + \gamma_2 (\ln y_t^{d*} - \ln \bar{y}_t) + \gamma_3 (\ln x_{1t}^{s*} - \ln \bar{x}_{1t}) \qquad (1.12)$$

we get for the effective supply of the average number of hours of work from substituting (1.12) into the effective supply, derived from (1.3) and taking account of the Walrasian commodity demand in (1.3) :

$$\ln x_{2t}^{s'} = \ln \quad \beta_2 \; -\left\{\frac{1-\alpha_3}{1-\alpha_2} + \gamma_2 \frac{(1-\alpha_3)^2}{(1-\alpha_1)(1-\alpha_2)}\right\} \ln \beta_3 + \gamma_2 \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_1$$

$$- \frac{1}{1-\alpha_2} \ln \left\{ \frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\right\} + \frac{1-\alpha_3}{1-\alpha_2} \left\{1+\gamma_2 \frac{1-\alpha_3}{1-\alpha_1}\right\} \ln \left(\frac{M_t}{p_{c,t}}\right)^*$$

$$+ \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln x_{1t}^s - \gamma_2 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{y}_t - \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{x}_{1t} \qquad (1.13)$$

If the consumer is rationed in the commodity market and on the average number of hours, there is no influence on the average number of workers, since it is assumed exogenous.

1.3.3. The consumer is rationed on the number of workers and on the average number of hours of work

Assuming that :

$$\ln (\frac{M_t}{p_{c,t}})' = \ln (\frac{M_t}{p_{c,t}})^* + \gamma_4 (\ln x_{1t}^{s*} - \ln \bar{x}_{1t}) + \gamma_5 (\ln x_{2t}^{s*} - \ln \bar{x}_{2t}) \quad (1.14')$$

the effective commodity demand can be written from substitution of (1.14) into the effective demand analogue of (1.3), taking account of the Walrasian number of hours-supply in (1.3) as :

$$\ln y_t^{d'} = \ln \beta_1 + \gamma_5 \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_2 - \{\frac{1-\alpha_3}{1-\alpha_1} + \gamma_5 \frac{(1-\alpha_3)^2}{(1-\alpha_1)(1-\alpha_2)}\} \ln \beta_3$$

$$- \gamma_5 \frac{1-\alpha_3}{(1-\alpha_1)(1-\alpha_2)} \ln \{\frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\} + \frac{1-\alpha_3}{1-\alpha_1} \{1 + \gamma_5 \frac{1-\alpha_3}{1-\alpha_2}\} \ln (\frac{M_t}{p_{c,t}})^*$$

$$+ \gamma_4 \frac{1-\alpha_3}{1-\alpha_1} \ln x_{1t}^s - \gamma_4 \frac{1-\alpha_3}{1-\alpha_1} \ln \bar{x}_{1t} - \gamma_5 \frac{1-\alpha_3}{1-\alpha_1} \ln \bar{x}_{2t} \quad (1.15)$$

1.3.4. The producer is rationed in the commodity market

Then, the effective commodity supply coincides with the Walrasian supply and the effective demands for workers and for the number of working hours can be found by solving the following programming problem, assuming that the producer knows his restrictions on the commodity market (then, he will produce without inventory (see (1.4-5)))(*)

(*) In section 3 of this paper we will also assume that inventories are allowed (e.g. when the producer does not exactly know his restrictions on the commodity market).

$$\underset{\{x_{1t},x_{2t}\}}{\text{Max.}} \sum_{t=0}^{\infty} \kappa_t(1-u_t)\{p_t y_t - w_t(1+s_t)x_{1t}x_{2t} - c_t\}$$
$$\text{s.t. } y_t = Ae^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta)\, x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \qquad (1.16)$$
$$y_t = \bar{y}_t$$

This yields as effective factor demand equations:

$\ln x_{1t}^{d'} = \frac{1}{\mu} \ln \bar{y}_t - \frac{1}{\mu} \ln A + \frac{1}{\rho} \ln 2 + \frac{1}{\rho} \ln \delta - \frac{\lambda}{\mu} t$	(1.17)
$\ln x_{2t}^{d'} = \frac{1}{\mu} \ln \bar{y}_t - \frac{1}{\mu} \ln A + \frac{1}{\rho} \ln 2 + \frac{1}{\rho} \ln (1-\delta) - \frac{\lambda}{\mu} t$	(1.18)
$\ln y_t^{s'} = \ln y_t^{s^*}$	(1.6)

so that the spill-over elasticities are equal for both labour demands (= $1/\mu$).

1.3.5. The producer is rationed on the number of workers

Then the effective demand for workers is equal to the Walrasian demand and the effective demand for the average number of hours of work per worker and the effective commodity supply are found from the solution of :

$$\underset{\{y_t, x_{2t}\}}{\text{Max}} \sum_{t=0}^{\infty} \kappa_t(1-u_t)\ \{p_t y_t - w_t(1+s_t)x_{1t}\, x_{2t} - c_t)\}$$
$$\text{s.t. } y_t = A\, e^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \qquad (1.19)$$
$$x_{1t} = \bar{x}_{1t}$$

which leads to the following expressions :

$$w_t(1+s_t)x_{2t} = p_t A e^{\lambda t}\mu\delta\ \{\delta\ x_{1t}^{-\rho} + (1-\delta)\ x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}}\ \bar{x}_{1t}^{-\rho-1} \qquad -\sigma_2$$
$$w_t(1+s_t)\bar{x}_{1t} = p_t\ A\ e^{\lambda t}\ \mu(1-\delta)\ \{\delta\ \bar{x}_{1t}^{-\rho} + (1-\delta)\ x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}}\ x_{2t}^{-\rho-1}$$
$$y_t = Ae^{\lambda t}\{\delta \bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \qquad (1.20)$$

We will use the last two equations to find $y_t^{s'}$ and $x_{2t}^{d'}$. Therefore, first order approximations for

$$\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}} \text{ and for } \{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}}$$

should be utilized.

(i) Approximation for $\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{\frac{-\mu+\rho}{\rho}}$

$$-\frac{\mu+\rho}{\rho} \ln \{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\} = (\mu+\rho) \ln x_{2t} - \frac{\mu+\rho}{\rho} \ln \{\delta(\frac{\bar{x}_{1t}}{x_{2t}})^{-\rho} + 1-\delta\}$$

Let

$$f(\ln\frac{x_{1t}}{x_{2t}}) = -\frac{\mu+\rho}{\rho} \ln \{\delta(\frac{\bar{x}_{1t}}{x_{2t}})^{-\rho} + 1-\delta\}$$

A first order Taylor approximation around $\rho=0$ (Cobb-Douglas case : Kmenta-approximation) of this function gives :

$$f(\ln\frac{\bar{x}_{1t}}{x_{2t}}) \cong \delta(\mu+\rho) \ln \frac{\bar{x}_{1t}}{x_{2t}}$$

This yields

$$\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}} \cong \bar{x}_{1t}^{\delta(\mu+\rho)} \; x_{2t}^{(1-\delta)(\mu+\rho)} \quad (1.21)$$

(ii) Approximation for $\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}}$

$$-\frac{\mu}{\rho} \ln\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\} = \mu \ln x_{2t} - \frac{\mu}{\rho} \ln\{\delta(\frac{\bar{x}_{1t}}{x_{2t}})^{-\rho} + 1-\delta\}$$

$$= \mu \ln x_{2t} + g(\ln \frac{\bar{x}_{1t}}{x_{2t}})$$

A first order Taylor approximation of g(.) around $\rho = 0$ gives

$$g(\ln \frac{\bar{x}_{1t}}{x_{2t}}) \cong \delta\mu \ln \frac{\bar{x}_{1t}}{x_{2t}}$$

Hence, we get :

$$\{\delta \bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \cong \bar{x}_{1t}^{\mu\delta} x_{2t}^{\mu(1-\delta)} \qquad (1.22)$$

Writing equalities, we have to solve the following two equations :

$$w_t(1+s_t)\bar{x}_{1t} = p_t A\mu(1-\delta)\bar{x}_{1t}^{\delta(\mu+\rho)} x_{2t}^{-\rho-1} + (1-\delta)\ (\mu+\rho)$$

$$y_t = Ae^{\lambda t}\bar{x}_{1t}^{\mu\delta}\ x_{2t}^{\mu(1-\delta)} \qquad (1.23)$$

This yields :

$$\ln x_{2t}^{d'} = \frac{1}{a} \ln w_t + \frac{1}{a} \ln (1+s_t) - \frac{1}{a} \ln p_t - \frac{1}{a} \ln A - \frac{1}{a} \ln \mu - \frac{\lambda}{a} t + \frac{1-\delta(\mu+\rho)}{a} \ln \bar{x}_{1t} \qquad (1.24)$$

$$\ln y_t^{s'} = -\frac{1+\rho\delta}{a} \ln A - \frac{(1-\delta)\mu}{a} \ln \mu + \frac{(1-\delta)\mu}{a} \ln w_t + \frac{(1-\delta)\mu}{a} \ln (1+s_t) - \frac{(1-\delta)\mu}{a} \ln p_t - \lambda\frac{1+\rho\delta}{a}t + \frac{\mu\{1-\delta(\rho+2)\}}{a} \ln \bar{x}_{1t} \qquad (1.25)$$

with $a: = (\mu+\rho)(1-\delta) - \rho - 1$

Hence, the spill-over elasticities for the effective demand for the average hours of work per worker and for the effective commodity supply are respectively :

$$\frac{1-\delta(\mu+\rho)}{(\mu+\rho)(1-\delta)-\rho-1} \quad \text{and} \quad \frac{\mu\{1-\delta(\rho+2)\}}{(\mu+\rho)(1-\delta)-\rho-1}$$

1.3.6. The producer is rationed on the average number of hours of work per worker.

Using the same approximations as under 1.3.5. we find :

$$\ln x_{1t}^{d'} = -\frac{1}{b}\ln A - \frac{1}{b}\ln\mu - \frac{1}{b}\ln\delta - \frac{\lambda}{b}t + \frac{1}{b}\ln w_t + \frac{1}{b}\ln(1+s_t) - \frac{1}{b}\ln p_t + \frac{1-(1-\delta)(\mu+\rho)}{b}\ln\bar{x}_{2t} \qquad (1.26)$$

$$\ln y_t^{s'} = \frac{\rho(\delta-1)-1}{b}\ln A - \frac{\delta\mu}{b}\ln\mu - \frac{\delta\mu}{b}\ln\delta + \lambda'\frac{\rho(\delta-1)-1}{b}t + \frac{\mu\delta}{b}\ln w_t + \frac{\mu\delta}{b}\ln(1+s_t) - \frac{\mu\delta}{b}\ln p_t + \frac{\mu(\delta\rho+2\delta-\rho-1)}{b}\ln\bar{x}_{2t} \qquad (1.27)$$

with

$$b := \delta(\mu+\rho) - \rho - 1$$

so that the spill-over elasticities are respectively:

$\frac{1-(1-\delta)(\mu+\rho)}{\delta(\mu+\rho)-\rho-1}$ on labour demand and $\frac{\mu(\delta\rho+2\delta-\rho-1)}{\delta(\mu+\rho)-\rho-1}$ on commodity supply.

If the producer is rationed on 2 markets, the C.E.S.-production function uniquely determines the effective quantity. Using Kmenta's approximation (Taylor series expansion around ρ=o), the effective commodity supply can, e.g., be written as a Cobb-Douglas specification of the rationed labour quantities :

$$\ln y_t^{s'} = \ln A + \lambda t + \mu\delta \ln \bar{x}_{1t} + \mu(1-\delta) \ln \bar{x}_{2t}, \quad (1.28)$$

where the spill-over elasticities are simply the (linearized) output elasticities with respect to the number of workers and the average number of hours of work per worker respectively.

In the next section, the joint likelihood function for the 3-markets QRM will be derived.

2. The joint likelihood function under "known" market conditions"

2.1. Introduction

Because we are working with a model where three markets are allowed, and because there is an excess demand or an excess supply in each market, eight different disequilibrium regimes are possible. We cannot deny the theoretical possibility of the underconsumption regime, where the producer is constrained on all markets.

But according to Malinvaud (1977) this regime, where the producers would like to attract more people than they are currently supplied with, notwithstanding the fact that they will not be able to increase sales (due to insufficient demand) this re-

gime only makes sense in multi-period setting, where stocks of finished, but as yet unsold, products can be carried over to the next period. When we use the effective relationships of the first section, where there is no inventory formation, the above problem cannot occur. That is the reason why we have excluded this regime and the two relates ones, with changing opposite disequilibria on the labour markets. So, there still remain five disequilibrium regimes. The first regime is recognized as a Keynesian unemployment with general excess supply - see I-PCC in table 1 -, the second as a classical unemployment with excess commodity demand (regime V-CCC) and the third regime as a repressed inflation regime with general excess demand (regime VIII-CPP).
Due to the splitting of the labour market, the 4th and the 5th regimes are typical, i.e., the VI-CCP and the VII-CPC-regimes of table 1.
Now, the QRM's of the 5 remaining regimes will be stochastically specified and the relating joint likelihood function will be derived.

2.2. The stochastic specification of the QRM

In principle we follow the procedure proposed by Kooiman and Kloek (1981) and Artus, Laroque and Michel (1982). Differences, however, occur owing to the introduction of a CES-production function and the consideration of a three market disequilibrium model.

