STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

THE RELATION BETWEEN ALTERNATIVE BENEFIT MEASURES FOR QUANTITY CONSTRAINED PRICE SUBSIDIES

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Abstract

In this paper we analyse the relation between Hicks' equivalent and compensating variations and Marshallian consumer's surplus for quantity constrained price subsidies. It turns out that the relative magnitude of the alternative benefit measures crucially depends on the level of the constraints on the subsidized good. Different cases are considered. For each case we investigate the relation between the different measures of welfare change.

We show that Marshallian benefit may substantially overestimate the equivalent variation. Moreover, we provide sufficient conditions under which the relation between the three measures is the same as in the case of a pure price subsidy.

0. INTRODUCTION

Substantial progress has recently been made with respect to measuring the change in welfare due to a pure price subsidy. Procedures have been developed to calculate the Hicksian measures of welfare change on the basis of observable market demand functions {Hausman (1981), Vartia (1983)}. It follows that the use of Marshallian consumer's surplus can no longer be justified in practice, even though in many cases it provides a reasonable approximation to the exact measures of welfare change {see Willig (1976)}.

The focus of much of the literature on the case of pure price subsidies is somewhat surprising. It must be realized that most government programs do not change the choice set of participating households by simply rotating their budget constraint. Many in-kind subsidy programs offer eligible families an all-or-nothing choice to consume a given quantity of the subsidized commodity at a unit price determined by the government. Examples include rent control, food stamps and public housing.

Only very recently several authors have considered the welfare implications for cases other than pure price subsidies. For changes in imposed quantities Lankford (1983) shows that Marshallian benefits do not yield a close approximation to Hicksian surplus measures, contrary to the original claim by Randall and

Stoll (1980). Moreover, he extends the procedures developed by Vartia and Hausman for this particular case. He shows how to derive the exact Hicksian measures on the basis of ordinary demand functions.

In this paper we do not deal with pure price subsidies nor with changes in imposed quantities. We focus on the measurement of welfare changes due to the introduction of in-kind subsidy programs like e.g. public housing. Measuring these effects is one of the main purposes of this dissertation.

We will define Hicksian and Marshallian benefit measures for the type of government program we have in mind. Moreover, we will investigate the relation between the equivalent and compensating variations (EV and CV, respectively) and Marshallian consumer's surplus (MB) for this specific case. A famous theorem in welfare economics asserts that for a pure price subsidy $EV \geq MB \geq CV$ Blaug (1978) . Although it has been indicated in the literature that this series of weak inequalities does not necessarily hold when quantity constraints are imposed {Murray (1976), Cornes and Albon (1981)}, it has not been studied how the relation between the alternative benefit measures is affected by the level of the constraint. This will be analyzed below.

The paper is organized as follows: in the first section we briefly review the derivation of the alternative benefit measures used in the literature in the case of pure price subsidies. We then show in section 2 how it is possible to define CV, MB and EV in an analogous way in the case of quantity-constrained price subsidies by the appropriate choice of a set of shadow prices for the constrained good. The results are used in section 3 to investigate the relation between the benefit measures. We consider four cases depending upon the quantity at which the government program fixes the consumption of the constrained good. A final section summarizes the main conclusions.

1. WELFARE ANALYSIS FOR PURE PRICE SUBSIDIES: A SUMMARY REVIEW

In this section we briefly revew the derivation of alternative benefit measures and show graphically how they are related for the case of a pure price subsidy. We assume all commodities can be aggregated into two goods: the subsidized commodity H and a composite X intended to capture 'all other goods'[1].

Consider the initial situation where the consumer faces market prices p_H^m and p_X^m, respectively. Given the budget constraint he maximizes utility and attains a utility level u^m. In equilibrium he consumes H^m and X^m of the two goods, as indicated on figure 1a, see point A. Suppose a pure price subsidy results in a lower price p_H^1. The consumer's new optimum is at point B where H^1 and X^1 are consumed and utility level u^1 is achieved.

The compensating variation asks for the amount the consumer would be willing to give up at the lower price p_H^1 and still be as well off as at the initial optimum. In terms of the indirect utility function $v(p_H, p_X, y)$, it is implicitly defined by the following equation

$$v(p_H^1, p_X^m, y - CV) = v(p_H^m, p_X^m, y)$$

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[1] This simplification allows graphical illustration of the benefit measures which will be extremely useful below. Dealing with a vector of other goods rather than with a composite X would be a straightforward generalization. We have to concentrate on a single subsidized good, however, because Marshallian surplus is not path independent in the case of multiple price changes {Richter (1977)}.

where we assume that the subsidy on H does not affect household income and market prices[1]. Defining the expenditure function $e(p_H, p_X, u)$ as the minimum expenditures necessary to attain utility level u when facing prices p_H and p_X, we should obviously have

$$v\{p_H^1, p_X^m, e(p_H^1, p_X^m, u^m)\} = v(p_H^m, p_X^m, y)$$

where: $u^m = v(p_H^m, p_X^m, y)$

Consequently we find that

$$CV = y - e(p_H^1, p_X^m, u^m) = e(p_H^m, p_X^m, u^m) - e(p_H^1, p_X^m, u^m)$$

The compensating variation of a pure price subsidy is just the difference in expenditures before and after the price change necessary to reach the intilial utility level u^m. In terms of X it can be represented by the distance CE on figure 1a.

Hicks equivalent variation on the other hand looks for the amount we should pay the consumer facing prices p_H^m and p_X^m in order to make him as well off as at the subsidized price p_H^1. It evaluates the welfare change as the difference in expenditures before and after the subsidy in order to reach utility u^1. Indeed, by definition

$$v(p_H^m, p_X^m, y + EV) = v(p_H^1, p_X^m, y)$$

Remembering that $u^1=v(p_H^1, p_X^m, y)$ we should have the following

[1] It is common in the literature to treat X as a numeraire good with unit price one, without loss in generality. Although this has certain advantages, especially for the graphical interpretation of the Hicksian benefit measures, we will keep p_X as an explicit argument in indirect utility, expenditure and demand functions throughout this paper.