For the producer we get the following set of general formulae, where the $\underline{\varepsilon}_i$'s are random error terms with expectation zero and constant variances :

(i) $\ln \underline{x}_1^d = \ln x_1^d(z) + \underline{\varepsilon}_1$

(ii) $\ln \underline{x}_2^d = \ln x_2^d(z) + \underline{\varepsilon}_2$

$$\begin{aligned}
&\text{(iii)}\quad \ln \underline{y}^s = \ln f(x_1^d(z),\ x_2^d(z)) + \underline{\varepsilon}_3 &\quad (2.1)\\
&\text{(iv)}\quad \ln \underline{x}_1^{d'} = \ln \underline{x}_1^d - \eta_1(\ln \underline{y}^s - \ln \bar{\underline{y}}) - \eta_2(\ln \underline{x}_2^d - \ln \bar{\underline{x}}_2)\\
&\text{(v)}\quad \ln \underline{x}_2^{d'} = \ln \underline{x}_2^d - \eta_3(\ln \underline{y}^s - \ln \bar{\underline{y}}) - \eta_4(\ln \underline{x}_1^d - \ln \bar{\underline{x}}_1)\\
&\text{(vi)}\quad \ln \underline{y}^{s'} = \ln \underline{y}^s - \kappa_1(\ln \underline{x}_1^d - \ln \bar{\underline{x}}_1) - \kappa_2(\ln \underline{x}_2^d - \ln \bar{\underline{x}}_2)\\
&\qquad \text{when } 0 \leqq \eta_i \leqq 1 \text{ and } 0 \leqq \kappa_i \leqq 1 \text{ for all } i
\end{aligned}$$

The producer demands labour and supplies commodities. The stochastic notional labour demand functions are represented by (i) and (ii)(*) and are explained by their deterministic parts $\ln n_1^d(z)$ and $\ln x_2^d(z)$, where the vector z summarizes all exogenous variables in the model. The functions $\ln x_1^d(z)$ and $\ln x_2^d(z)$ can be either effective or Walrasian according to the kind of rationing regime considered and are given in the first section, $\bar{y}$, $\bar{x}_1$ and $\bar{x}_2$ being the market transactions; $\underline{\varepsilon}_1$ and $\underline{\varepsilon}_2$ are standing for stochastic deviations from deterministic quantities which should be valid if the agent would not be constrained in other markets; these deterministic quantities have been derived from economic theory (as e.g. in the previous section). All error terms will be assumed to be independently normally distributed with zero means and constant variances.

The notional supply of consumption goods is determined by a CES production function. The constrained or effective demands for labour are displayed in equations (iv) and (v) which can, theoretically, be influenced by a spill-over from the commodity market and the other labour market. The effective supply of commodities is represented by equation (vi) where quantity rationings from the labour markets are possible.

(*) To simplify notation, the * pointing to a Walrasian or notional quantity is dropped henceforth.

Similarly, for the consumer we have :

$$
\begin{aligned}
&\text{(vii)} \quad \ln \underline{x}_1^s = \ln x_1^s(z) + \underline{\varepsilon}_4 \\
&\text{(viii)} \quad \ln \underline{x}_2^s = \ln x_2^s(z) + \underline{\varepsilon}_5 \\
&\text{(ix)} \quad \ln \underline{y}^d = \ln y^d(z) + \underline{\varepsilon}_6 \\
&\text{(x)} \quad \ln \underline{x}_1^{s'} = \ln \underline{x}_1^s - \zeta_1(\ln \underline{x}_2^s - \ln \bar{x}_2) - \zeta_2(\ln \underline{y}^d - \ln \bar{y}) \\
&\text{(xi)} \quad \ln \underline{x}_2^{s'} = \ln \underline{x}_2^s - \zeta_3(\ln \underline{x}_1^s - \ln \bar{x}_1) - \zeta_4(\ln \underline{y}^d - \ln \bar{y}) \qquad (2.2) \\
&\text{(xii)} \quad \ln \underline{y}^{d'} = \ln \underline{y}^d - \kappa_3(\ln \underline{x}_1^s - \ln \bar{x}_1) - \kappa_4(\ln \underline{x}_2^s - \ln \bar{x}_2)
\end{aligned}
$$

where $0 \leq \zeta_i \leq 1$ and $0 \leq \kappa_i \leq 1$ for all i.

The consumer demands commodities and delivers labour. Analogously to the producer the notional or Walrasian demand and supply functions are displayed by equations (vii), (viii) and (ix), while the effective expressions are represented by (x), (xi) and (xii).

The likelihood function of one observation on y, x_1 and x_2 can be derived as the sum of five likelihoods, each giving the contribution to the value of the likelihood function for being in either of the five regimes (compare Gouriéroux, Laffont and Monfort (1980), Ito (1980)):

$$L = L_1 + L_2 + L_3 + L_4 + L_5$$

where $L_1 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}_1}^{\infty} \int_{\ln \bar{x}_2}^{\infty} g_1(\ln \bar{y}, \ln \bar{x}_1, \ln \bar{x}_2, \ln y^s, \ln x_1^s, \ln x_2^s) \, d \ln y^s \, d \ln x_1^s \, d \ln x_2^s$

$$L_2 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}_1}^{\infty} \int_{\ln \bar{x}_2}^{\infty} g_2(\ln \bar{y}, \ln \bar{x}_1, \ln \bar{x}_2, \ln y^d, \ln x_1^s, \ln x_2^s) \, d \ln y^d \, d \ln x_1^s \, d \ln x_2^s$$

$$\text{(2.3)} \quad L_3 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}_1}^{\infty} \int_{\ln \bar{x}_2}^{\infty} g_3(\ln \bar{y}, \ln \bar{x}_1, \ln \bar{x}_2, \ln y^d, \ln x_1^d, \ln x_2^d) \, d \ln y^d \, d \ln x_1^d \, d \ln x_2^d$$

$$L_4 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}_1}^{\infty} \int_{\ln \bar{x}_2}^{\infty} g_4(\ln \bar{y}, \ln \bar{x}_1, \ln \bar{x}_2, \ln y^d, \ln x_1^s, \ln x_2^d) \, d \ln y^d \, d \ln x_1^s \, d \ln x_2^d$$

$$L_5 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}_1}^{\infty} \int_{\ln \bar{x}_2}^{\infty} g_5(\ln \bar{y}, \ln \bar{x}_1, \ln \bar{x}_2, \ln y^d, \ln x_1^d, \ln x_2^s) \, d \ln y^d \, d \ln x_1^d \, d \ln x_2^s$$

The joint density functions g_1 through g_5 of the supply and demand variables relevant to the regime can be obtained by describing it as a product of the conditional density functions. To facilitate our notation we introduce the symbol $n(v;\Sigma)$ to denote the joint normal density function of $\underline{v}$ with zero mean vector and covariance matrix Σ and we use the symbol $N(v;\sigma^2)$ to denote the cumulative normal distribution function of the variate $\underline{v}$ with mean zero and variance σ^2. According to the given sets of equations for consumers and producers in (2.1) and (2.2) we define the following stochastic deviations from transactions :

$$
\begin{aligned}
\ln \underline{u}_1 &:= \ln \bar{\underline{x}}_1 - \ln x_1^d(z) \\
\ln \underline{u}_2 &:= \ln \bar{\underline{x}}_2 - \ln x_2^d(z) \\
\ln \underline{u}_3 &:= \ln \bar{\underline{y}} - \ln f(x_1^d(z), x_2^d(z)) \\
\ln \underline{u}_4 &:= \ln \bar{\underline{x}}_1 - \ln x_1^s(z) \qquad (2.4) \\
\ln \underline{u}_5 &:= \ln \bar{\underline{x}}_2 - \ln x_2^s(z) \\
\ln \underline{u}_6 &:= \ln \bar{\underline{y}} - \ln y^d(z)
\end{aligned}
$$

In this study we deal with the specifications of spillovers on the production side of the economy, i.e. the effective goods supply and the labour demand functions. For the <u>first regime</u> (Keynesian Unemployment), where we have a general excess supply, we get the following set of equations for the observed quantities by convenient substitution in (2.1) and (2.2) :

(i) $\ln \bar{\underline{x}}_1 = \ln \bar{\underline{x}}_1^{d'} = \ln x_1^d(z) + \underline{\varepsilon}_1 - \eta_1(\ln f(x_1^d(z), x_2^d(z)) + \underline{\varepsilon}_3 - \ln \bar{\underline{y}})$

(i), (iii), (iv) of (2.1)

or

$$\ln \bar{\underline{x}}_1 = \ln x_1^d(z) + \eta_1(\ln \bar{\underline{y}} - \ln f(x_1^d(z), x_2^d(z))) + \underline{\varepsilon}_1 - \eta_1\underline{\varepsilon}_3$$

(ii) $\ln \bar{\underline{x}}_2 = \ln \underline{x}_2^{d'} = \ln x_2^d(z) + \underline{\varepsilon}_2 - \eta_3(\ln f(x_1^d(z), x_2^d(z)) + \underline{\varepsilon}_3 - \ln \bar{\underline{y}})$

(ii),(iii), (v) of (2.1)

or

$$\ln \bar{\underline{x}}_2 = \ln x_2^d(z) + \eta_3(\ln \bar{\underline{y}} - \ln f(x_1^d(z), x_2^d(z))) + \underline{\varepsilon}_2 - \eta_3\underline{\varepsilon}_3$$

$$\text{(iii)} \quad \ln \underline{y}^{s'} = \ln \underline{y}^{s} = \ln f(x_1^d(z),\ x_2^d(z)) + \underline{\varepsilon}_3 \qquad (2.5)$$

$$\text{(iv)} \quad \ln \underline{x}_1^{s'} = \ln x_1^s(z) + \underline{\varepsilon}_4 - \zeta_1(\ln x_2^s(z) + \underline{\varepsilon}_5 - \ln \bar{\underline{x}}_2)$$

(vii), (viii), (x) of (2.2)

or

$$\ln \underline{x}_1^{s'} = \ln x_1^s(z) + \zeta_1(\ln \bar{\underline{x}}_2 - \ln x_2^s(z)) + \underline{\varepsilon}_4 - \zeta_1\underline{\varepsilon}_5$$

$$\text{(v)} \quad \ln \underline{x}_2^{s'} = \ln x_2^s(z) + \underline{\varepsilon}_5 - \zeta_3\ (\ln x_1^s(z) + \underline{\varepsilon}_4 - \ln \bar{\underline{x}}_1)$$

(vii), (viii), (xi) of (2.2)

or

$$\ln \underline{x}_2^{s'} = \ln x_2^s(z) + \zeta_3(\ln \bar{\underline{x}}_1 - \ln x_1^s(z)) + \underline{\varepsilon}_5 - \zeta_3\underline{\varepsilon}_4$$

$$\text{(vi)} \quad \ln \bar{\underline{y}} = \ln \underline{y}^{d'} = \ln y^d(z) + \underline{\varepsilon}_6 - \kappa_3(\ln x_1^s(z) + \underline{\varepsilon}_4 - \ln \bar{\underline{x}}_1)$$

$$- \kappa_4(\ln x_2^s(z) + \underline{\varepsilon}_5 - \ln \bar{\underline{x}}_2)$$

(vii), (viii), (ix), (xii) of (2.2)

or

$$\ln \bar{\underline{y}} = \ln y^d(z) + \kappa_3(\ln \bar{\underline{x}}_1 - \ln x_1^s(z)) + \kappa_4(\ln \bar{\underline{x}}_2 - \ln x_2^s(z)) +$$

$$+ \underline{\varepsilon}_6 - \kappa_3\underline{\varepsilon}_4 - \kappa_4\underline{\varepsilon}_5$$

After elaborating the joint density function g_1 as shown in Appendix A, and performing the integration according to the general excess supply regime as indicated in (2.3), we obtain the following expression for L_1 :

$$L_1 = n(\ln u_6 - \kappa_3 \ln u_4 - \kappa_4 \ln u_5;\ \sigma_6^2 + \kappa_3^2\sigma_4^2 + \kappa_4^2\sigma_5^2)$$

$$n(\ln u_4 - \zeta_1 \ln u_5;\ \sigma_4^2 + \zeta_1^2\sigma_5^2)\, n(\ln u_5 - \zeta_3 \ln u_4;\ \sigma_5^2 + \zeta_3^2\sigma_4^2)$$

$$\{1-N(\ln u_1;\ \sigma_2^2)\}$$

$$\{1-N(\ln u_2;\sigma_2^2)\}\{1-N(\ln u_3 + \frac{\eta_1\sigma_2^2\sigma_3^2}{\sigma_1^2\sigma_2^2+\eta_3^2\sigma_1^2\sigma_3^2+\eta_1^2\sigma_2^2\sigma_3^2}(\ln u_1$$

$$-\eta_1 \ln u_3) + \frac{\eta_3\sigma_1^2\sigma_3^2}{\sigma_1^2\sigma_2^2+\eta_3^2\sigma_1^2\sigma_3^2+\eta_1^2\sigma_2^2\sigma_3^2}(\ln u_2 - \eta_3 \ln u_3); \qquad (2.6)$$

$$\sigma_3^2 - \frac{\eta_1^2\sigma_2^2\sigma_3^4 + \eta_3^2\sigma_1^2\sigma_3^4}{\sigma_1^2\sigma_2^2 + \eta_3^2\sigma_1^2\sigma_3^2 + \eta_1^2\sigma_2^2\sigma_3^2}))\}$$

The other integrations can be performed analogously to the first regime (see appendix B).

The joint sample likelihood function on the observable quantities (the transactions) can then be written for T observations as :

$$L := \prod_{t=1}^{T} h(\ln \bar{y}_t,\ \ln \bar{x}_{1t},\ \ln \bar{x}_{2t}) = \prod_{t=1}^{T} \left(\sum_{i=1}^{5} L_{i,t}\right) \qquad (2.7)$$

so that its logarithm : $\ln L = \sum_{t=1}^{T} \ln\left(\sum_{i=1}^{5} L_{i,t}\right)$ has to be

maximized w.r.t. all unknown parameters.

3. A statistically identified 3-markets QRM without any a priori known constraints

If producers and consumers do not know a priori the (theoretical) constraints perceived on the other markets, so that, in this case, there is not a utility function nor a production function involved, and assuming, as in the previous sections, that the realized transactions are equal to the minimum of effective demand and effective supply, a statistically identified QRM for 3 not necessarily clearing markets will be derived.

Noticing that, in general, the supply of workers (LFP) is assumed to be endogenous, the effective demand and supply functions are specified as functions of the realized transactions according to the different quantity rationing regimes <u>in a first step</u>, so that we have for each regime i(i=1, 2, ..., 8)(*).

$$\begin{bmatrix} y^{d'}_{it} \\ y^{s'}_{it} \\ x^{d'}_{i1t} \\ x^{s'}_{i1t} \\ x^{d'}_{i2t} \\ x^{s'}_{i2t} \end{bmatrix} = F_i(\bar{y}_t, \bar{x}_{1t}, \bar{x}_{2t}) + \begin{bmatrix} \delta_{oit} \\ \lambda_{oit} \\ \delta_{1it} \\ \lambda_{1it} \\ \delta_{2it} \\ \lambda_{2it} \end{bmatrix} \tag{3.1}$$

with $\bar{y}_t$, $\bar{x}_{1t}$ and $\bar{x}_{2t}$ the quantities transacted of commodities, workers and working hours respectively, and δ_{jit} and λ_{jit} ($j = 0, 1, 2;\ i = 1, 2, \ldots 8;\ t = 1, 2, \ldots, T$) functions of prices (as, e.g., the nominal wage rate w_t, the wholesale price

(*) Note that, in contrast to the previous sections, underconsumption-types of regimes may occur now since inventories might arise when neither the consumer(s), nor the producer(s) know the constraints beforehand.

index p_t, the consumer price index $p_{c,t}$, the producer's and consumer's relative contributions to social security s_t and q_t, the average personal income tax rate u_t, etc....) and other exogenous variables (as, e.g., income, weather indicators, instrumental variables, etc....).

Linearizing the functions F_i, δ_{ji} and λ_{ji} ($j = 0, 1, 2$; $i = 1, 2, \ldots, 8$) around the corresponding Walrasian quantities and the exogenously given prices and variables respectively, the right-hand side vector of (3.1) can be written in simplifying notation as :

$$\begin{bmatrix} \underline{\delta}_{jit} \\ \underline{\lambda}_{jit} \end{bmatrix} \quad j = 0, 1, 2 \quad = B_i X_{it} + \underline{u}_{it} \quad , \qquad (3.2)^{(*)}$$

where B_i is a matrix of unknown parameters, X_{it} is a vector of (known) exogenous variables including prices and $\underline{u}_{it}$ is a 6-dimensional random vector which density g.

In the general excess supply regime (Keynesian Unemployment), where the (representative) consumer is rationed on the labour markets and the producer on the commodity market, we may write a linearized version of (3.1) as :

$$\begin{aligned} y^{d'} &= a_{11}\bar{x}_1 = a_{21}\bar{x}_2 + \delta_{01} \\ y^{s'} &= y^s \\ x_1^{d'} &= c_{01}\bar{y} + \delta_{11} \\ x_1^{s'} &= d_{21}\bar{x}_2 + \lambda_{11} \end{aligned} \qquad (3.3)$$

(*) Note that 0 stands for the commodity market, 1 for the workers' market and 2 for the number of working hours' market.

$x_2^{d'} = e_{01}\bar{y} + \delta_{21}$

$x_2^{s'} = f_{11}\bar{x}_1 + \lambda_{21}$

with $\bar{x}_1 := x_1^{d'}$; $\bar{x}_2 := x_2^{d'}$ and $\bar{y} := y^{d'}$.

The Walrasian demands and supplies are found by replacing the realized transactions by the Walrasian quantities in the effective demand (and supply) functions (fundamental identity property).

For the Walrasian commodity supply we may write :

$$y^s = b_{11}x_1^d + b_{21}x_2^d + \lambda_{01} \qquad (3.4)$$

so that we have

$$\begin{aligned}
y^d &= a_{11}x_1^s + a_{21}x_2^s + \delta_{01} \\
y^s &= b_{11}x_1^d + b_{21}x_2^d + \lambda_{01} \\
x_1^d &= c_{01}y^s + \delta_{11} \\
x_1^s &= d_{21}x_2^s + \lambda_{11} \\
x_2^d &= e_{01}y^s + \delta_{21} \\
x_2^s &= f_{11}x_1^s + \lambda_{21}
\end{aligned} \qquad (3.5)$$

Solving for y^s gives

$$y^s = \frac{b_{11}\delta_{11} + b_{21}\delta_{21} + \lambda_{01}}{1 - b_{11}c_{01} - b_{21}e_{01}} \qquad (3.6)$$

Substituting (3.6) into (3.3) and solving for $\begin{Bmatrix} \underline{\delta}_{j1} \\ \underline{\lambda}_{j1} \end{Bmatrix}_{j = 0, 1, 2}$

yields

$$\begin{Bmatrix} \underline{\delta}_{j1} \\ \underline{\lambda}_{j1} \end{Bmatrix}_{j = 0, 1, 2} = A_1 . \begin{Bmatrix} \underline{y}^{d'} \\ \vdots \\ \underline{x}_2^{s'} \end{Bmatrix} \qquad (3.7)$$

with

$$A_1 = \begin{bmatrix} 1 & 0 & -a_{11} & 0 & -a_{21} & 0 \\ (b_{11}c_{01}+b_{21}e_{01})(1-b_{11}c_{01}-b_{21}e_{01}) & -b_{11} & 0 & -b_{21} & 0 \\ -c_{01} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -d_{21} & 0 \\ -e_{01} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -f_{11} & 0 & 0 & 1 \end{bmatrix}$$

$$|A_1| = (1 - b_{11}c_{01} - b_{21}e_{01})(1 - a_{21}e_{01} - a_{11}c_{01}) \qquad (3.8)$$

In Meersman&Plasmans (1982), pp. 8 - 14, the evaluations of the A_i-matrices (i = 2, 3, ..., 8) with corresponding determinants are given. The first necessity of the resulting model is the proof of statistical identifiability. We will prove in appendix C of this paper that the corresponding simpler 2-market model (see Gouriéroux, Laffont&Monfort (1980)) is not identified. This is, by analogy, also true for the above 3-market model, which is not worked out in detail, however, in order to avoid unnecessary elaborations.

Since the identification problem is caused by the similarity between some regimes, it is argued in the appendix that a sufficient condition to obtain statistical identifiability of the 3-market QRM is the exogenisation of the labour force participation rate.

In the sequel of this section, the 3-market QRM will be re-specified with exogenous supply of the number of workers.

In a <u>second step</u>, the whole set of equations (3.2) is solved for exogenous labour force participation (LFP) :

$$\begin{bmatrix} \underline{\delta}_{oit} \\ \underline{\lambda}_{oit} \\ \underline{\delta}_{1it} \\ \underline{\delta}_{2it} \\ \underline{\lambda}_{2it} \end{bmatrix} = C_i \begin{bmatrix} \underline{y}_t^{d'} \\ \underline{y}_t^{s'} \\ \underline{x}_{1t}^{d'} \\ \underline{x}_{2t}^{d'} \\ \underline{x}_{2t}^{s'} \end{bmatrix} + k_i LFP_t = \Lambda_i X_{it} + \underline{\varepsilon}_{it} \quad (3.9)$$

$$(i = 1, 2, \ldots, 8)$$

with $LFP_t : x_{1t}^{s} = x_{1t}^{s'}$, and where

$\underline{\delta}_{oit}$, $\underline{\lambda}_{oit}$, $\underline{\delta}_{1it}$, $\underline{\delta}_{2it}$, $\underline{\lambda}_{2it}$ are functions of prices and other variables,

C_i are (5 x 5)-matrices of unknown parameters,

k_i are 5-dimensional vectors of unknown parameters belonging to the impact of the workers' supply,

Λ_i is a matrix of unknown parameters,

X_{it} is a vector of exogenous variables, including prices

and $\underline{\varepsilon}_{it}$ is a 5-dimensional random vector with probability density g(.).

In a <u>third</u> step, the joint density of the effective quantities, and from this, the simultaneous density of the transactions ($\bar{y}_t$, $\bar{x}_{1t}$, $\bar{x}_{2t}$) can be derived for each observation period as :

$$
\begin{aligned}
h_t(\bar{y}, \bar{x}_1, \bar{x}_2) = \int_{\bar{y}}^{\infty}\int_{\bar{x}_1}^{\infty}\int_{\bar{x}_2}^{\infty} \Big[& g\{ C_1.(\bar{y},y,\bar{x}_1,\bar{x}_2,x_2)' + k_1 LFP - \Lambda_1 X\} \, |C_1| \\
& + g\{ C_2.(\bar{y},y,\bar{x}_1,x_2,\bar{x}_2)' + k_2 LFP - \Lambda_2 X\} \, |C_2| \\
& + g\{ C_3.(\bar{y},y,x_1,\bar{x}_2,x_2)' + k_3 LFP - \Lambda_3 X\} \, |C_3| \\
& + g\{ C_4.(\bar{y},y,x_1,x_2,\bar{x}_2)' + k_4 LFP - \Lambda_4 X\} \, |C_4| \\
& + g\{ C_5.(y,\bar{y},\bar{x}_1,\bar{x}_2,x_2)' + k_5 LFP - \Lambda_5 X\} \, |C_5| \\
& + g\{ C_6(y,\bar{y},\bar{x}_1,x_2,\bar{x}_2)' + k_6 LFP - \Lambda_6 X\} \, |C_6| \\
& + g\{ C_7(y,\bar{y},x_1,\bar{x}_2,x_2)' + k_7 LFP - \Lambda_7 X\} \, |C_7| \\
& + g\{ C_8(y,\bar{y},x_1,x_2,\bar{x}_2)' + k_8 LFP - \Lambda_8 X\} \, |C_8| \Big] \\
& dy \; dx_1 \; dx_2 \qquad (3.10),
\end{aligned}
$$

with C_1 derived from (3.8), by deleting the fourth row and the fourth column (i.e., that of $x^{s'}_{1t} = LFP_t$) of the A_1-matrix :

$$
C_1 = \begin{bmatrix} 1 & 0 & -a_{11} & -a_{21} & 0 \\ b_{11}c_{01}+b_{21}e_{01} & (1-b_{11}c_{01}-b_{21}e_{01}) & -b_{11} & -b_{21} & 0 \\ -c_{01} & 0 & 1 & 0 & 0 \\ -e_{01} & 0 & 0 & 1 & 0 \\ 0 & 0 & -f_{11} & 0 & 1 \end{bmatrix}, \; k_1 = \{0\} \qquad (3.11)
$$

with

$$|C_1| = (1 - b_{11}c_{01} - b_{21}e_{01})\,(1 - a_{21}e_{01} - a_{11}c_{01}).$$

and, proceeding in a similar way as above, for the other 7 regimes :

$$C_2=\begin{bmatrix} 1 & 0 & -a_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 & -b_{22} \\ -c_{02} & 0 & 1 & 0 & -c_{22} \\ -e_{01} & 0 & 0 & 1 & 0 \\ 0 & 0 & -f_{11} & 0 & 1 \end{bmatrix}, k_2=\{0\} \qquad (3.12)$$

with

$|C_2| = 1 - f_{11}c_{22} - a_{12}c_{02}$,

$$C_3=\begin{bmatrix} 1 & 0 & 0 & -a_{23} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -c_{01} & 0 & 1 & 0 & 0 \\ -e_{03} & 0 & 0 & 1 & 0 \\ -f_{03} & 0 & 0 & f_{03}a_{23} & 1-f_{03}a_{23} \end{bmatrix}, k_3=\begin{bmatrix} 0 \\ -b_{13} \\ 0 \\ -e_{23} \\ -f_{13} \end{bmatrix} \qquad (3.13)$$

with

$|C_3|= (1 - a_{23}f_{03}) (1 - a_{23}e_{03})$,

$$C_4=\begin{bmatrix} 1 & 0 & 0 & 0 & -a_{24} \\ 0 & 1 & 0 & 0 & -b_{24} \\ -c_{02} & 0 & 1 & 0 & -c_{22} \\ -e_{03} & 0 & 0 & 1 & 0 \\ -f_{03} & 0 & 0 & 0 & 1 \end{bmatrix}, \quad k_4=\begin{bmatrix} -a_{14} \\ -b_{14} \\ 0 \\ -e_{13} \\ -f_{13} \end{bmatrix} \quad (3.14)$$

with

$|C_4| = 1 - f_{03}a_{24}$,

$$C_5=\begin{bmatrix} 1 & 0 & -a_{11} & -a_{21} & 0 \\ 0 & 1 & -b_{11} & -b_{21} & 0 \\ 0 & -c_{05} & 1 & -c_{25} & 0 \\ 0 & -e_{05} & -e_{15} & 1 & 0 \\ 0 & -f_{05} & -f_{15} & 0 & 1 \end{bmatrix}, \quad k_5=\{0\} \quad (3.15)$$

with

$|C_5| = 1 - b_{11}c_{25}e_{05} - c_{05}e_{25}b_{21} - b_{21}e_{05} - b_{11}c_{05} - c_{25}e_{15}$,

$$C_6=\begin{bmatrix} 1 & 0 & -a_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 & -b_{22} \\ 0 & 0 & 1 & 0 & -c_{26} \\ 0 & -e_{05} & -e_{15} & (1-e_{05}b_{22}-e_{15}c_{26}) & (e_{05}b_{22}+e_{15}c_{26}) \\ 0 & -f_{05} & -f_{15} & 0 & 1 \end{bmatrix} \quad (3.16)$$

$k_6 = \{0\}$, with $|C_6| = (1 - e_{05}b_{22} - e_{15}c_{26})\ (1 - b_{22}f_{05} - c_{26}\ f_{15})$,

$$C_7 = \begin{bmatrix} 1 & 0 & 0 & -a_{23} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -c_{05} & (1-c_{05}b_{13}-c_{25}e_{17}) & -c_{25} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -f_{07} & 0 & 0 & 1 \end{bmatrix}, k_7 = \begin{bmatrix} 0 \\ -b_{13} \\ c_{05}b_{13}+c_{25}e_{17} \\ -e_{17} \\ 0 \end{bmatrix} \quad (3.17)$$

with

$|C_7| = 1 - c_{05}b_{13} - c_{25}e_{17}$

$$C_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & -a_{24} \\ 0 & 1 & 0 & 0 & -b_{24} \\ 0 & 0 & 1 & 0 & -c_{26} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -f_{07} & 0 & 0 & 1 \end{bmatrix}, \; k_8 = \begin{bmatrix} -a_{14} \\ -b_{14} \\ 0 \\ -e_{17} \\ 0 \end{bmatrix} \quad (3.18)$$

with

$|C_8| = 1 - f_{07}b_{24}$

If we choose the parameter values such that all the above determinants are different from zero, we say that the resulting system of equations is complete (since, in general, the coherency conditions are fulfilled, see Gouriéroux, Laffont & Monfort (1980). We already know from appendix C that the above system is statistically identifiable in that case.