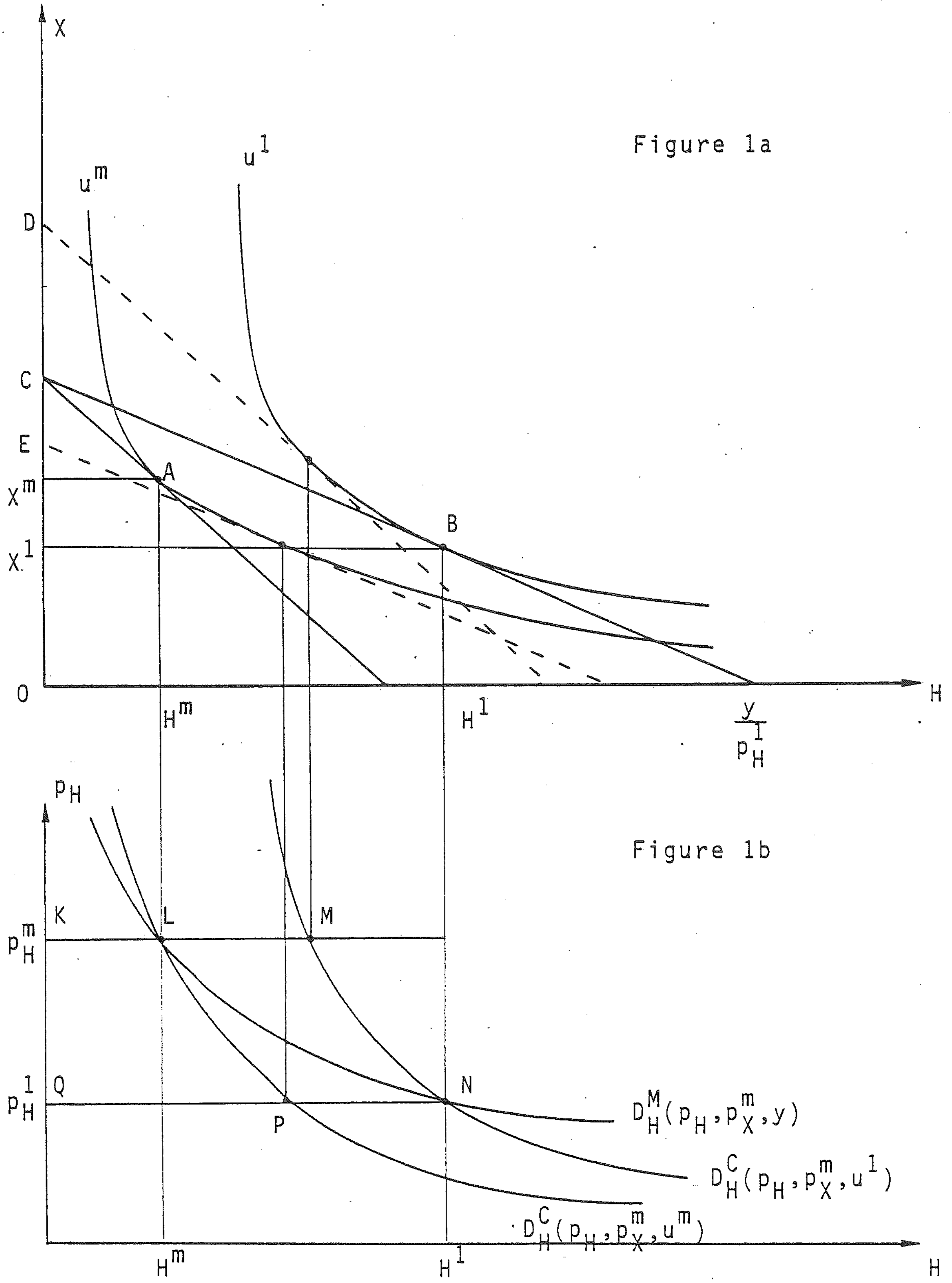

Figure 1: derivation of alternative benefit measures for a pure price subsidy.

$$v\{p_H^m, p_X^m, e(p_H^m, p_X^m, u^1)\} = v(p_H^1, p_X^m, y)$$

which implies

$$EV = e(p_H^m, p_X^m, u^1) - y = e(p_H^m, p_X^m, u^1) - e(p_H^1, p_X^m, u^1)$$

In terms of commodity X the equivalent variation is indicated on figure 1a as the distance DC.

Using the properties of expenditure functions it is straightforward to show that CV and EV are related to areas under the appropriate compensated demand functions D_H^c for the subsidized good H (see, e.g. Varian (1980), p.210):

$$CV = e(p_H^m,p_X^m,u^m)-e(p_H^1,p_X^m,u^m) = -\int_{p_H^m}^{p_H^1} D_H^c(p_H,p_X^m,u^m)dp_H$$

$$EV = e(p_H^m,p_X^m,u^1)-e(p_H^1,p_X^m,u^1) = -\int_{p_H^m}^{p_H^1} D_H^c(p_H,p_X^m,u^1)dp_H$$

Finally, Marshallian consumer's surplus MB is defined as the area to the left of the Marshallian demand curve D_H^M for H between the pre and post-subsidy prices:

$$MB = -\int_{p_H^m}^{p_H^1} D_H^M(p_H,p_X^m,y)dp_H.$$

The three benefit measures are represented on an ordinary price-quantity diagram in figure 1b which is derived from 1a in the usual way. The compensating and equivalent variation correspond to the areas KLPQ and KMNQ respectively. Marshallian benefit equals the area KLNQ.

The relation between the three benefit measures is obvious from figure 1b. As long as the subsidized good is a normal good it will be the case that[1]:

$$EV \geqslant MB \geqslant CV$$

This result has often implicitly been used as a justification for calculating MB as an approximation to the Hicksian measures. In section 3 we will show that the relation does not necessarily hold for quantity-constrained price subsidies.

2. WELFARE ANALYSIS FOR IN-KIND SUBSIDIES

In this section we show how Hicks' equivalent and compensating variations of an in-kind subsidy can be expressed in terms of areas around compensated demand functions for the subsidized good by appropriate selection of a set of shadow prices. Again, the analysis assumes that H is the subsidized good.

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[1] See e.g. Blaug (1978, pp. 374-383). The result directly follows from the Slutsky equation

$$\frac{\delta D_H^M(p_H,p_X^m,y)}{\delta p_H} = \frac{\delta D_H^C(p_H,p_X^m,u)}{\delta p_H} - \frac{\delta D_H^M(p_H,p_X^m,y)}{\delta y} D_H^M(p_H,p_X^m,y)$$

If H is normal, i.e. if $\delta D_H^M(p_H,p_X^m,y)/\delta y \geqslant 0$, the slope of the compensated demand curves is at least as steep as the slope of the Marshallian demand curve implying the above weak inequalities. Note that a zero income elasticity for H yields EV=CV=MB as Hicksian and Marshallian demand curves coincide.

Consider a household buying H and X on the private market at prices p_H^m and p_X^m, respectively. Suppose the utility maximizing household consumes in equilibrium H^m and X^m of both commodities, attaining a utility level u^m. Now assume that the household is selected to participate in a government program offering a given quantity H^s of the first good at a subsidized price p_H^s. The program yields utility u^s, $u^s > u^m$.

Participating households face the restriction $H = H^s$ in addition to the budget constraint. Therefore, in what follows we will use some ideas developed in the theory of demand under quantity restrictions, see e.g. Pollak (1969), Howard (1977), Latham (1980) and especially Neary and Roberts (1980).

First we introduce some definitions and notation. The constrained expenditure function $\tilde{e}(H^s, p_H, p_X, u)$ gives the minimum expenditures necessary to reach a given utility level u, when facing prices p_H and p_X under the restriction that consumption of good H is constrained at H^s, i.e.

$$\tilde{e}(H^s, p_H, p_X, u) = \underset{X}{\mathrm{Min}}\{p_X X + p_H H^s / u(X, H^s) \geq u\}$$

$$= p_H H^s + \underset{X}{\mathrm{Min}}\{p_X X / u(X, H^s) \geq u\}$$

Note that this expenditure function is only defined if it is possible to reach utility level u under the quantity constraint, i.e. if there exists an X such that $u(X, H^s) \geq u$. Inverting yields a constrained indirect utility function, $\tilde{v}(H^s, p_H, p_X, y)$, giving maximum attainable utility as a function of prices, income and the quantity constraint. As in the unconstrained case, the compensated demands for both commodities - given the constraint $H=H^S$ - are found by differentiation of the expenditure function $\tilde{e}(H^S,p_H,p_X,u)$

$$\frac{\delta\tilde{e}(H^S,p_H,p_X,u)}{\delta p_X} = \tilde{D}_X^c(H^S,u)$$

$$\frac{\delta\tilde{e}(H^S,p_H,p_X,u)}{\delta p_H} = H^S$$

Note that in a two-commodity world the conditional compensated demand functions are independent of prices[1].

Using the previous tools the Hicksian benefit measures can be defined. The equivalent variation EV is the amount one should pay the household facing prices p_H^m and p_X^m in order to make it as well off as under the program. It is the solution to the implicit equation

$$v(p_H^m,p_X^m,y+EV) = \tilde{v}(H^S,p_H^S,p_X^m,y) \qquad (1)$$

where, as before, v(.) is the unconditional indirect utility function. Since by definition

$$u^S = \tilde{v}(H^S,p_H^S,p_X^m,y) = v\{p_H^m,p_X^m, e(p_H^m,p_X^m,u^S)\}$$

the solution to (1) yields

$$EV = e(p_H^m,p_X^m,u^S)-y$$

[1] Conditional compensated demand functions are always independent of the price of the constrained good because this only appears in the fixed cost term of the expenditure function, see Deaton and Muellbauer (1980, p.110). In the two-good case these demand functions are independent of both prices.

or alternatively

$$EV = e(p_H^m, p_X^m, u^s) - \tilde{e}(H^s, p_H^s, p_X^m, u^s) \qquad (2)$$

where e(.) is the unconditional expenditure function.