These (as yet unknown) parameters are econometrically estimated then from the maximization of a sample likelihood function, e.g., under the assumption that the observation vector $(\bar{y}_t, \bar{x}_{1t}, \bar{x}_{2t})$ is independently normally distributed for all sample observations $t = 1, 2, \ldots, T$, the parameters are found from the maximization of :

$$L = \prod_{t=1}^{T} h_t \, (\bar{y}, \bar{x}_1, \bar{x}_2) \qquad (3.19)$$

taking account of (3.10 - 18).

4. A brief description of a companion 2-markets QRM (*)

Assuming that a (representative) consumer maximizes a Johansen-type utility function as in (1.1) subject to budget constraint (1.2), the consumer's Walrasian and effective demand and supply functions in a 2-markets setting (commodities and working hours) are given by (1.3), (1.10) and (1.15) with $\gamma_4 := 0$. Alternatively, a (representative) producer is assumed to maximize expected profits (1.5) subject to the short run production function :

$$\underline{y}_t = \alpha \, x_t^{\rho} + \underline{\varepsilon}_t = y_t + \underline{\varepsilon}_t, \qquad (4.1)$$

with $x_t := x_{1t} \cdot x_{2t}$ the (total) number of hours of work during period t and with $\alpha > 0$, $0 < \rho < 1$, from which follow the Walrasian commodity supply and labour demand :

$$\ln y_t^s : -\frac{1}{\rho-1} \ln \alpha + \frac{\rho}{\rho-1} \ln(w_t(1+s_t)) - \frac{\rho}{\rho-1} \ln(p_t(1-u_t).\alpha\rho)$$

(*) For the case in which the agents cannot observe the constraints occurring in the 2 markets, the reader is referred to Gouriéroux, Laffont and Monfort (1980) and to appendix C.

and (4.2)

$$\ln x_t^d = \frac{1}{\rho-1} \ln (w_t(1+s_t)) - \frac{1}{\rho-1} \ln (p_t(1-u_t)\ \alpha\rho)$$

respectively.
The effective demand and supply functions satisfy :

$$\ln y_t^{s'} = \ln \alpha + \rho \ln \bar{x}_t \tag{4.3}$$

when the producer is rationed in the labour market and

$$\ln x_t^{d'} = -\frac{1}{\rho} \ln \alpha + \frac{1}{\rho} \ln \bar{y}_t \tag{4.4}$$

where the producer is rationed in the commodity market.

In this 2 market problem, we can define three different regimes as presented in table 4.5 (*) :

Regime	Commodity market	Labour market
Keynesian Unemployment(K.U.)	$y^{d'} < y^{s'}$	$x^{d'} < x^{s'}$
Classical Unemployment(C.U.)	$y^{d'} > y^{s'}$	$x^{d'} < x^{s'}$
Repressed Inflation (R.I.)	$y^{d'} > y^{s'}$	$x^{d'} > x^{s'}$

Table 4.5 : Regime definitions

(*) The underconsumption regime ($y^{d'} < y^{s'}$ and $x^{d'} > x^{s'}$) is the border case between the K.U.- and R.I.-regimes in the (w,p)-plane (see Meersman and Plasmans ('82)), and since inventories do not occur when each agent perceives the other agent's quantities, as is the case here, this regime is excluded because of efficiency reasons.

The sample likelihood function is derived in a way similar to that followed in section 2 of this paper.

Therefore, according to (2.1), the producer's behaviour is described by the Walrasian and effective functions :

$$
\begin{aligned}
&(i) \quad \ln \underline{x}^d = \ln x^d(z) + \underline{\varepsilon}_1 \\
&(ii) \quad \ln \underline{y}^s = \ln f(x^d(z)) + \underline{\varepsilon}_2 \\
&(iii) \quad \ln \underline{x}^{d'} = \ln \underline{x}^d - \eta_1(\ln \underline{y}^s - \ln \bar{\underline{y}}) \qquad (4.6) \\
&(iv) \quad \ln \underline{y}^{s'} = \ln \underline{y}^s - \eta_2(\ln \underline{x}^d - \ln \bar{\underline{x}})
\end{aligned}
$$

$$\text{with} \quad 0 \leqq \eta_i \leqq 1 \ (i = 1, 2),$$

with an interpretation similar as that given for (2.1).

Similarly, the consumer's set of Walrasian and effective relationships is according to (2.2) :

$$
\begin{aligned}
&(v) \quad \ln \underline{x}^s = \ln x^s(z) + \underline{\varepsilon}_3 \\
&(vi) \quad \ln \underline{y}^d = \ln y^d(z) + \underline{\varepsilon}_4 \qquad (4.7) \\
&(vii) \quad \ln \underline{x}^{s'} = \ln \underline{x}^s - \eta_3(\ln \underline{y}^d - \ln \bar{\underline{y}}) \\
&(viii) \quad \ln \underline{y}^{d'} = \ln \underline{y}^d - \eta_4(\ln \underline{x}^s - \ln \bar{\underline{x}})
\end{aligned}
$$

$$\text{with } 0 \leqq \eta_i \leqq 1, \ i = 3, 4$$

The likelihood function of one observation on y and x can be derived as the sum of three likelihood values to be in either of the three regimes (compare also Gouriéroux, Laffont and Monfort (1980), Ito (1980) and (2.3)) :

$$L = L_1 + L_2 + L_3 \qquad (4.8)$$

where :

$$L_1 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}}^{\infty} g_1(\ln \bar{y}, \ln \bar{x}, \ln y^s, \ln x^s) d \ln y^s \, d \ln x^s$$

$$L_2 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}}^{\infty} g_2(\ln \bar{y}, \ln \bar{x}, \ln y^d, \ln x^s) \, d \ln y^d \, d \ln x^s$$

$$L_3 = \int_{\ln \bar{y}}^{\infty} \int_{\ln \bar{x}}^{\infty} g_3(\ln \bar{y}, \ln \bar{x}, \ln y^d, \ln x^d) \, d \ln y^d \, d \ln x^d$$

After elaborating the joint density functions introducing stochastic residuals and performing the integrations according to the considered regime (see(2.4 - 2.5) and appendix A), we obtain the following expressions :

$$L_1 = [1 - N(\ln u_1; \sigma_1^2)] \; [1 - N(\ln u_4; \sigma_4^2)]$$

$$[1 - N(\frac{\sigma_1^2}{\sigma_1^2+\eta_1^2\sigma_2^2} \ln u_2 + \frac{\eta_1\sigma_2^2}{\sigma_1^2+\eta_1^2\sigma_2^2} \ln u_1; \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\eta_1^2\sigma_2^2})] \qquad (4.9)$$

$$[1 - N(\frac{\sigma_4^2}{\sigma_4^2+\eta_4^2\sigma_3^2} \ln u_3 + \frac{\eta_4\sigma_3^2}{\sigma_4^2+\eta_4^2\sigma_3^2} \ln u_4; \frac{\sigma_3^2\sigma_4^2}{\sigma_4^2+\eta_4^2\sigma_3^2})]$$

$$L_2 = \left[1-N\left(\frac{(1-\eta_1\eta_2)\sigma_2^2}{\sigma_2^2+\eta_2^2\sigma_1^2}\ln u_1 + \frac{(1-\eta_1\eta_2)\eta_2\sigma_1^2}{\sigma_2^2+\eta_2^2\sigma_1^2}\ln u_2;\right.\right.$$

$$\left.\left.\frac{(1+\eta_1^2\eta_2^2)\sigma_1^2\sigma_2^2}{\sigma_2^2+\eta_2^2\sigma_1^2}\right)\right] \quad \left[1 - N\left(\frac{(1-\eta_1\eta_2)\sigma_1^2}{\sigma_1^2+\eta_1^2\sigma_2^2}\ln u_2 + \right.\right. \tag{4.10}$$

$$\left.\left.\frac{(1-\eta_1\eta_2)\eta_1\sigma_2^2}{\sigma_1^2+\eta_1^2\sigma_2^2}\ln u_1; \frac{(1+\eta_1^2\eta_2^2)\sigma_1^2\sigma_2^2}{\sigma_1^2+\eta_1^2\sigma_2^2}\right)\right]$$

$$n(\ln u_3; \sigma_3^2) \;.\; n(\ln u_4; \sigma_4^2)$$

$$L_3 = \left[1 - N\left(\frac{\sigma_2^2}{\sigma_2^2+\eta_2^2\sigma_1^2}\ln u_1 + \frac{\eta_2\sigma_1^2}{\sigma_2^2+\eta_2^2\sigma_1^2}\ln u_2; \frac{\sigma_1^2\sigma_2^2}{\sigma_2^2+\eta_2^2\sigma_1^2}\right)\right]$$

$$\left[1 - N(\ln u_2; \sigma_2^2)\right] \left[1 - N(\ln u_3; \sigma_3^2)\right] \tag{4.11}$$

$$\left[1 - N\left(\frac{\sigma_3^2}{\sigma_3^2+\eta_3^2\sigma_4^2}\ln u_4 + \frac{\eta_3\sigma_4^2}{\sigma_3^2+\eta_3^2\sigma_4^2}\ln u_3; \frac{\sigma_3^2\sigma_4^2}{\sigma_3^2+\eta_3^2\sigma_4^2}\right)\right],$$

where n(.,.) and N(.,.) are normal density and normal cumulative density notations as in section 2.

Finally, the unknown parameters are estimated by maximizing : $\sum_{t=1}^{T} \ln(L_{1t} + L_{2t} + L_{3t})$ w.r.t. all the unknown parameters and assuming that all error terms are independently and normally distributed.

5. The "aggregate" versus the "disaggregate" approach : theory and empirical results

Up to now, it has not yet been decided whether the QRM's should be applied either to an aggregate (manufacturing) sector or to homogeneous micro-markets, where a firm (or group of firms) produces a homogeneous output and utilizes (homogeneous) labour input for it. In the latter case, it may be assumed that the aggregate economy (the manufacturing sector) consists of a large number of micro-markets, in each of which the min(imum)-condition prevails. Then, the aggregate economy would be described in a much more continuous way and a simultaneous occurrence of unemployment and job vacancies and/or of undesired idle capacities (stocks) and unfilled orders would be possible.

In this section we will first discuss the aggregation problem (paragraph 5.1) and will present econometric estimates for two types of QRM's (paragraph 5.2), i.e., firstly, when applying the FIML-procedures of sections 2-4 directly to quarterly data of the aggregate Belgian manufacturing sector for the period 1963^{III}-1979^{IV} (the "aggregate" approach, where the min-condition is assumed to hold on the aggregate level), and, secondly, when disaggregating over a large number of micro-markets, using the "smoothing by aggregation"-approach, proposed by Muellbauer (1978) and Malinvaud (1980) and already applied by Kooiman (1982) to annual data of the Dutch manufacturing sector and by Lambert (1984) to annual data of the Belgian manufacturing sector (the "disaggregate" approach, where the min-condition is assumed to hold on the micro-level).

5.1. The "smoothing by aggregation" approach

In this approach, each aggregate market is considered as a continuum of micro-markets in disequilibrium, each of which is characterized by excess demand or excess supply according to a min-condition rationing on this micro-market.

If it is assumed that a 3-markets economy consists of a large number N of micro-markets, a natural extension to the stochastic specification of the QRM, as mentioned in paragraph 2.2, is that the smooth continuous density of the micro-level demands and supplies may be described by a 6-dimensional lognormal density

$$g(x^d_{1j}, x^s_{1j}, x^d_{2j}, x^s_{2j}, y^d_j, y^s_j) \quad \text{or}$$

$(\underline{\varepsilon}^d_{1j}, \underline{\varepsilon}^s_{1j}, \underline{\varepsilon}^d_{2j}, \underline{\varepsilon}^s_{2j}, \underline{\varepsilon}^d_{3j}, \underline{\varepsilon}^s_{3j})' \overset{iid}{\sim} N_6(0,\Sigma)$. Assuming that the covariance-matrix Σ is blockdiagonal : $\Sigma = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{bmatrix}$ (*)

$$\text{with } \Sigma_i := \begin{bmatrix} \sigma^2_{\varepsilon^d_i} & \rho\sigma_{\varepsilon^d_i \varepsilon^s_i} \\ \rho\sigma_{\varepsilon^d_i \varepsilon^s_i} & \sigma^2_{\varepsilon^s_i} \end{bmatrix} \qquad (i = 1,2,3),$$

and that the min-condition is active in each micro-market, or :

$$\ln \bar{\underline{x}}_{ij} = \min(\ln \underline{x}^d_{ij}, \ln \underline{x}^s_{ij}) \qquad (i = 1,2) \quad \text{and}$$

$$(j = 1,2,\ldots,N) \qquad (5.1)$$

$$\ln \bar{\underline{y}}_j = \min(\ln \underline{y}^d_j, \ln \underline{y}^s_j),$$

(*) This assumption allows us to perform a similar procedure separately for each aggregate market.

and using the "aggregation by integration" technique (see Muellbauer (1978) and Malinvaud (1980)), the three aggregate transaction functions can be found to be of the very simple C.E.S.-type (see Lambert (1984), Appendix A, pp. 119 - 130) :

$$\bar{\underline{X}}_i = \sum_{j=1}^{N} \bar{\underline{x}}_{ij} = \int_0^\infty \int_{Nx_i^d}^\infty N x_i^d \, g_i(x_i^d, x_i^s) \, d\, x_i^d \, d\, x_i^s$$

$$+ \int_0^\infty \int_{Nx_i^s}^\infty N x_i^s \, g_i(x_i^d, x_i^s) \, d\, x_i^d \, d\, x_i^s \qquad (i = 1,2) \quad (5.2)$$

$$= \left[(\underline{X}_i^d)^{-\rho_i^*} + (\underline{X}_i^s)^{-\rho_i^*} \right]^{-1/\rho_i^*} \leqq \min(\underline{X}_i^d, \underline{X}_i^s)$$

and

$$\bar{\underline{Y}} := \sum_{j=1}^{N} \bar{\underline{y}}_j = \int_0^\infty \int_{Ny^d}^\infty N y^d g_3(y^d, y^s) \, d\, y^d \, d\, y^s + \int_0^\infty \int_{Ny^s}^\infty N y^s g_3(y^d, y^s) \, d\, y^d \, d\, y^s$$

$$= \left[(\underline{Y}^d)^{-\rho_3^*} + (\underline{Y}^s)^{-\rho_3^*} \right]^{-1/\rho_3^*} \leqq \min(\underline{Y}^d, \underline{Y}^s), \qquad (5.3)$$

where $\underline{X}_i^d := \sum_{j=1}^{N} \underline{x}_{ij}^d$, $\underline{X}_i^s := \sum_{j=1}^{N} \underline{x}_{ij}^s$ $(i=1,2)$, $\underline{Y}^d := \sum_{j=1}^{N} \underline{y}_j^d$ and

$\underline{Y}^s := \sum_{j=1}^{N} \underline{y}_j^s$ are the aggregate quantities demanded and supplied on the labour markets and on the commodity market respectively, $g_i(.)$ $(i = 1,2,3)$ are bivariate lognormal density functions $(g(.) := g_1(.) \cdot g_2(.) \cdot g_3(.))$ and where ρ_i^* $(i = 1,2,3)$ are "dispersion parameters", being inversely related to the standard deviations σ_i^* :

$$\rho_i^* = -1 + \frac{2}{\sigma_i^*} \frac{n(-\frac{1}{2}\sigma_i^*)}{N(-\frac{1}{2}\sigma_i^*)} \qquad (i=1,2,3) \qquad (5.4)$$

with n(.) the standard normal density function, N(.) the cumulative standard normal density function for a stochastic $\underline{z}$ (here : $\underline{z} - \frac{1}{2}\sigma_i^*$) and σ_i^* being the standard deviation of the micro-distribution of excess log demands (i.e. the "degree of stochastic mismatch") in market i = 1,2,3 :

$$\sigma_i^* = \sqrt{\text{var}\ (\underline{\varepsilon}_i^d - \underline{\varepsilon}_i^s)} = \sqrt{\sigma^2_{\varepsilon_i^d} + \sigma^2_{\varepsilon_i^s} - 2\rho_i \sigma_{\varepsilon_i^d} \sigma_{\varepsilon_i^s}} \quad (*) \qquad (5.5)$$

This "aggregation by integration" approach encompasses the aggregate min-approach as a degenerate case because from (5.2-3) :

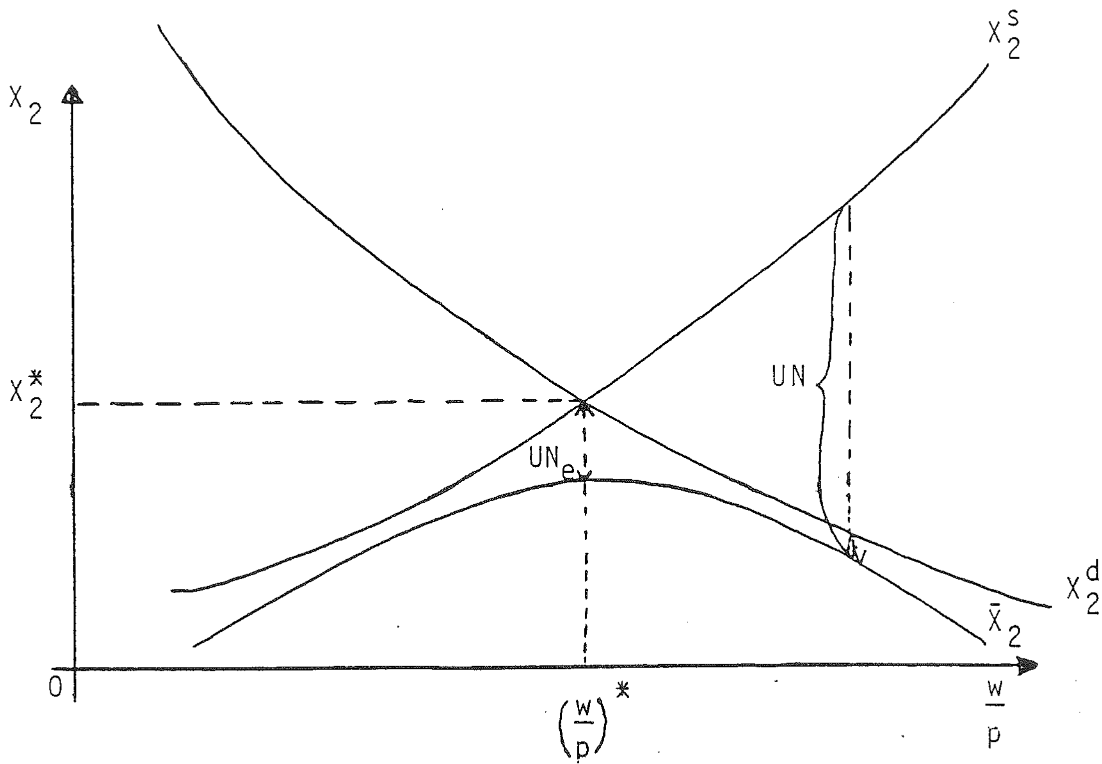
$$\lim_{\rho_i^* \to \infty} \bar{\underline{X}}_i = \min(\underline{X}_i^d, \underline{X}_i^s)\ (i=1,2) \text{ and } \lim_{\rho_3^* \to \infty} \bar{Y} = \min(\underline{Y}^d, \underline{Y}^s) \qquad (5.6)$$

(Leontief-type of transaction functions).

The aggregate transaction function of the (average) number of working hours per period exhibits a shape like that of the smooth curve in figure 5.7. A decrease of ρ_2^* moves the transaction curve away from the broken line segment: min $\{\underline{X}_2^s , \underline{X}_2^d\}$.

(*) Notice that the parameters $\sigma^2_{\varepsilon_i^d}$, $\sigma^2_{\varepsilon_i^s}$ and ρ_i are not identified separately. From (5.4), it directly follows that $\lim_{\sigma_i^* \to \infty} \rho_i^* = 0 (i=1,2,3)$.

figure 5.7



In particular, the "inefficiency at equilibrium", i.e. the discrepancy between $\bar{\underline{X}}_2$ and $\underline{X}_2^d = \underline{X}_2^s = X_2^*$ is computed from (5.2) as :

$$\frac{\bar{\underline{X}}_2}{\underline{X}_2^d} = \frac{\bar{\underline{X}}_2}{\bar{\underline{X}}_2^s} = 2^{-1/\rho_2^*} \qquad (5.8)$$

so that the degree of frictional unemployment at equilibrium depends only on ρ_2^* :

$$\underline{UN}_e = 1 - \frac{\bar{\underline{X}}_2}{X_2^*} = 1 - 2^{-1/\rho_2^*} \qquad (5.9)$$

The weighted shares of micro-markets in excess demand, where each micro-market is weighted by its contribution to the aggregate transaction, can similarly be derived as :

$$P_W(x_i^d \geqq x_i^s) = P_{X_i} = \frac{1}{\bar{X}_i} \int_o^\infty \int_{Nx_i^s}^\infty N\, x_i^s\, g_i(x_i^d, x_i^s)\, d\, x_i^d\, d\, x_i^s \tag{5.10}$$

$$= \frac{1}{1 + \left(\frac{X_i^d}{X_i^s}\right)^{-\rho_i^*}} \qquad (i=1,2) \qquad (*)$$

and

$$P_W(y^d \geqq y^s) = P_Y = \frac{1}{\bar{Y}} \int_o^\infty \int_{Ny^s}^\infty N\, y^s\, g_3(y^d, y^s)\, d\, y^d\, d\, y^s \tag{5.11}$$

$$= \frac{1}{1 + \left(\frac{Y^d}{Y^s}\right)^{-\rho_3^*}} \qquad (*)$$

so that from (5.4)

$$\lim_{\sigma_i^* \to o} P_W(x_i^d \geqq x_i^s) = \lim_{\sigma_3^* \to o} P_W(y^d \geqq y^s) = 1 \quad (i=1,2) \tag{5.13}$$

Alternatively, the weighted proportions of micro-markets being in excess supply amount to :

$$P_W(x_i^d < x_i^s) = 1 - P_W(x_i^d \geqq x_i^s) = \frac{1}{1 + \left(\frac{X_i^d}{X_i^s}\right)^{\rho_i^*}} \quad (i=1,2) \tag{5.14}$$

(*) Inserting (5.10) and (5.11) into the aggregate transaction functions (5.2) and (5.3), we get

$$\underline{\bar{X}}_i = \underline{X}_i^s (P_{X_i})^{\frac{1}{\rho_i^*}} \quad (i=1,2) \text{ and } \underline{\bar{Y}} = \underline{Y}^s (P_Y)^{\frac{1}{\rho_3^*}} \quad \text{respectively.} \tag{5.12}$$

and

$$P_w(y^d < y^s) = 1 - P_w(y^d \geqq y^s) = \frac{1}{1 + \left(\frac{Y^d}{Y^s}\right)^{\rho_3^*}} \tag{5.15}$$

with

$$\lim_{\sigma_i^* \to o} P_w(x_i^d < x_i^s) = \lim_{\sigma_3^* \to o} P_w(y^d < y^s) = o \quad (i=1,2) \tag{5.16}$$

There are two possibilities now :

- either one assumes log-linear micro-spillover terms, so that each micro-market is described by a producer's set of relations as in (2.1) for a 3-markets QRM or as in (4.6) for a 2-markets QRM, and where an aggregate short-run production function is derived afterwards, and by a consumer's set of relations as in (2.2) or (4.7) respectively (see Kooiman (1982)),
- or one assumes log-linear aggregate spillover terms where effective quantities are only defined on the aggregate level (see Lambert (1984)).

Under the first assumption, all the quantities in (5.1 - 5.16) are effective quantities demanded and supplied, and should be provided by a'. The aggregate production function, however, is difficult to obtain in this case (see Kooiman (1982)), and, therefore, the second possibility of aggregate spillover terms will be followed in this section. In this case, the relationships (iv), (v), (vi) of (2.1), (x), (xi), (xii) of (2.2) or (iii), (iv) of (4.6) and (vii), (viii) of (4.7) are valid for the aggregate quantities in the case of the 3-markets model or the 2-markets model respectively (the aggregates being indicated by capital figures $\bar{Y}$, Y^d, Y^s, $\bar{X}_i$, X_i^d, X_i^s (i=1,2)).

By way of example, the "disaggregate" 2-markets model can be fully specified as (see (4.6-7) and (5.2-15)) :

$$\ln \underline{X}^{d'} = \ln \underline{X}^{d} - \eta_1(\ln \underline{Y}^{s} - \ln \bar{\underline{Y}})$$

$$\ln \underline{Y}^{s'} = \ln \underline{Y}^{s} - \eta_2(\ln \underline{X}^{d} - \ln \bar{\underline{X}})$$

$$\ln \underline{X}^{s'} = \ln \underline{X}^{s} - \eta_3(\ln \underline{Y}^{d} - \ln \bar{\underline{Y}}) \quad (*)$$

$$\ln \underline{Y}^{d'} = \ln \underline{Y}^{d} - \eta_4(\ln \underline{X}^{s} - \ln \bar{X})$$

$$(5.17)$$

$$\bar{\underline{X}} = \left[(\underline{X}^{d})^{-\rho_L^*} + (\underline{X}^{s})^{-\rho_L^*}\right]^{-1/\rho_L^*} = \underline{X}^{s}(\underline{P}_X)^{1/\rho_L^*} = \underline{X}^{d}(1 - \underline{P}_X)^{1/\rho_L^*}$$

$$\underline{P}_X = \left[1 + \left(\frac{\underline{X}^{d}}{\underline{X}^{s}}\right)^{-\rho_L^*}\right]^{-1}$$

$$\bar{\underline{Y}} = \left[(Y^{d})^{-\rho_G^*} + (\underline{Y}^{s})^{-\rho_G^*}\right]^{-1/\rho_G^*} = \underline{Y}^{s}(\underline{P}_Y)^{1/\rho_G^*} = \underline{Y}^{d}(1 - \underline{P}_Y)^{1/\rho_G^*}$$

$$\underline{P}_Y = \left[1 + \left(\frac{\underline{Y}^{d}}{\underline{Y}^{s}}\right)^{-\rho_G^*}\right]^{-1}$$

with ρ_L^* and ρ_G^* being the "dispersion parameters" on the labour market and the goods market respectively, which are assumed to be approximately constant over time.

(*) Because of the open character of the Belgian economy and the large substitution possibilities among domestic and foreign goods, the Belgian households are not severely rationed on their commodity market, so that η_3 will be expected to be small or negligible.

Assuming that the aggregate notional quantities can be expressed from (4.6-7) as :

$$\ln \underline{X}^d = \ln X^d(Z) + \underline{u}_1$$

$$\ln \underline{Y}^s = \ln Y^s(Z) + \underline{u}_2$$

$$\ln \underline{X}^s = \ln X^s(Z) + \underline{u}_3 \qquad (5.18)$$

$$\ln \underline{Y}^d = \ln Y^d(Z) + \underline{u}_4$$

with Z being a vector of exogenous variables, having an impact on the aggregate quantities, and $(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4)'$ being a jointly (normally) distributed stochastic error vector, and provided that initial estimates for the weighted shares $\underline{p}_X$ and $\underline{p}_Y$ of micro-markets in excess demand (e.g, through business surveys) are available, (5.17) and (5.18) can be rewritten straightforwardly in structural form, suitable for econometric estimation (see also Lambert (1984), Chapter 2).

5.2. An empirical application to the Belgian manufacturing sector

5.2.1. Application of the "aggregate" approach

Provided that the rationing min-condition is assumed to hold on the aggregate level, the FIML-procedures of sections 2, 3 and 4 are directly applied to quarterly data of the Belgian manufacturing sector for the period 1963^{III} - 1979^{IV}. Hence, the period of estimation covers 66 quarters. Time series data have been obtained from the "Cahiers Economiques de Bruxelles" (Dulbêa) and from various monthly and quarterly publications of the Belgian National Institute of Statistics. The data base can be obtained from the authors by simple request.

To save space, only the FIML-estimates for the 3-markets QRM of section 2, obtained by maximizing the joint sample likelihood function (2.7), taking account of the expressions in (2.6) and appendix B, and the FIML-estimates for the companion 2-markets QRM of section 4, obtained by maximizing the relating joint sample likelihood function taking account of (4.9-11), will be reported in this paper.