The constraint $H=H^s$ implies that one can think of several possibilities for the definition of the compensating variation, depending on the situation one associates with utility level u^s under the program[1]. We have chosen what seemed to be the most natural approach by taking the actual situation under the program as a starting point, taking into account both the subsidized price p_H^s and the restriction on consumption. The compensating variation was simply defined as the amount the household would be willing to give up, given its situation under the program, and still be as well off as in its absence. Consequently:

$$\tilde{v}(H^s, p_H^s, p_X^m, y-CV) = v(p_H^m, p_X^m, y) \qquad (3)$$

Noting that

$$u^m = v(p_H^m, p_X^m, y) = \tilde{v}\{H^s, p_H^s, p_X^m, \tilde{e}(H^s, p_H^s, p_X^m, u^m)\}$$

we immediately derive that

$$CV = y - \tilde{e}(H^s, p_H^s, p_X^m, u^m) = e(p_H^m, p_X^m, u^m) - \tilde{e}(H^s, p_H^s, p_X^m, u^m) \qquad (4)$$

[1] In principle, the compensating variation could be defined as the amount that could be taken away from consumers facing prices p_H^s and p_X^m under the program and still leave them as well off as in the absence of the program. This seems undesirable, however, because it neglects the constraint and implicitly assumes that H^s is the optimal quantity, given the price p_H^s. Alternatively, one could define CV by considering the price of H that would have yielded u^s as optimal utility level, again not explicitly dealing with the constraint on H. This is the approach taken by Cornes and Albon (1981). Moreover, these authors note that the relation $EV \geq MB \geq CV$ does not necessarily hold when quantity restrictions are imposed and redefine an 'Ideal Marshallian Consumer's Surplus' so that the series of weak inequalities will always be satisfied. Their proposed measure has some desirable properties, but it is not the Marshallian benefit that has been used in the literature. We concentrate on the Marshallian measure encountered in the literature.

Both CV and EV are illustrated on a simple diagram, see figure 2. Application of expressions (2) and (4) implies that, in terms of good X, the equivalent and compensating variations are indicated by the distances AB and CD, respectively. Note that CV, due to the constraint on H, is just the vertical distance between u^s and u^m at H^s.

Both benefit measures may be expressed in terms of <u>unconstrained</u> compensated demand functions through the use of shadow prices for good H. First consider the equivalent variation defined by (2). We construct a shadow price p_H' such that the bundle (H^s,X^s) would have been the expenditure minimizing bundle to reach utility level u^s, had this shadow price been the market price. Graphically p_H' is derived in figure 3 by drawing a tangent line to the indifference curve corresponding to u^s at the bundle (H^s,X^s). The minimum expenditures are $y'=e(p_H',p_X^m,u^s)$.

It is important to note that the shadow price p_H' may be below or above the initial price p_H^m. Suppose $p_H'=p_H^m$. This would imply that the bundle (H^s,X^s) consumed under the program is on the income-consumption path for income $e(p_H',p_X^m,u^s)$. In this case the household would have consumed exactly the bundle (H^s,X^s) if it were given an unrestricted cash grant equal to the 'subsidy', i.e. the difference in total market value of all goods consumed with and without the program. Now suppose $p_H'<p_H^m$. This implies the bundle (H^s,X^s) is to the right of the income-consumption path: the household would consume less H when given a grant equal to the subsidy. Conversely, if $p_H'>p_H^m$ the household consumes at a point to the left of the income-consumption path. It would consume more H when given the aforementioned grant.

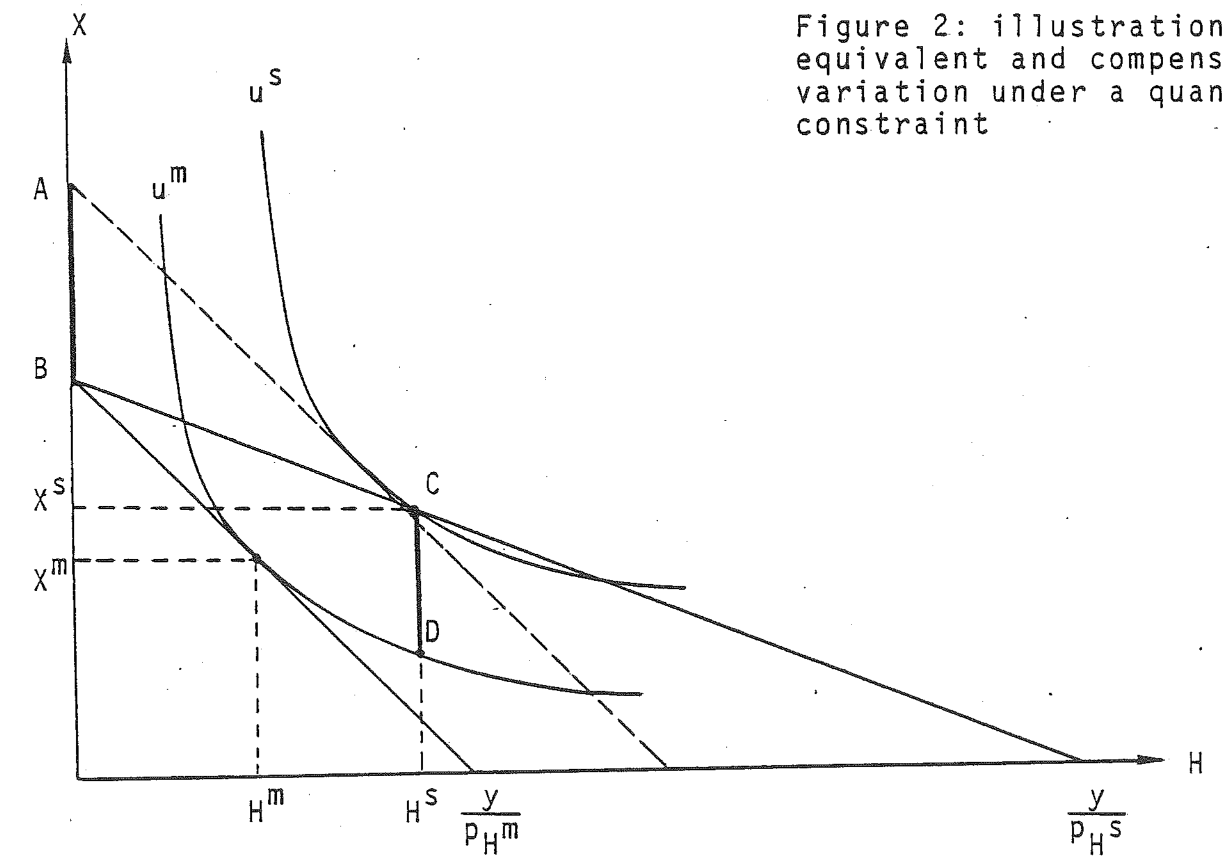

Figure 2: illustration of the equivalent and compensating variation under a quantity constraint