The computations were run on an ICL-2966 computer at the University of Tilburg (*). Since most of the parameters are constrained within lower and upper bounds (as, e.g., the spill-over coefficients), we utilized a constrained mathematical programming procedure. After some initial experiments, we stuck to the E04 JBB-maximization routine, which is a comprehensive quasi-Newton algorithm allowing for inequality constraints. This algorithm evaluates numerically the first order partial derivatives and an approximation to the (inverse) Hessian of the joint sample likelihood function. It is said to converge if both the function values and the parameter estimates converge.

Making use of various subprograms for sensitivity analysis, computing, e.g. the relative change of the likelihood function value (and the values of some key parameters) owing to a change in one of the parameters, we could reduce the 35 parameters of the 3-markets QRM to 28 parameters. This was also done by analytic reasoning. For example, since η_1 and η_3 denote the spillover elasticities from the commodity market on the effective demands for workers and (the average number of) working hours respectively (see relationships (iv) and (v) of (2.1)), it may be assumed a priori that both elasticities do not differ too much in the case of producers' rationing. A similar reasoning might be followed for the producers' spillover elasticities η_2 and η_4 from either labour market to the other labour market, for the producers' spillover elasticities κ_1 and κ_2 of the both labour markets to the commodity market, and for the consumers' spillover elasticities ξ_1 and ξ_3, ξ_2 and ξ_4, κ_3 and κ_4 and γ_4

(*) We thank Mr. Theo De Beer of the Tilburg Computer Center for his help in performing the laborious non-linear estimations.

and γ_5 (in(1.14)). All these assumptions have been tested experimentally and have been found to satisfy approximately. Henceforth, η_1 and η_3 will be denoted by η_{13}, η_2 and η_4 by η_{24}, κ_1 and κ_2 by κ_{12}, etc....

When running the minimization procedure, we noticed that various parameters rapidly converged to one of their bounds. On the other side, utilizing the sensitivity analysis subprograms, a fixed value has been assigned to some parameters, because, in a first stage estimation, these parameter values yielded the largest increase of the likelihood function value. In the following table, lower and upper bounds imposed on the parameters and constrained optimal values of these parameters are summarized; note that parameters being fixed are indicated by an asterisk*. It is directly verified that 17 out of the 28 parameters converged to one of their bounds. Hence, estimates of the asymptotic standard deviations, however small for the free parameters, are not reported here.

Table 5.19 : Bounds and optimal values for the parameters of the 3-markets QRM.

	Parameter	Lower bound	Upper bound	Optimal parameter value		Parameter	Lower bound	Upper bound	Optimal parameter value
1	μ	0.1	1.5	1.41	15	κ_{34}	0.2	0.99	0.2
2	λ	0.0001	0.1	0.0001	16	ζ_{13}	0.2	0.99	0.2
3	δ	0.1	1.0	0.798	17	ζ_{24}	0.2	0.99	0.2
4	ρ	-1.0	1.0	0.6	18	α_1	0.2	0.98	0.962
5	A	10	150	10	19	α_2	0.2	0.99	0.64*
6	σ_1^2	10	150	81.91	20	α_3	0.5	0.99	0.56*
7	σ_2^2	50	300	300	21	α_4	0.05	1.0	1.0
					22	β_1	0.1	1.0	1.0
8	σ_3^2	13	150	13	23	β_2	0.1	1.0	1.0
9	σ_4^2	10	150	150	24	β_3	0.1	1.0	0.201
10	σ_5^2	10	150	10	25	γ_1	0.2	0.8	0.8
11	σ_6^2	50	300	50	26	γ_2	0.2	0.8	0.413
12	η_{13}	0.4	0.9	0.4	27	γ_3	0.1	1.0	1.0
13	η_{24}	0.4	0.9	0.83	28	γ_{45}	0.2	0.8	0.2
14	κ_{12}	0.4	0.9	0.47					

From the above estimation results it follows that the elasticity of substitution between (the number of) workers and (the average number of) working hours is below unity ($\sigma \simeq 0.625$). In the short-run production surroundings analyzed, it is also observed that there exist increasing returns to scale in the labour inputs of Belgian manufacturing sector, and that, during the sample period, the "worker intensity" was approximately four times higher than the "working hours intensity"(*), so that the production elasticities of workers are found to be much more important than those of working hours.

Moreover, the spillover elasticity η_{24} of the producers' excess labour demands on the effective labour demands doubles κ_{12}, the spillover elasticity of the producers' excess commodity supply on these effective labour demands.

When maximizing the companion 2-markets joint sample likelihood function of section 4, and experimenting with the sensitivity analysis subprogram, 10 parameters are found to vary freely, and the other 8 are fixed since they imply little contribution to the likelihood function value, so that varying values of the latter 8 parameters yielded approximately the same objective value of the complete sample. Therefore, these 8 parameters, all belonging to the consumers' Walrasian and effective demand and supply functions (1.3), (1.10) and (1.15), which are similar for the 3- and 2-markets QRM's (with $\gamma_4 := o$), have been set at their optimal 3-markets values (denoted by an asterisk*). In the following table the imposed lower and upper bounds and the optimal values of the parameters lying within these bounds have been summarized for the 2-markets QRM.

(*) Notice also that technical progress is found to be negligible which is due to the high negative collinearity between time and the average number of working hours.

Table 5.20 : Bounds and optimal values for the parameters of the 2-markets QRM.

	Parameter	Lower bound	Upper bound	Optimal parameter value		Parameter	Lower bound	Upper bound	Optimal parameter value
1	α	0.001	1.0	0.193	11	α_1	0.1	0.99	0.962*
2	ρ	0.1	0.99	0.837	12	α_2	0.1	0.99	0.64*
3	σ_1^2	0.4	500	29.986	13	α_3	0.1	0.99	0.56*
4	σ_2^2	0.4	500	9.946	14	β_1	0.1	1.0	1.0*
5	σ_3^2	0.4	500	10.002	15	β_2	0.1	1.0	1.0*
6	σ_4^2	0.4	500	9.993	16	β_3	0.1	1.0	0.201*
7	η_1	0.1	0.99	0.675	17	γ_1	0.2	0.8	0.8*
8	η_2	0.1	0.99	0.900	18	γ_5	0.2	0.8	0.2*
9	η_3	0.1	0.99	0.554					
10	η_4	0.1	0.99	0.841					

The above solution has been obtained after 10 to 20 iterations each time when utilizing various starting values for the parameters. Hence, we may decide that the above solution is the optimal solution for the 2-markets QRM, which, as compared with the results in table 5.19, yields much lower residual variances. Notice, equally, that the postwar spillover elasticities of the manufacturing commodity market on the labour market are much lower than the spillover elasticities of the labour market on this commodity market. Hence, excess demand on the labour market is penalized more severely than excess supply on the commodity-market.
When computing the probabilities of regime occurrence, we found that the Keynesian Unemployment-regime is being predominant since the oil crisis (from 73^{III} on), and that the Classical Unemployment regime was predominant during 69^{II} - 73^{II} . The qualification that "the sixties were a golden age" was verified by the Repressed Inflation regime being dominant during 63^{III} - 66^{II}.

5.2.2. Application of the "disaggregate" approach

Following ideas put forward in section 5.1, the Belgian manufacturing sector can be considered as an aggregate over a large number of micro-markets, each of which is characterized by a particular disequilibrium situation. Assuming log-linear aggregate spillover terms, the effective demands and supplies are supposed to be valid on the aggregate level only and the QRM (5.17) can be statistically estimated, together with its 3-markets counterpart. Following Lambert (1984, Chapter 3), (5.17) can be estimated by a FIML-procedure, provided that the error terms in (5.18) are jointly normally distributed. Assuming that the notional quantities are expressed as in section 4 (and section 1), the spillover elasticities of (5.17) can be evaluated after 10 to 15 iterations, depending on the starting values as (estimates of the asymptotic standard deviations between brackets) :

$$\hat{\eta}_1 = \underset{(0.14)}{0.513} \; , \; \hat{\eta}_2 = \underset{(0.21)}{0.947} \; , \; \hat{\eta}_3 = \underset{(0.11)}{0.480} \; , \; \hat{\eta}_4 = \underset{(0.24)}{0.847} .$$

Comparing with the spillover elasticities in table 5.20, a remarkable resemblance is verified, so that both the "aggregate" and the "disaggregate" 2-markets QRM's point to more or less similar rationing impacts. Notice that, despite of the very open character of the Belgian economy and the a priori assumed large substitution possibilities between domestic and foreign goods, the spillover elasticity η_3 is, although being estimated as the smallest of the four spillover elasticities, not negligible at all, so that Belgian households are also found to be rationed on their domestic market of manufacturing goods. This, at the first sight somewhat surprising, result can be explained from the phenomenon that many Belgian manufacturing traders have (had) long term export contracts during the sixties and the seventies (*), so that the substitutability among domestically and foreignly manufactured goods was rather limited during the sample period. The "disaggregate" 3-markets QRM could not yet be estimated appropriately.

(*) The deterioration of the Belgian balance of trade in manufactured goods started (in a significant way) from the second half of the seventies onwards (and lasted until '83). Hence, this deterioration took place mainly after our sample period.

Appendix A

We start here with the equations derived in (2.5) for the Keynesian Unemployment regime :

$$\ln \bar{\underline{x}}_1 = \ln x_1^d(z) + \eta_1(\ln \bar{\underline{y}} - \ln f(x_1^d(z), x_2^d(z))) + \underline{\varepsilon}_1 - \eta_1\underline{\varepsilon}_3$$

$$\ln \bar{\underline{x}}_2 = \ln x_2^d(z) + \eta_3(\ln \bar{\underline{y}} - \ln f(x_1^d(z), x_2^d(z))) + \underline{\varepsilon}_2 - \eta_3\underline{\varepsilon}_3$$

$$\ln \underline{y}^s = \ln f(x_1^d(z), x_2^d(z)) + \underline{\varepsilon}_3$$

$$\ln \underline{x}_1^{s'} = \ln x_1^s(z) + \zeta_1(\ln \bar{\underline{x}}_2 - \ln x_2^s(z)) + \underline{\varepsilon}_4 - \zeta_1\underline{\varepsilon}_5$$

$$\ln \underline{x}_2^{s'} = \ln x_2^s(z) + \zeta_3(\ln \bar{\underline{x}}_1 - \ln x_1^s(z)) + \underline{\varepsilon}_5 - \zeta_3\underline{\varepsilon}_4$$

$$\ln \bar{\underline{y}} = \ln y^d(z) + \kappa_3(\ln \bar{\underline{x}}_1 - \ln x_1^s(z)) + \kappa_4(\ln \bar{\underline{x}}_2 - \ln x_2^s(z)) + \underline{\varepsilon}_6 - \kappa_3\underline{\varepsilon}_4 - \kappa_4\underline{\varepsilon}_5$$

Hence, the joint probability density function

$g_1(\ln \bar{y}, \ln \bar{x}_1, \ln \bar{x}_2, \ln y^{s'}, \ln x_1^{s'}, \ln x_2^{s'})$ can be

factorized as :

$$g_1 = g^1(\ln \bar{y} | \ln \bar{x}_1, \ln \bar{x}_2), g^2(\ln \bar{x}_1 | \ln \bar{y}), g^3(\ln \bar{x}_2 | \ln \bar{y})$$

$$g^4(\ln y^s | \ln \bar{y}), g^5(\ln x_1^{s'} | \ln \bar{x}_1), g^6(\ln x_2^s | \ln \bar{x}_2)$$

Since all error terms are assumed to be mutually independent, g^1, g^5 and g^6 can be written from (2.4.5) as :

$$n(\ln u_6 - \kappa_3 \ln u_4 - \kappa_4 \ln u_5;\ \sigma_6^2 + \kappa_3^2\sigma_4^2 + \kappa_4^2\sigma_5^2),$$

$$n(\ln u_4 - \zeta_1 \ln u_5;\ \sigma_4^2 + \zeta_1^2\sigma_5^2) \qquad \text{and}$$

$$n(\ln u_5 - \zeta_3 \ln u_4;\ \sigma_5^2 + \zeta_3^2\sigma_4^2)$$

respectively.

The remaining factors taken together constitute the joint density function of $\ln \bar{x}_1$, $\ln \bar{x}_2$ and $\ln y^s$. It is obtained from (i), (ii) and (iii) of (2.5) with mean vector :

$$\begin{bmatrix} \ln x_1^d(z) + \eta_1(\ln\bar{y} - \ln f(x_1^d(z), x_2^d(z))) \\ \ln x_2^d(z) + \eta_3(\ln\bar{y} - \ln f(x_1^d(z), x_2^d(z))) \\ \ln f(x_1^d(z), x_2^d(z)) \end{bmatrix}$$

and with variance-covariance matrix :

$$\begin{bmatrix} \sigma_1^2+\eta_1^2\sigma_3^2 & \eta_1\eta_3\sigma_3^2 & -\eta_1\sigma_3^2 \\ \eta_1\eta_3\sigma_3^2 & \sigma_2^2+\eta_3^2\sigma_3^2 & -\eta_3\sigma_3^2 \\ -\eta_1\sigma_3^2 & -\eta_3\sigma_3^2 & \sigma_3^2 \end{bmatrix}$$

From the formulae for conditional means and variances for a multivariate-normal distribution (*) the conditional normal density functions for the equations of the above Keynesian unemployment system can be computed as :

$$n(\ln u_1-\eta_1 \ln u_3-[\eta_1\eta_3\sigma_3^2, -\eta_1\sigma_3^2]\begin{bmatrix}\sigma_2^2+\eta_3^2\sigma_3^2 & -\eta_3\sigma_3^2\\ -\eta_3\sigma_3^2 & \sigma_3^2\end{bmatrix}^{-1}$$

$$\begin{bmatrix}\ln u_2-\eta_3 \ln u_3\\ \ln y^s-\ln f(x_1^d(z), x_2^d(z))\end{bmatrix}; \sigma_1^2+\eta_1^2\sigma_3^2 - [\eta_1\eta_3\sigma_3^2, -\eta_1\sigma_3^2]$$

$$\begin{bmatrix}\sigma_2^2+\eta_3^2\sigma_3^2 & -\eta_3\sigma_3^2\\ -\eta_3\sigma_3^2 & \sigma_3^2\end{bmatrix}^{-1}\begin{bmatrix}\eta_1\eta_3\sigma_3^2\\ -\eta_1\sigma_3^2\end{bmatrix}) \qquad \text{for } g^2,$$

$$n(\ln u_2-\eta_3 \ln u_3-[\eta_1\eta_3\sigma_3^2, -\eta_3\sigma_3^2]\begin{bmatrix}\sigma_1^2+\eta_1^2\sigma_3^2 & -\eta_1\sigma_3^2\\ -\eta_1\sigma_3^2 & \sigma_3^2\end{bmatrix}^{-1}$$