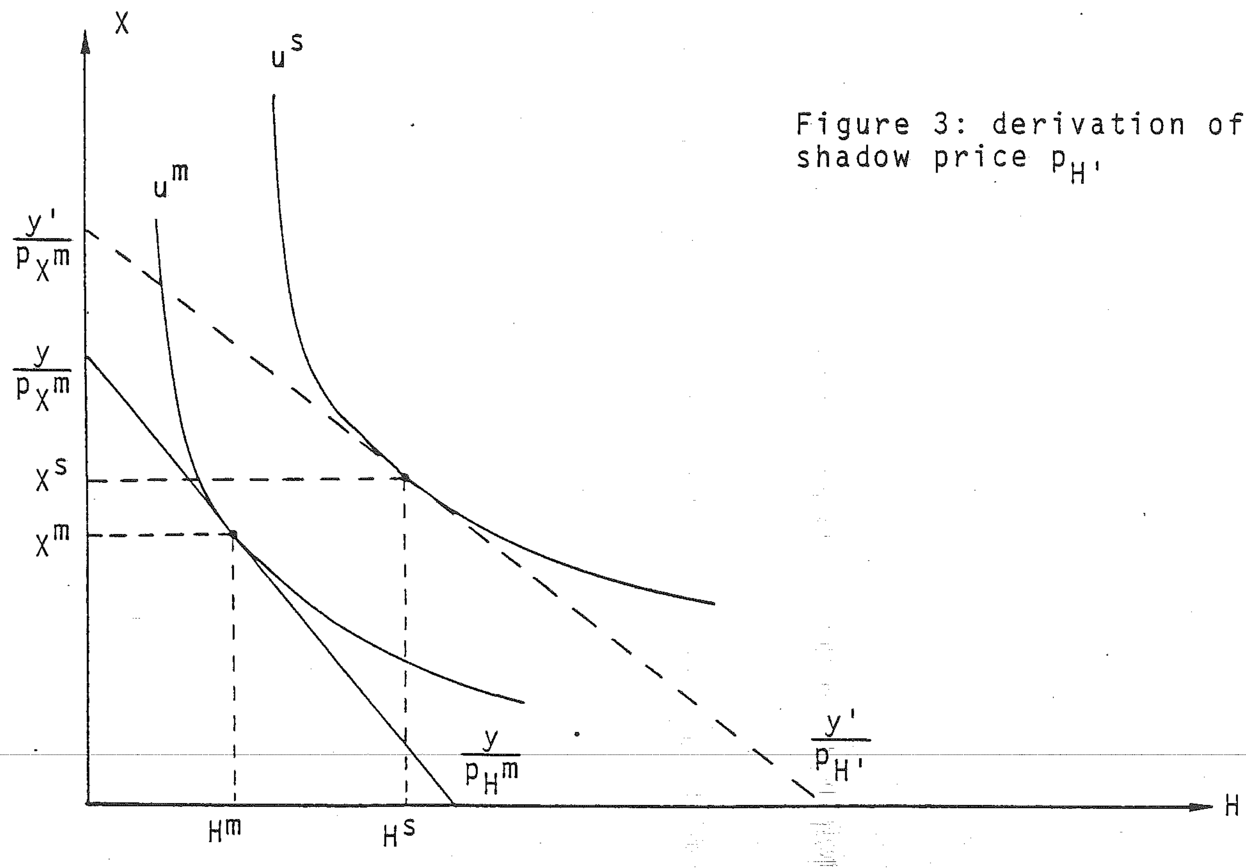

Figure 3: derivation of the shadow price $p_{H'}$

By definition of $p_H^{'}$ the unconstrained compensated demands at prices $p_H^{'}$ and p_X^m equal the constrained compensated demands, all demand functions evaluated at u^s:

$$D_H^c(p_H^{'},p_X^m,u^s) = H^s \tag{5}$$

$$D_X^c(p_H^{'},p_X^m,u^s) = \tilde{D}_X^c(H^s,u^s) \tag{6}$$

where $D_H^c(.)$ and $D_X^c(.)$ are the unconditional compensated demand functions. Using these results together with the definitions of constrained expenditure and demand functions we can rewrite the final term in the expression for EV given in (2):

$$\begin{aligned}\tilde{e}(H^s,p_H^s,p_X^m,u^s) &= p_X^m\tilde{D}_X^c(H^s,u^s) + p_H^sH^s \\ &= p_X^mD_X^c(p_H^{'},p_X^m,u^s) + p_H^sH^s \\ &= p_X^mD_X^c(p_H^{'},p_X^m,u^s) + p_H^{'}H^s+(p_H^s-p_H^{'})H^s\end{aligned}$$

which finally yields

$$\tilde{e}(H^s,p_H^s,p_X^m,u^s) = e(p_H^{'},p_X^m,u^s) + (p_H^s-p_H^{'})H^s \tag{7}$$

This result is a special case of the relation between constrained and unconstrained expenditure functions as derived by Neary and Roberts (1980,p.30). The final term on the right-hand side is a 'correction factor' due to the imposition of the constraint $H=H^s$. If the shadow price were equal to the subsidized price, conditional and unconditional expenditures would coincide, since in that case the household would have chosen the bundle (H^s, X^s) when freely maximizing utility at prices p_H^s and p_X^m.

Substitution of (7) in (2) yields

$$EV = e(p_H^m, p_X^m, u^s) - e(p_H', p_X^m, u^s) - (p_H^s - p_H')H^s$$

$$= - \int_{p_H^m}^{p_H'} D_H^c(p_H, p_X^m, u^s)dp_H - (p_H^s - p_H')H^s \qquad (8)$$

The importance of this result is that it allows us, unlike expression (2), to interpret EV in terms of areas to the left of an unconditional compensated demand function.

The derivation of the equivalent variation is illustrated on figure 4. The Hicksian demand curve is derived by considering the tangent lines at E and F, reflecting prices p_H^m and p_H', respectively. Applying (8) we find that EV is the area corresponding to KLMNP in the price-quantity diagram in the lower portion of the figure.

The definition of the compensating variation in (4) may be transformed in an expression similar to (8). First observe that

$$\tilde{e}(H^s, p_H^s, p_X^m, u^m) = p_X^m \tilde{D}_X^c(H^s, u^m) + p_H^s H^s \qquad (9)$$

We define a shadow price p_H'' such that the unconditional compensated demands evaluated at u^m would have been equal to the conditional compensated demands, had p_H'' been the market price. Consequently:

$$D_H^c(p_H'', p_X^m, u^m) = H^s \qquad (10)$$

$$D_X^c(p_H'', p_X^m, u^m) = \tilde{D}_X^c(H^s, u^m) \qquad (11)$$

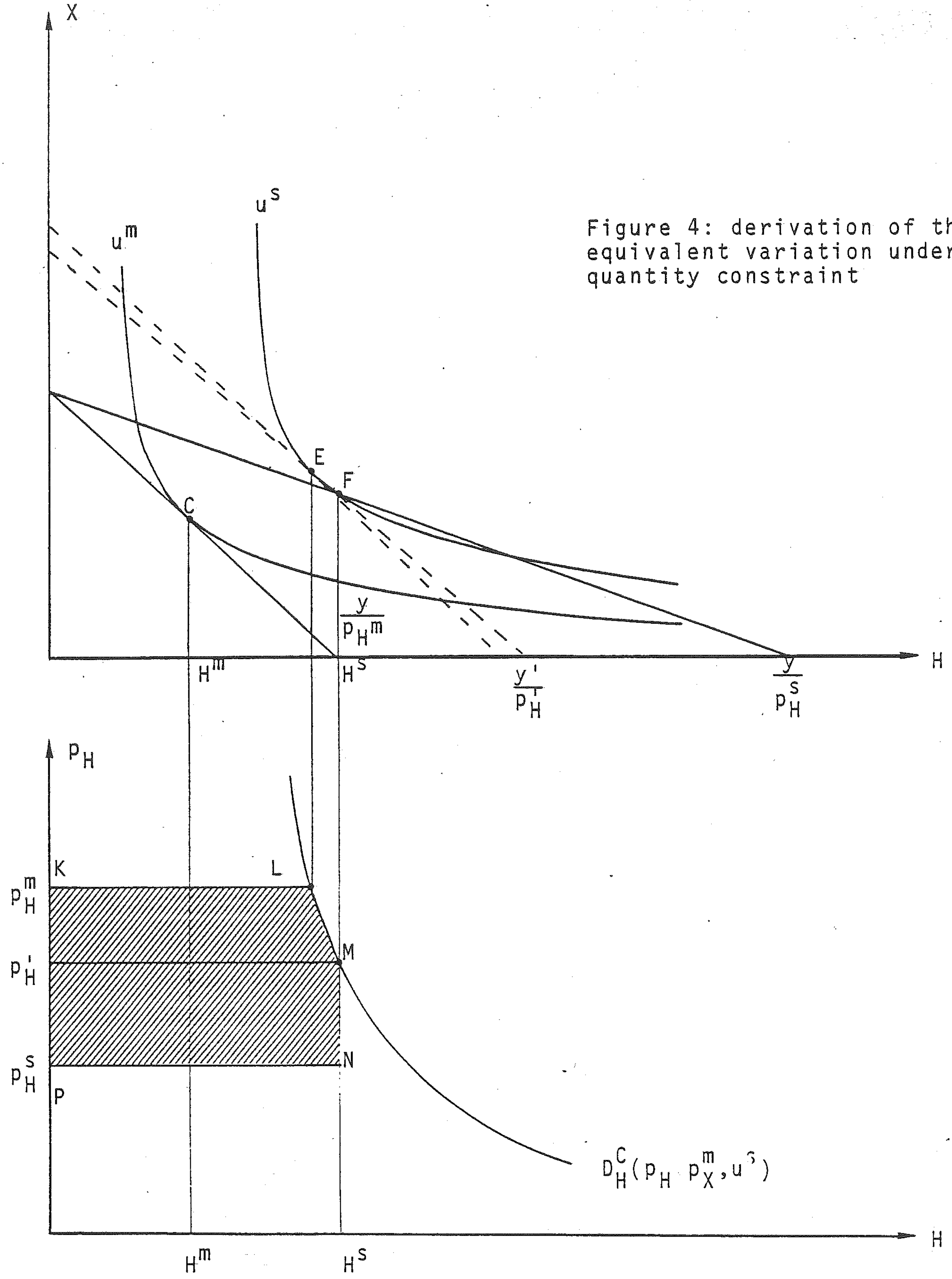

Figure 4: derivation of the equivalent variation under quantity constraint

If prices were $p_H^{"}$ and p_X^m, the minimum expenditures to reach u^m would lead the household to consume H^S of good H. These minimum expenditures are given by $y"=e(p_H^{"},p_X^m,u^m)$. Graphically the price $p_H^{"}$ is derived on figure 5 by drawing a tangent line to the indifference curve corresponding to u^m at the constrained quantity H^S.

Using (10) and (11) together with the definition

$$e(p_H^{"},p_X^m,u^m) = p_X^m D_X^c(p_H^{"},p_X^m,u^m) + p_H^{"}H^S$$

we can rewrite (9) as follows

$$\tilde{e}(H^S,p_H^S,p_X^m,u^m) = e(p_H^{"},p_X^m,u^m) + (p_H^S-p_H^{"})H^S$$

$$= - \int_{p_H^m}^{p_H^{"}} D_H^c(p_H,p_X^m,u^m)dp_H - (p_H^S-p_H^{"})H^S \quad (12)$$

The latter expression allows us to interpret CV in terms of areas to the left of the appropriate compensated demand function. A graphical derivation is in figure 6. The compensating variation equals area KLMNP on the price quantity diagram, applying (12)[1].

Finally consider Marshallian consumer's surplus under quantity constraint. It is simply defined as the difference in consumer's surplus with and without the government program. They allow for easy

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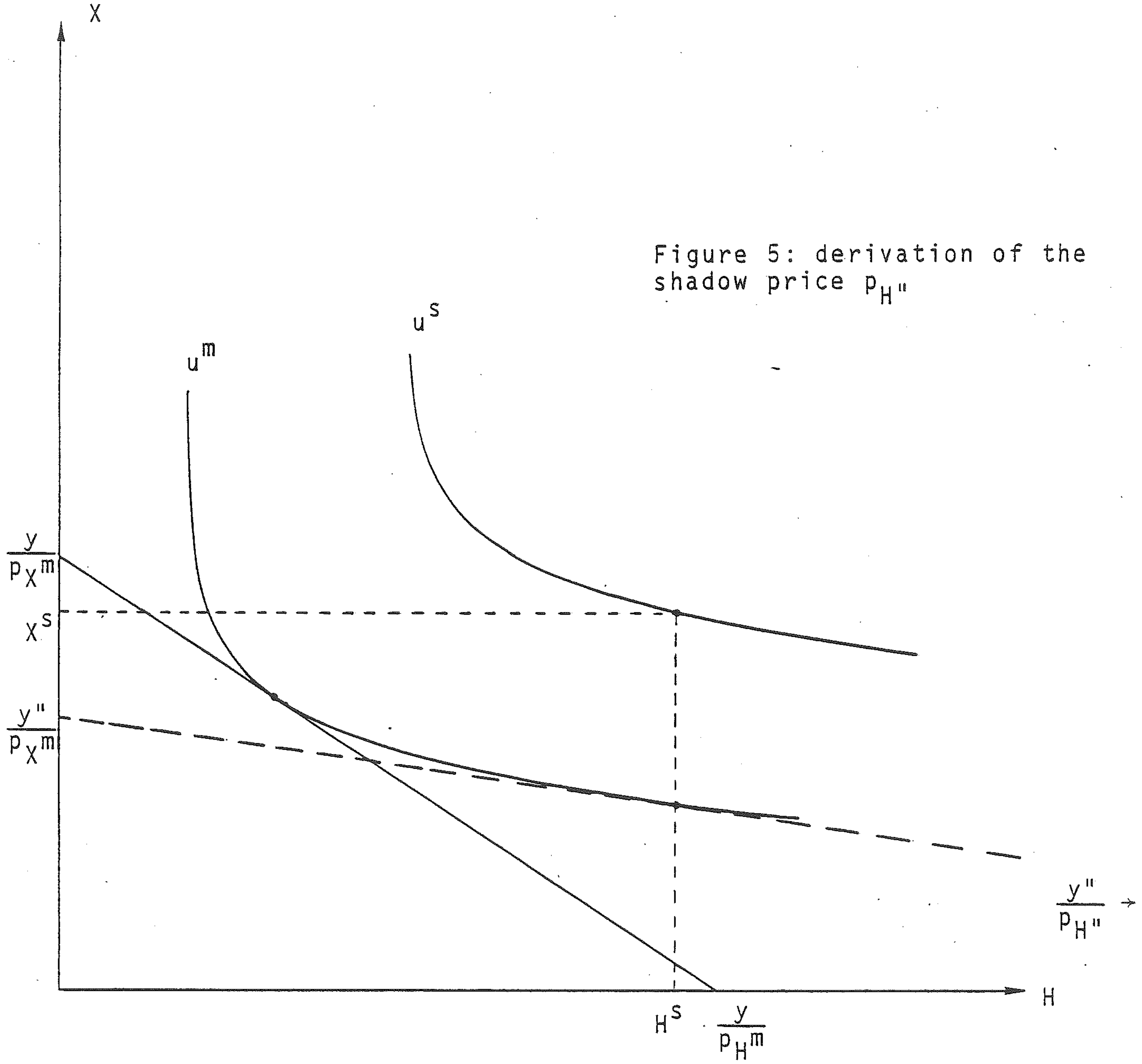
[1] Note that there is no reason why $p_{H"}$ should be greater than p_H^S. Depending upon the restricted quantity and the slope of the indifference curves, $p_{H"}$ may be greater or smaller than p_H^S. Similarly, it is perfectly possible that $p_{H"}>p_H^m$. This will happen if the program offers $H^S<H^m$, by definition of the shadow price.

Figure 5: derivation of the shadow price $p_{H''}$



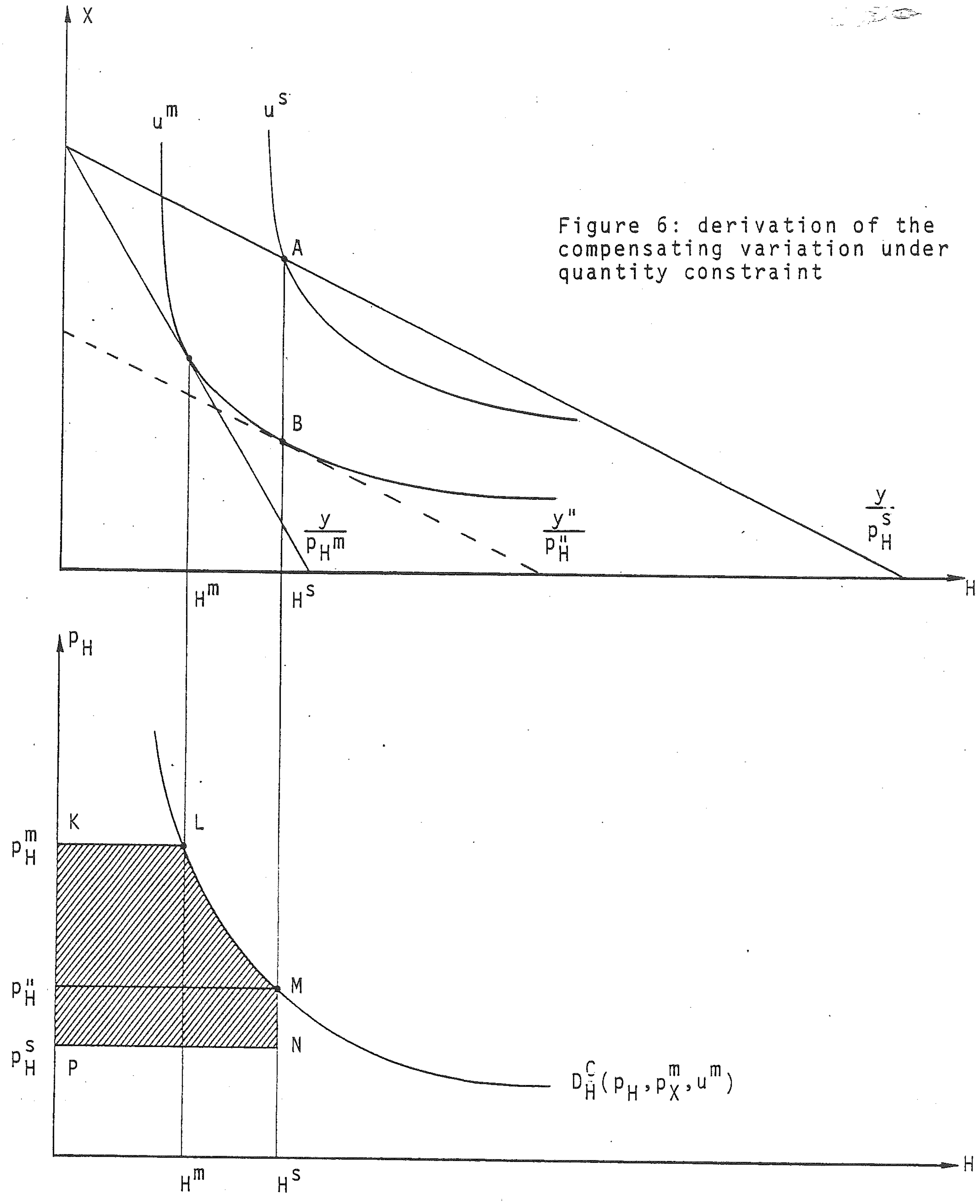


Figure 6: derivation of the compensating variation under quantity constraint

comparison with the Hicksian measures we use a different expression than the one usually found in the literature, although they yield the same result[1]. Again we define a shadow price, p_H'', such that H^S would have been the utility maximizing quantity, had this price been the observed market price. Consequently, the price-quantity combination (p_H'', H^S) is a point on the Marshallian demand curve:

$$D_H^M(p_H'', p_X^m, y) = H^S$$

where $D_H^M(.)$ is the Marshallian demand curve. Moreover, if we denote the utility level reached at prices p_H'' and p_X^m by u'' then it is obviously the case that

$$D_H^C(p_H'', p_X^m, u'') = H^S$$

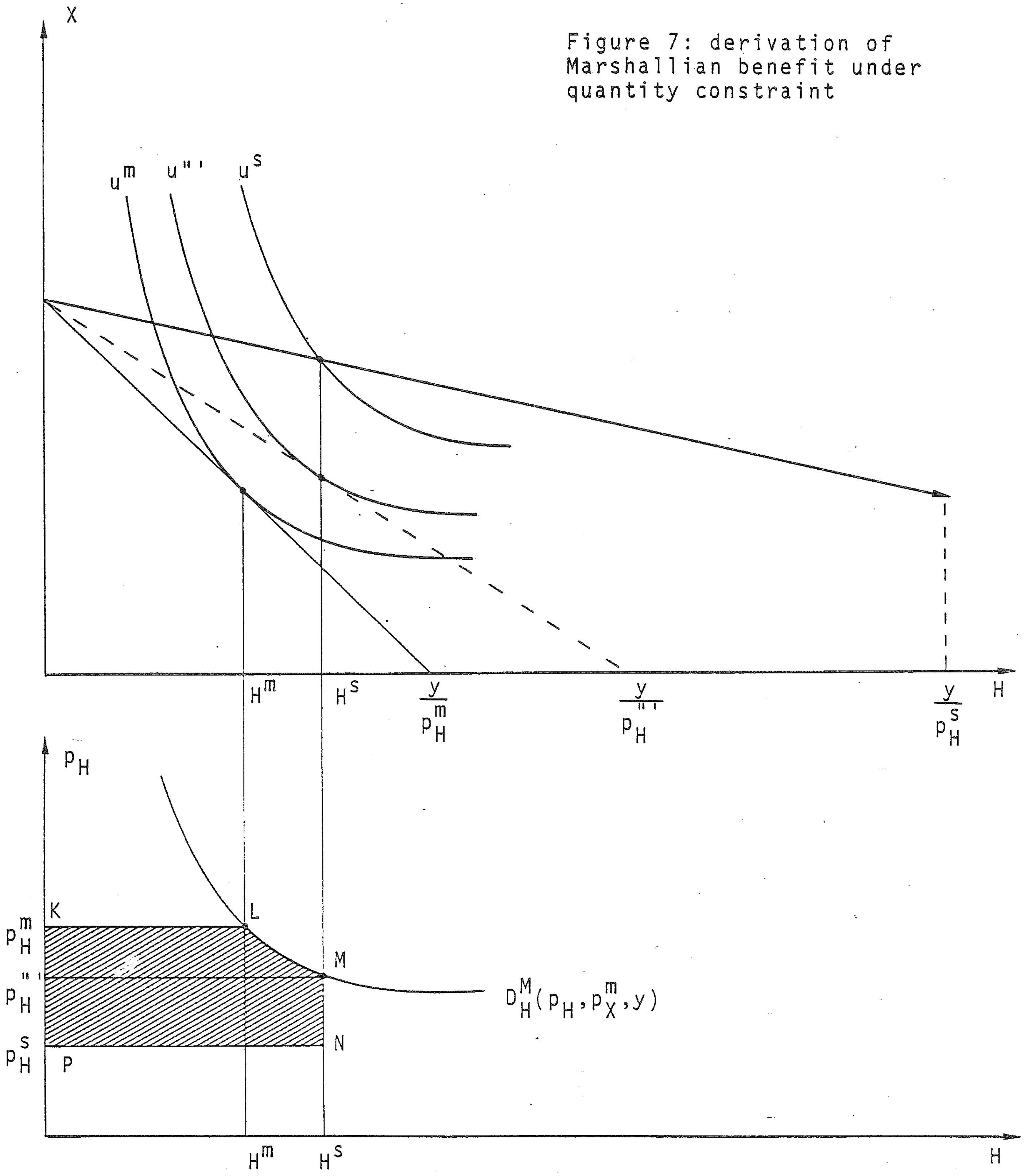
Construction of p_H'' and Marshallian consumer's surplus are illustrated on figure 7. It equals the area KLMNP and can be written as

$$MB = - \int_{p_H^m}^{p_H''} D_H^M(p_H, p_X^m, y)\,dp_H - (p_H^S - p_H'')H^S \qquad (13)$$

[1] In a recent study Olsen and York (1982) e.g. calculate the appropriate area under the Marshallian demand curve as

$$MB = \{\int_{H^m}^{H^S} p_H(H)dH\} + p_H H^m - p_H^S H^S$$, where $p_H(H)$ is the inverse Marshallian demand curve for good H. Graphical inspection shows the equivalence of this expression and (13), see figure 7 below. It is important to note that the integrals in expressions (8) and (12) for EV and CV can be transformed into expressions involving integration of inverse compensated demand functions over quantities as well. Formulas for these transformations can be derived using the results given by W. Pauwels (1977, p.52). This may have certain conceptual advantages because the shadow prices are endogeneous, unlike quantities. For the comparison of the different benefit measures, it is more appropriate to use the formulation involving integration over prices, see below.

Figure 7: derivation of Marshallian benefit under quantity constraint



3. COMPARING ALTERNATIVE MEASURES

In this section we investigate the relation between MB, CV and EV using the formulas previously derived. The relative magnitude of the three measures crucially depends on the level of the constraint $H=H^s$. The value H^s as compared to two well-defined quantities will turn out to be extremely important, viz. the quantity chosen in the absence of the program, H^m, and the optimal quantity the household would have bought at the subsidized price p_H^s in the absence of the constraint. The latter quantity will be denoted H^1.

In what follows different cases will be considered. Depending upon the amount H^s the relation between the prices

$$p_H^m,\ p_H^s,\ p_H^{'},\ p_H^{''} \text{ and } p_H^{'''}$$

will change, which in turn affects the relation between the alternative benefit measures.

CASE 1: $H^s<H^m<H^1$

Suppose the government program offers a restricted quantity H^s, which is less than the amount the household would have consumed in the absence of the program. The inequality $H^s<H^m$ implies $p_H^{'''}>p_H^m$ by definition of $p_H^{'''}$. This in turn implies $u^{'''}<u^m$. Since for a participating household $u^s>u^m$ it follows that

$$u^{'''}<u^m<u^s \qquad (14)$$

Note, however, that by definition of the shadow prices the following series of equalities hold:

$$H^s = D_H^c(p_H', p_X^m, u^s) = D_H^c(p_H'', p_X^m, u^m) = D_H^c(p_H''', p_X^m, u''') \qquad (15)$$

As long as H is a normal good, (14) and (15) imply $p_H''' < p_H'' < p_H'$ because compensated demand functions are non-increasing in own price[1]. Consequently, the complete relation between the different prices is:

$$p_H^s < p_H^m < p_H''' < p_H'' < p_H'$$

The situation is depicted on figure 8. If we rewrite the equations (8), (12) and (13) as:

$$EV = -\int_{p_H^m}^{p_H'} D_H^c(p_H, p_X^m, u^s)\,dp_H + (p_H' - p_H^m)H^s + (p_H^m - p_H^s)H^s$$

$$CV = -\int_{p_H^m}^{p_H''} D_H^c(p_H, p_X^m, u^m)\,dp_H + (p_H'' - p_H^m)H^s + (p_H^m - p_H^s)H^s$$

$$MB = -\int_{p_H^m}^{p_H'''} D_H^M(p_H, p_X^m, y)\,dp_H + (p_H''' - p_H^m)H^s + (p_H^m - p_H^s)H^s,$$

[1] Alternatively, the relation between the shadow prices can be understood as follows. Remember that p_H'', p_H' and p_H''' were graphically constructed by drawing tangent lines at indifference curves corresponding to u^m, u^s and u''' at the constrained quantity H^s, respectively. As long as H is a normal good the slope of indifference curves at a fixed quantity increases with utility. Consequently relation (14) directly implies $p_H''' < p_H'' < p_H'$.