(*) If a k-vector x is assumed to be normally distributed with mean vector μ and variance-covariance matrix Ω, then the conditional probability density of $(x_1|x_2)$, where x_1 is a 1-subvector of x and x_2 is the resulting (k-1)-subvector of x, is also normal with mean vector $\mu_1+\Omega_{12}\Omega_{22}^{-1}(x_2-\mu_2)$ and variance-covariance matrix $\Omega_{11}-\Omega_{12}\Omega_{22}^{-1}\Omega_{21}$, where μ_i and Ω_{ij} (i,j=1,2) are the correspondingly partitioned vectors and matrices of μ and Ω (See Mood and Graybill (1963), chapter 9).

$$\begin{bmatrix} \ln u_1 - \eta_1 \ln u_3 \\ \ln y^s - \ln f(x_1^d(z), x_2^d(z)) \end{bmatrix} ; \; \sigma_2^2 + \eta_3^2\sigma_3^2 - [\eta_1\eta_3\sigma_3^2, \; -\eta_3\sigma_3^2]$$

$$\begin{bmatrix} \sigma_1^2 + \eta_1^2\sigma_3^2 & -\eta_1\sigma_3^2 \\ -\eta_1\sigma_3^2 & \sigma_3^2 \end{bmatrix}^{-1} \begin{bmatrix} \eta_1\eta_3\sigma_3^2 \\ -\eta_3\sigma_3^2 \end{bmatrix} \Bigg) \quad \text{for } g^3,$$

and $n(\ln y^s - \ln f(x_1^d(z), x_2^d(z)) - [-\eta_1\sigma_3^2, -\eta_3\sigma_3^2] \begin{bmatrix} \sigma_1^2 + \eta_1^2\sigma_3^2 & \eta_1\eta_3\sigma_3^2 \\ \eta_1\eta_3\sigma_3^2 & \sigma_2^2 + \eta_3^2\sigma_3^2 \end{bmatrix}^{-1}$

$$\begin{bmatrix} \ln u_1 - \eta_1 \ln u_3 \\ \ln u_2 - \eta_3 \ln u_3 \end{bmatrix} ; \; \sigma_3^2 - [-\eta_1\sigma_3^2, \; -\eta_3\sigma_3^2] \begin{bmatrix} \sigma_1^2 + \eta_1^2\sigma_3^2 & \eta_1\eta_3\sigma_3^2 \\ \eta_1\eta_3\sigma_3^2 & \sigma_2^2 + \eta_3^2\sigma_3^2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} -\eta_1\sigma_3^2 \\ -\eta_3\sigma_3^2 \end{bmatrix} \Bigg) \quad \text{for } g^4.$$

The joint density function g_1 can then be written as:

$$n(\ln u_6 - \kappa_3 \ln u_4 - \kappa_4 \ln u_5;$$

$$\sigma_6^2 + \kappa_3^2\sigma_4^2 + \kappa_4^2\sigma_5^2)\, n(\ln u_4 - \zeta_1 \ln u_5; \sigma_4^2 + \zeta_1^2\sigma_5^2)$$

$$n(\ln u_5 - \zeta_3 \ln u_4; \sigma_5^2 + \zeta_3^2\, \sigma_4^2)$$

$$n(\ln u_1 - \eta_1 \ln u_3 - [\eta_1\eta_3\sigma_3^2, \; -\eta_1\sigma_3^2] \begin{bmatrix} \sigma_2^2 + \eta_3^2\sigma_3^2 & -\eta_3\sigma_3^2 \\ -\eta_3\sigma_3^2 & \sigma_3^2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \ln u_2 - \eta_3 \ln u_3 \\ \ln y^s - \ln f(x_1^d(z), x_2^d(z)) \end{bmatrix} ; \; \sigma_1^2 + \eta_1^2\sigma_3^2 - [\eta_1\eta_3\sigma_3^2, \; -\eta_1\sigma_3^2]$$

$$\begin{bmatrix} \sigma_2^2+\eta_3^2\sigma_3^2 & -\eta_3\sigma_3^2 \\ -\eta_3\sigma_3^2 & \sigma_3^2 \end{bmatrix}^{-1} \begin{bmatrix} \eta_1\eta_3\sigma_3^2 \\ -\eta_1\sigma_3^2 \end{bmatrix} \Bigg)$$

$$n(\ln u_2-\eta_3\ln u_3 - [\eta_1\eta_3\sigma_3^2, -\eta_3\sigma_3^2] \begin{bmatrix} \sigma_1^2+\eta_1^2\sigma_3^2 & -\eta_1\sigma_3^2 \\ -\eta_1\sigma_3^2 & \sigma_3^2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \ln u_1-\eta_1\ln u_3 \\ \ln y^s-\ln f(x_1^d(z), x_2^d(z)) \end{bmatrix} ; \sigma_2^2+\eta_3^2\sigma_3^2 - [\eta_1\eta_3\sigma_3^2, -\eta_3\sigma_3^2]$$

$$\begin{bmatrix} \sigma_1^2+\eta_1^2\sigma_3^2 & -\eta_1\sigma_3^2 \\ -\eta_1\sigma_3^2 & \sigma_3^2 \end{bmatrix}^{-1} \begin{bmatrix} \eta_1\eta_3\sigma_3^2 \\ -\eta_3\sigma_3^2 \end{bmatrix} \Bigg)$$

$$n(\ln y^s-\ln f(x_1^d(z),x_2^d(z))-[-\eta_1\sigma_3^2, -\eta_3\sigma_3^2] \begin{bmatrix} \sigma_1^2+\eta_1^2\sigma_3^2 & \eta_1\eta_3\sigma_3^2 \\ \eta_1\eta_3\sigma_3^2 & \sigma_2^2+\eta_3^2\sigma_3^2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \ln u_1-\eta_1\ln u_3 \\ \ln u_2-\eta_3\ln u_3 \end{bmatrix} ; \sigma_3^2-[-\eta_1\sigma_3^2, -\eta_3\sigma_3^2] \begin{bmatrix} \sigma_1^2+\eta_1\sigma_3^2 & \eta_1\eta_3\sigma_3^2 \\ \eta_1\eta_3\sigma_3^2 & \sigma_2^2+\eta_3^2\sigma_3^2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} -\eta_1\sigma_3^2 \\ \eta_3\sigma_3^2 \end{bmatrix} \Bigg).$$

Appendix B

Proceeding for the other 4 regimes in a similar way as in section 2 and appendix A, we get for the corresponding likelihood functions :

$$L_2 = n(\ln u_3;\sigma_3^2)\,n(\ln u_1;\sigma_1^2)\,n(\ln u_2;\sigma_2^2)$$

$$\{1-N(\ln u_4-\zeta_1\ln u_5-\zeta_2\ln u_6;\ \sigma_4^2+\zeta_1^2\sigma_5^2+\zeta_2^2\sigma_6^2)\}$$

$$\{1-N(\ln u_5-\zeta_3\ln u_4-\zeta_4\ln u_6;\ \sigma_5^2+\zeta_3^2\sigma_4^2+\zeta_4^2\sigma_6^2)\}$$

$$\{1-N(\ln u_6-\kappa_3\ln u_4-\kappa_4\ln u_5;\ \sigma_6^2+\kappa_3^2\sigma_4^2+\kappa_4^2\sigma_5^2)\}$$

$$L_3 = \{1-N(\ln u_4-\zeta_2\ln u_6;\sigma_4^2+\zeta_2^2\sigma_6^2)\}\{1-N(\ln u_6;\sigma_6^2)\}$$

$$\{1-N(\ln u_5-\zeta_4\ln u_6;\sigma_5^2+\zeta_4^2\sigma_6^2)\}\{1-N(\ln u_1-\eta_2\ln u_2-$$

$$\frac{1}{B}\{(\ln u_2-\eta_4\ln u_1)(\eta_4\sigma_1^2\sigma_3^2-\eta_2\sigma_2^2\sigma_3^2+\eta_4\kappa_2^2\sigma_1^2\sigma_2^2-\eta_2\kappa_1^2\sigma_1^2\sigma_2^2$$

$$-\kappa_1\kappa_2\sigma_1^2\sigma_2^2+2\eta_4\kappa_1^2\sigma_1^4-\eta_2\eta_4\kappa_1\kappa_1\sigma_1^2\sigma_2^2)-(\ln u_3-\kappa_1\ln u_1-\kappa_2\ln u_2)$$

$$(\eta_4\kappa_2\sigma_1^2\sigma_2^2+\eta_2\eta_4\kappa_1\sigma_1^2\sigma_2^2-\kappa_1\sigma_1^2\sigma_2^2+\eta_4^2\eta_2\kappa_2\sigma_1^2\sigma_2^2-2\eta_4^2\kappa_1\sigma_1^4)\};$$

$$\sigma_1^2+\eta_2^2\sigma_2^2+\frac{1}{B}\{(\eta_4\sigma_1^2+\eta_2\sigma_2^2)(\eta_4\sigma_1^2\sigma_3^2-\eta_2\sigma_2^2\sigma_3^2+\eta_4\kappa_2^2\sigma_1^2\sigma_2^2-$$

$$\eta_2\kappa_1^2\sigma_1^2\sigma_2^2-\kappa_1\kappa_2\sigma_1^2\sigma_2^2+2\eta_4\kappa_1^2\sigma_1^4-\eta_2\eta_4\kappa_1\kappa_2\sigma_1^2\sigma_2^2)+$$

$$(\kappa_1\sigma_1^2-\eta_2\kappa_2\sigma_2^2)(\eta_4\kappa_2\sigma_1^2\sigma_2^2-\eta_2\eta_4\kappa_1\sigma_1^2\sigma_2^2-\kappa_1\sigma_1^2\sigma_2^2-2\eta_4^2\kappa_1\sigma_1^4$$

$$+\eta_4^2\eta_2\kappa_2\sigma_1^2\sigma_2^2)\}$$

$$(\text{with } B := \sigma_2^2\sigma_3^2+\kappa_1^2\sigma_1^2\sigma_2^2+\eta_4^2\sigma_1^2\sigma_3^2+\eta_4^2\kappa_2^2\sigma_1^2\sigma_2^2+2\eta_4\kappa_1\kappa_2\sigma_1^2\sigma_2^2)$$

$$\{1-N(\ln u_2-\eta_4\ln u_1+\tfrac{1}{C}\{(\eta_4\sigma_1^2\sigma_3^2+\eta_4\kappa_2^2\sigma_1^2\sigma_2^2+\eta_2\sigma_2^2\sigma_3^2$$

$$+\eta_2\kappa_1^2\sigma_1^2\sigma_2^2+\kappa_2\sigma_1^2\sigma_2^2-\eta_2\eta_4\kappa_1\kappa_2\sigma_1^2\sigma_2^2)(\ln u_1-\eta_2\ln u_2)\ -$$

$$(\eta_2\eta_4\kappa_2\sigma_1^2\sigma_2^2-\eta_2\kappa_1\sigma_1^2\sigma_2^2-\kappa_2\sigma_1^2\sigma_2^2+\eta_2^2\eta_4\kappa_1\sigma_1^2\sigma_2^2)$$

$$(\ln u_3-\kappa_1\ln u_1-\kappa_2\ln u_2)\};\sigma_2^2+\eta_4^2\sigma_1^2-\tfrac{1}{C}\{(\eta_4\sigma_1^2\sigma_3^2+$$

$$\eta_4\kappa_2^2\sigma_1^2\sigma_2^2+\eta_2\sigma_2^2\sigma_3^2+\eta_2\kappa_1^2\sigma_1^2\sigma_2^2+\kappa_2\sigma_1^2\sigma_2^2-\eta_2\eta_4\kappa_1\kappa_2\sigma_1^2\sigma_2^2)$$

$$(\eta_4\sigma_1^2+\eta_2\sigma_2^2)+(\eta_2\eta_4\kappa_2\sigma_1^2\sigma_2^2-\eta_2\kappa_1\sigma_1^2\sigma_2^2-\kappa_2\sigma_1^2\sigma_2^2+\eta_2^2\eta_4\kappa_1\sigma_1^2\sigma_2^2)$$

$$(\kappa_2\sigma_2^2-\eta_4\kappa_1\sigma_1^2)\}$$

$$(\text{with } C=\sigma_1^2\sigma_3^2+\kappa_2\sigma_1^2\sigma_2^2+\eta_2^2\sigma_2^2\sigma_3^2+\eta_2^2\kappa_1^2\sigma_1^2\sigma_2^2+2\eta_2\kappa_1\kappa_2\sigma_1^2\sigma_2^2)$$

$$\{1-N(\ln u_3-\kappa_1\ln u_1-\kappa_2\ln u_2-\tfrac{1}{D}\{(-\kappa_1\sigma_1^2\sigma_2^2+\eta_2\eta_4^2\kappa_2\sigma_2^4-\eta_4\kappa_2\sigma_1^2\sigma_2^2$$

$$+\eta_2\eta_4\kappa_1\sigma_1^2\sigma_2^2)(\ln u_1-\eta_2\ln u_2)-(-\eta_2\kappa_1\sigma_1^2\sigma_2^2+\eta_2\eta_4\kappa_2\sigma_1^2\sigma_2^2-\kappa_2\sigma_1^2\sigma_2^2$$

$$+\eta_2^2\eta_4\kappa_1\sigma_1^2\sigma_2^2)(\ln u_2-\eta_4\ln u_1)\};$$

$$\sigma_3^2+\kappa_1^2\sigma_1^2+\kappa_2^2\sigma_2^2-\tfrac{1}{D}\{(-\kappa_1\sigma_1^2\sigma_2^2+\eta_2\eta_4^2\kappa_2\sigma_2^4-\eta_4\kappa_2\sigma_1^2\sigma_2^2+\eta_2\eta_4\kappa_1\sigma_1^2\sigma_2^2)$$

$$(-\kappa_1\sigma_1^2+\eta_2\kappa_2\sigma_2^2)-(-\eta_2\kappa_1\sigma_1^2\sigma_2^2+\eta_2\eta_4\kappa_2\sigma_1^2\sigma_2^2-\kappa_2\sigma_1^2\sigma_2^2+\eta_2^2\eta_4\kappa_1\sigma_1^2\sigma_2^2)$$

$$(-\kappa_2\sigma_2^2+\eta_4\kappa_1\sigma_1^2)\}\quad(\text{with } D := \sigma_1^2\sigma_2^2+\eta_2^2\eta_4^2\sigma_1^2\sigma_2^2-2\eta_2\eta_4\sigma_1^2\sigma_2^2)$$

$$L_4 = \{1-N(\ln u_3-\kappa_2\ln u_2;\sigma_3^2+\kappa_2^2\sigma_2^2)\}\{1-N(\ln u_3;\sigma_3^2)\}$$

$$\{1-N(\ln u_5-\zeta_3\ln u_4-\zeta_4\ln u_6;\sigma_5^2+\zeta_3^2\sigma_4^2+\zeta_4^2\sigma_6^2)\}\{1-N(\ln u_1;\sigma_1^2)\}$$

$$\{1-N(\ln u_6-\kappa_3\ln u_4;\sigma_6^2+\kappa_3^2\sigma_4^2)\}\{1-N(\ln u_2+\tfrac{1}{E}\{\eta_2\sigma_2^2\sigma_3^2(\ln u_1-\eta_2\ln u_2)$$

$$+\kappa_2\sigma_1^2\sigma_2^2(\ln u_3-\kappa_2\ln u_2)\};\sigma_2^2-\tfrac{1}{E}\{\eta_2^2\sigma_2^4\sigma_3^2+\kappa_2^2\sigma_1^2\sigma_2^4\})\}$$

$$(\text{with } E := \sigma_1^2\sigma_3^2+\kappa_2^2\sigma_1^2\sigma_2^2+\eta_2^2\sigma_2^2\sigma_3^2)$$

$$L_5 = n(\ln E^d(x)+\Pi_1 \ln u_6+\Pi_2 \ln u_1;\sigma_7^2+\Pi_1^2\sigma_6^2+\Pi_2^2\sigma_1^2)$$

$$n(\ln I^d(x)-\Pi_4 \ln u_4-\Pi_5 \ln u_1;\sigma_8^2+\Pi_4^2\sigma_6^2+\Pi_5^2\sigma_1^2)$$

$$n(\ln u_4-\zeta_1 \ln u_5-\zeta_2 \ln u_6;\sigma_4^2+\zeta_1^2\sigma_5^2+\zeta_2^2\sigma_6^2)$$

$$\{1-N(\ln u_5-\zeta_4 \ln u_6;\sigma_5^2+\zeta_4^2\sigma_6^2)\}\{1-N(\ln u_2;\sigma_2^2)\}$$

$$\{1-N(\ln u_6-\kappa_4 \ln u_5;\sigma_6^2+\kappa_4^2\sigma_5^2)\}\{1-N(\ln u_3;\sigma_3^2)\}$$

$$\{1-N(\ln u_1+\tfrac{1}{F}\{\eta_3\sigma_1^2\sigma_3^2(\ln u_2-\eta_3 \ln u_1)+\kappa_1\sigma_1^2\sigma_2^2(\ln u_3-\kappa_1 \ln u_1)\};$$
$$\sigma_1^2-\tfrac{1}{F}\{\eta_3^2\sigma_1^4\sigma_3^2+\kappa_1^2\sigma_1^4\sigma_2^2\})\}$$

(with $F := \sigma_2^2\sigma_3^2+\kappa_1^2\sigma_1^2\sigma_2^2+\eta_3^2\sigma_1^2\sigma_3^2$)

Appendix C: Statistical identifiability of the 2-market model in Gouriéroux, Laffont & Monfort (1980)

Gouriéroux, Laffont & Monfort are giving the following relationship in their notation:

$$\begin{pmatrix} \underline{\delta}_{1t} \\ \underline{\lambda}_{1t} \\ \underline{\delta}_{2t} \\ \underline{\lambda}_{2t} \end{pmatrix} = \sum_{i=1}^{4} A_i \mathbb{1}_{C_i} \begin{pmatrix} \underline{D}_{1t} \\ \underline{S}_{1t} \\ \underline{D}_{2t} \\ \underline{S}_{2t} \end{pmatrix} = \Lambda X_t + \underline{u}_t \quad \text{(see also (3.2) and (3.9))}$$

with

$\underline{\delta}_{it}$, $\underline{\lambda}_{it}$	functions of prices and other variables
X_t	a vector of exogenous variables
Λ	a matrix of unknown parameters
$\underline{u}_t$	a (4x1) random vector having a density g

$C_1 := \{D_1>S_1,\ D_2>S_2\}$ $\quad D_1$, S_1 effective demand and supply in market 1

$C_2 := \{D_1>S_1,\ D_2<S_2\}$

$C_3 := \{D_1<S_1,\ D_2<S_2\}$ $\quad D_2$, S_2 effective demand and supply in market 2

$C_4 := \{D_1<S_1,\ D_2>S_2\}$

$\mathbb{1}_{C_i}(x) = 1 \iff x \in C_i$

A_i are matrices of spill-over coefficients

$$A_1 = \begin{bmatrix} 1-\alpha_1\beta_2 & \alpha_1\beta_2 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\beta_1 \\ 0 & -\alpha_2 & 1-\alpha_2\beta_1 & \alpha_2\beta_1 \\ 0 & -\beta_2 & 0 & 1 \end{bmatrix} \quad \text{(compare with (3.8))}$$

$$A_2 = \begin{bmatrix} 1 & 0 & -\alpha_1 & 0 \\ 0 & 1 & -\beta_1 & 0 \\ 0 & -\alpha_2 & 1 & 0 \\ 0 & -\beta_2 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & -\alpha_1 & 0 \\ \alpha_2\beta_1 & 1-\alpha_2\beta_1 & -\beta_1 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\beta_2 & 0 & \alpha_1\beta_2 & 1-\alpha_1\beta_2 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\beta_1 \\ -\alpha_2 & 0 & 1 & 0 \\ -\beta_2 & 0 & 0 & 1 \end{bmatrix}$$

$Q_1 := \min \{D_1, S_1\}$
$Q_2 := \min \{D_2, S_2\}$

The density of u_t is given by $g(u_t)$
The density of (Q_1, Q_2) is

$$\begin{aligned} h_t(Q_1, Q_2) = \int_{Q_1}^{\infty} \int_{Q_2}^{\infty} \{ & |A_1| g(A_1\{Q_1, x, Q_2, y\}' - \Lambda X_t) \\ + & |A_2| g(A_2\{Q_1, x, y, Q_2\}' - \Lambda X_t) \\ + & |A_3| g(A_3\{x, Q_1, Q_2, y\}' - \Lambda X_t) \\ + & |A_4| g(A_4\{x, Q_1, y, Q_2\}' - \Lambda X_t)\} \, dxdy\} \end{aligned}$$

(see also (3.10))

The likelihood function $L_\Theta(X)$ satisfies:

$$L_\Theta(X) = \prod_{t=1}^{T} h_t(Q_1, Q_2)$$

with

$\Theta: = (\alpha_1, \alpha_2, \beta_1, \beta_2, \Lambda, \xi)' \in (H)$

$(H): = \{\theta | 1-\alpha_1\beta_2>0;\ 1-\alpha_2\beta_1>0;\ 1-\alpha_1\alpha_2>0;$
$1-\beta_1\beta_2>0;\ \alpha_1,\alpha_2,\beta_1,\beta_2>0;$ Λ is a
$4xk$ matrix with k the number of exogenous variables;$\xi \in Z\}$;

ξ is the parameter, defining the probability density of u_t.
Let $\mathcal{P}: = \{H_\Theta : \Theta\varepsilon\ (H)\}$, where H_Θ is the distribution of the stochastic vector with density $L_\Theta\ (X)$. The class, $\mathcal{P}$ is said to be identified by (H) if

$$\forall\ \theta_1, \theta_2 \in (H) : \theta_1 \neq \theta_2 \rightarrow L_{\theta_1}(X) \neq L_{\theta_2}(X)$$

$$<=>$$

$$h_{\theta_1}(Q_1, Q_2) \neq h_{\theta_2}(Q_1, Q_2)$$

When we write out $h_{\theta_1}(Q_1, Q_2)$ with

$\Lambda_1 X_t \rightarrow$ first row of ΛX_t

$\Lambda_2 X_t \rightarrow$ second row of ΛX_t

and $\theta_1: = (\alpha_1,\beta_1,\alpha_2,\beta_2,\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4,\xi)$

$h_{\theta_1}(Q_1,Q_2) =$

$$\int_{Q_1}\int_{Q_2} \{\ (1-\alpha_1\beta_2)(1-\beta_1\alpha_2)(1-\alpha_1\alpha_2)g\left\{\begin{bmatrix} Q_1-\alpha_1Q_2-\Lambda_1X_t \\ \alpha_2\beta_1Q_1+(1-\beta_1\alpha_2)x-\beta_1Q_2-\Lambda_2X_t \\ -\alpha_2Q_1+Q_2-\Lambda_3X_t \\ -\beta_2Q_1+\alpha_1\beta_2Q_2+(1-\beta_2\alpha_1)y-\Lambda_4X_t \end{bmatrix}\right\}$$

$$+(1-\alpha_1\beta_2)g\left\{\begin{bmatrix} Q_1-\alpha_1Q_2-\Lambda_1X_t \\ x-\beta_1Q_2-\Lambda_2X_t \\ -\alpha_2Q_1+y-\Lambda_3X_t \\ -\beta_2Q_1+Q_2-\Lambda_4X_t \end{bmatrix}\right\}$$

$$+(1-\alpha_2\beta_1)g\left\{\begin{bmatrix} x-\alpha_1Q_2-\Lambda_1X_t \\ Q_1-\beta_1Q_2-\Lambda_2X_t \\ -\alpha_2Q_1+Q_2-\Lambda_3X_t \\ -\beta_2Q_1+y-\Lambda_4X_t \end{bmatrix}\right\}$$

$$+(1-\alpha_1\beta_2)(1-\beta_1\alpha_2)(1-\beta_1\beta_2)g\left\{\begin{bmatrix} (1-\alpha_1\beta_2)x+\alpha_1\beta_2Q_1-\alpha_1Q_2-\Lambda_1X_t \\ Q_1-\beta_1Q_2-\Lambda_2X_t \\ -\alpha_2Q_1+(1-\alpha_2\beta_1)y+\alpha_2\beta_1Q_2-\Lambda_3X_t \\ -\beta_2Q_1+Q_2-\Lambda_4X_t \end{bmatrix}\right\}\Big\} dx\,dy$$

If we consider a parameter set $\theta_2 \neq \theta_1$

$$\theta_2 := (\beta_1,\alpha_1,\beta_2,\alpha_2,\Lambda_2,\Lambda_1,\Lambda_4,\Lambda_3)$$

Then we must have, in order to obtain a statistically identified model, that $h_{\theta_1}(Q_1,Q_2) \neq h_{\theta_2}(Q_1,Q_2)$.

However, $h_{\theta_1}(Q_1,Q_2) = h_{\theta_2}(Q_1,Q_2)$ as

$$h_{\theta_2}(Q_1,Q_2) = \int_{Q_1}\int_{Q_1}\Big\{(1-\beta_1\alpha_2)(1-\alpha_1\beta_2)(1-\beta_1\beta_2)g\left\{\begin{matrix} Q_1-\beta_1Q_2-\Lambda_2X_t \\ \beta_2\alpha_1Q_1+(1-\alpha_1\beta_2)x-\alpha_1Q_2-\Lambda_1X_t \\ -\beta_2Q_1+Q_2-\Lambda_4X_t \\ -\alpha_2Q_1+\beta_1\alpha_2Q_2+(1-\alpha_2\beta_1)y-\Lambda_3X_t \end{matrix}\right\}$$

$$+(1-\beta_1\alpha_2)g\begin{bmatrix} Q_1-\beta_1Q_2-\Lambda_2X_t \\ x-\alpha_1Q_2-\Lambda_1X_t \\ -\beta_2Q_1+y-\Lambda_4X_t \\ -\alpha_2Q_1+Q_2-\Lambda_3X_t \end{bmatrix}$$

$$+(1-\beta_2\alpha_1)g\begin{bmatrix} x-\beta_1Q_2-\Lambda_2X_t \\ Q_1-\alpha_1Q_2-\Lambda_1X_t \\ -\beta_2Q_1+Q_2-\Lambda_4X_t \\ -\alpha_2Q_1+y-\Lambda_3X_t \end{bmatrix}$$

$$+(1-\beta_1\alpha_2)(1-\alpha_1\beta_2)(1-\alpha_1\alpha_2)g\left\{\begin{bmatrix} (1-\beta_1\alpha_2)x+\beta_1\alpha_2Q_1-\beta_1Q_2-\Lambda_2X_t \\ Q_1-\alpha_1Q_2-\Lambda_1X_t \\ -\beta_2Q_1+(1-\beta_2\alpha_1)y+\beta_2\alpha_1Q_2-\Lambda_4X_t \\ -\alpha_2Q_1+Q_2-\Lambda_3X_t \end{bmatrix}\right\}\}dxdy$$

which is equal to $h_{\theta_1}(Q_1, Q_2)$, if we change the parameters in the argument of the error density g by permuting the role of the first and the second component on the one hand and the role of the third and fourth component on the other hand.
The conclusion is that the 2-market-model in Gouriéroux, Laffont and Monfort (1980) is not statistically identified. Hence, exchanging the demand and supply equations on each market, we have the same density for the realizations (Q_1, Q_2). A statistically identified QRM could, however, be obtained if the set of exogenous variables is not the same in both equations. This could, e.g. (a fortiori) be reached by making either endogenous variable exogenous.

Since a similar (but more elaborate) proof can be followed for our 3-markets QRM in section 2, we propose to exogenize the labour force participation rate determining the workers' supply of labour (which is already principally determined by exogenous demographic factors (see Meersman and Plasmans (1980), pp. 22-24). Note, however, that exogenizing the workers' supply is sufficient, but in no way necessary for statistical identification of the 3-markets QRM!

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