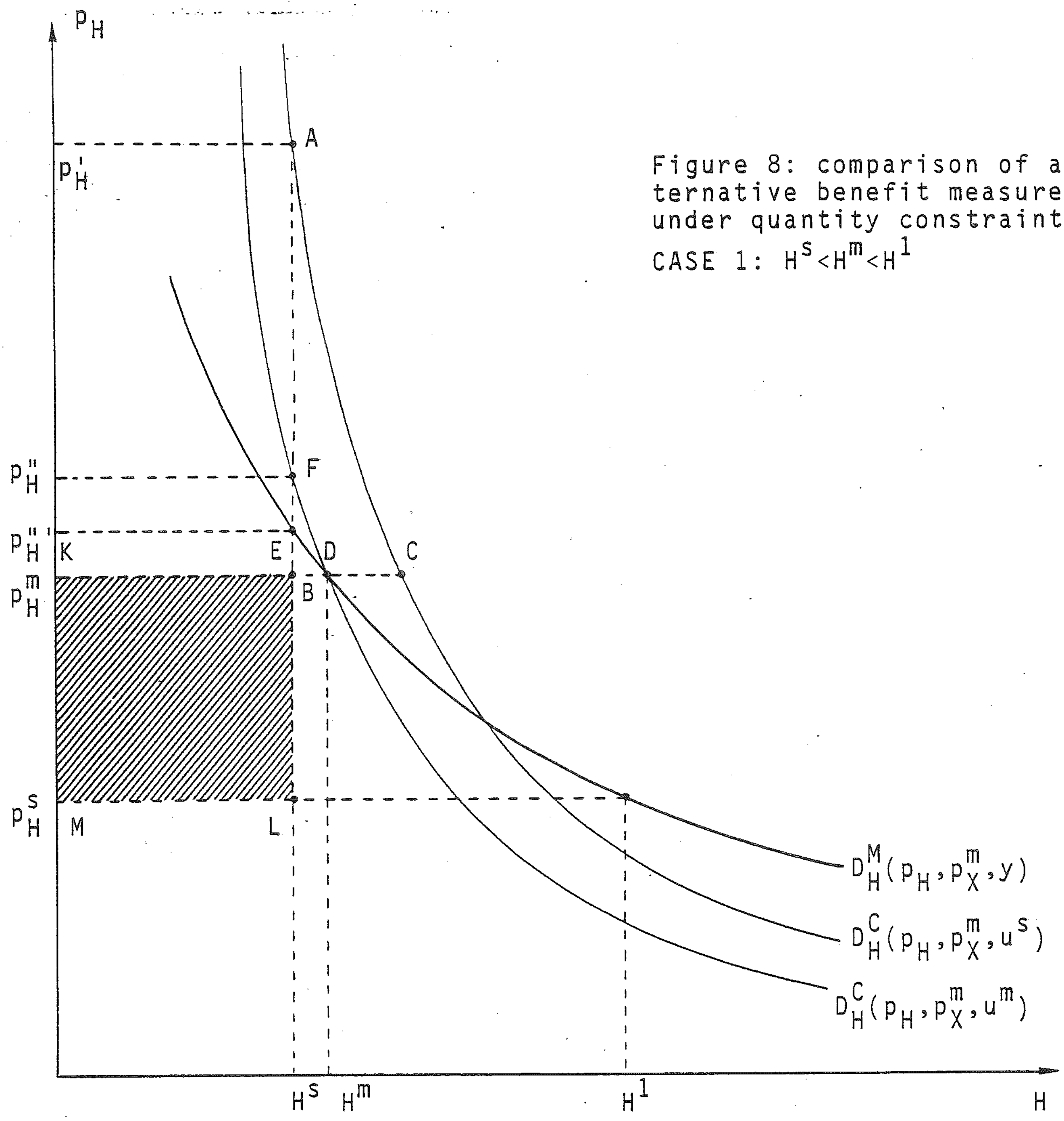


Figure 8: comparison of alternative benefit measures under quantity constraint

CASE 1: $H^s < H^m < H^1$

and apply these definitions we finally obtain:

$$CV = \text{area (KBLM)} - \text{area (FBD)}$$
$$EV = \text{area (KBLM)} - \text{area (ABC)}$$
$$MB = \text{area (KBLM)} - \text{area (EDB)}$$

where area (KBLM) corresponds to $(p_H^m - p_H^s)H^s$. Careful comparison leads to the conclusion

$$EV < CV < MB$$ [1]

CASE 2: $H^m<H^s<H^1$

This case is probably more realistic for many government programs. The quantity H^s offered under the program is between the unconstrained optima at prices p_H^m and p_H^s. It immediately follows that $p_H^m > p_H''' > p_H^s$. Since $H^s > H^m$ we also have $u^m < u'''$ by analogous reasoning to case 1. Moreover, using $p_H''' > p_H^s$ it also follows that $u''' < u^s$. Consequently $u^m < u''' < u^s$, which implies, using (15) and the properties of compensated demand functions:

$$p_H'' < p_H''' < p_H'$$

At this point, a further distinction is necessary which will turn out to have important consequences. First assume $p_H' > p_H^m$. As previously discussed the meaning of this assumption is that the program offers a quantity H^s which is less than the amount the household would have bought if it were given a cash grant with the same market value as the program. In this case the complete

[1] Note that MB=CV when $H^s=H^m$. In that case $p_H''=p_H^m=p_H'''$ which implies both MB and CV equal $(p_H^m-p_H^s)H^s$, see formulas (12) and (13).

relation between the prices for H is:

$$p_H^s \gtrless p_H'' < p_H''' < p_H^m < p_H'$$

Whether $p_H^s > p_H''$ or $p_H^s < p_H''$ does not affect the relation between the benefit measures.

A graphical illustration of this case is given in figure 9a. Applying the formulas (8), (12) and (13) we derive the following results:

CV = area (DKPGF)
EV = area (DLGF) - area (BCL)
MB = area (DKRGF)

Comparing these areas we can only conclude MB>CV. However, the relation of the equivalent variation with respect to the other welfare measures is indeterminate.

Next consider the case where $p_H^m > p_H'$. We then have the series of inequalities:

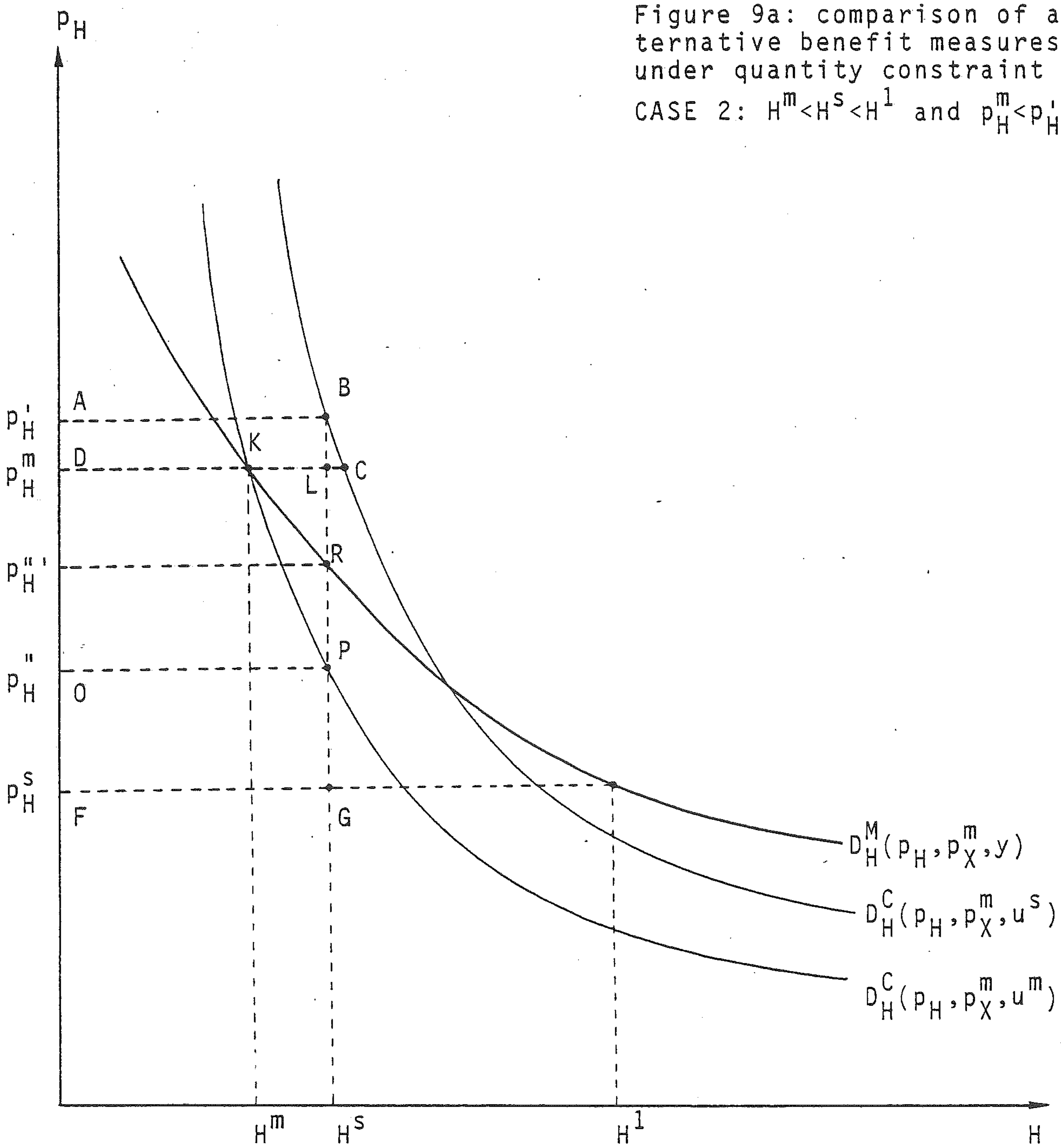
$$p_H^s \gtrless p_H'' < p_H''' < p_H' < p_H^m$$

Application of (8), (12) and (13) in figure 9b yields:

CV = area (ABFGK)
EV = area (ACDGK)
MB = area (ABEGK)

Consequently, we find that for this case EV>MB>CV. This is the classical relation between alternative benefit measures in the case of a pure price subsidy.

Figure 9a: comparison of alternative benefit measures under quantity constraint
CASE 2: $H^m < H^s < H^1$ and $p_H^m < p_H'$



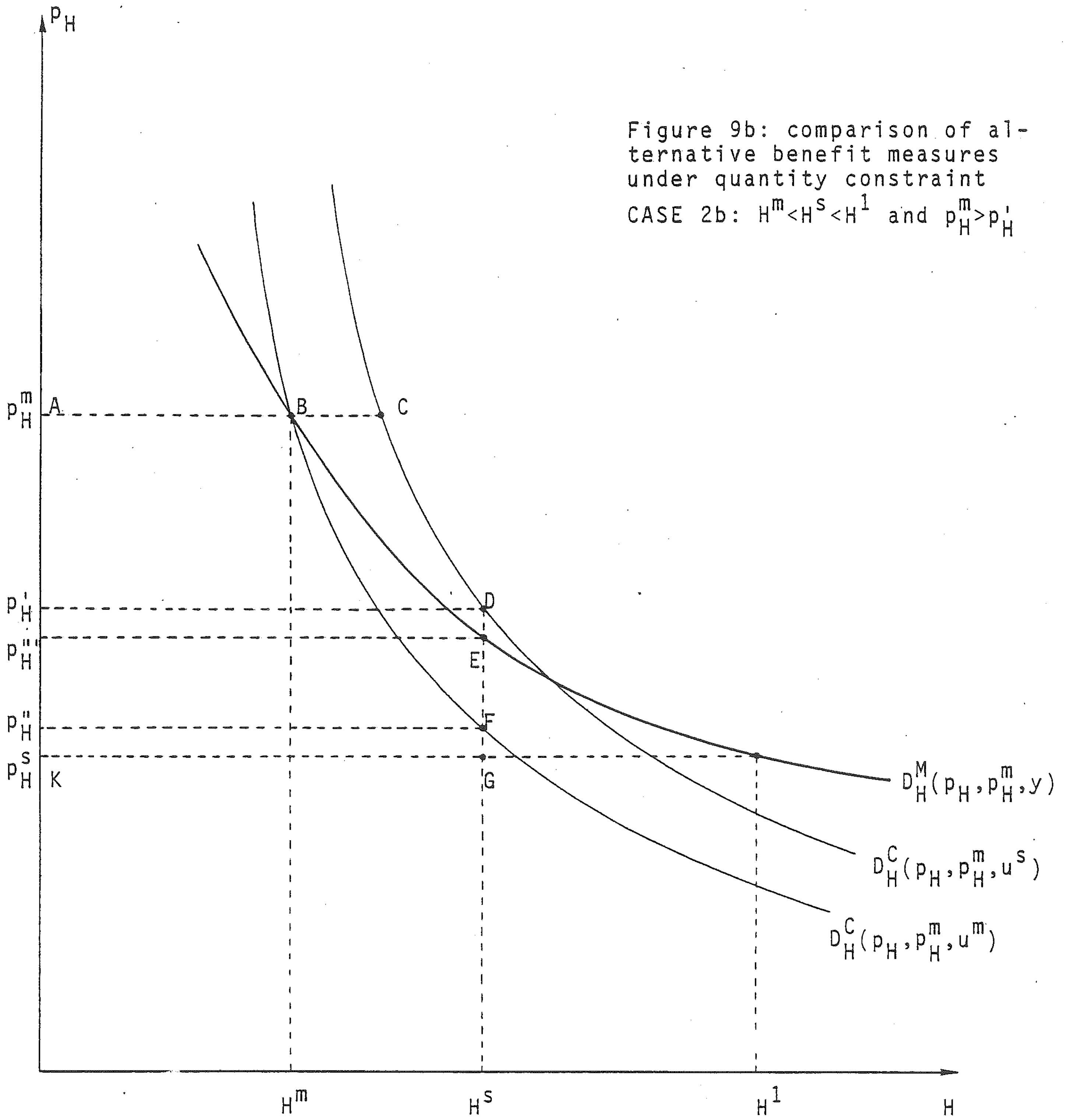


Figure 9b: comparison of alternative benefit measures under quantity constraint
CASE 2b: $H^m < H^s < H^1$ and $p_H^m > p_H'$

CASE 3: $H^m<H^1<H^s$

It is theoretically possible that the government program offers a quantity H^s which exceeds the amount the household would have bought if the market price were p_H^s. The reasoning applied in cases 1 and 2 may be used to show that $u^m<u^s<u''{}'$ which implies $p_H''<p_H'<p_H''{}'$. Moreover, $H^s>H^1$ also means $p_H''{}'<p_H^s$ so that the relation between the prices is:

$$p_H'' < p_H' < p_H''{}' < p_H^s < p_H^m$$

A graphical illustration is in figure 10. Applying the formulas, we find:

CV = area (ABFG) - area (MNFG) = area (ABZM) - area (ZNF)
EV = area (ACEK) - area (MNEK) = area (ACRM) - area (RNE)
MB = area (ABDL) - area (MNDL) = area (ABQM) - area (QND)

By careful comparison we can conclude that MB>CV and EV>CV. However, the relation between MB and EV is not a priori determined.

AN ILLUSTRATIVE EXAMPLE

The results thus far may be summarized as follows. First, only in two cases have we been able to derive a unique relation between CV, EV and MB. If the quantity H^s provided under the program is less than the optimal quantity in the absence of the program then we showed MB>CV>EV. If, on the other hand, $H^m<H^s<H^1$ and $p_H^m>p_H'$, we concluded that the usual relation EV>MB>CV holds, just as for pure price subsidies. In the two remaining cases we

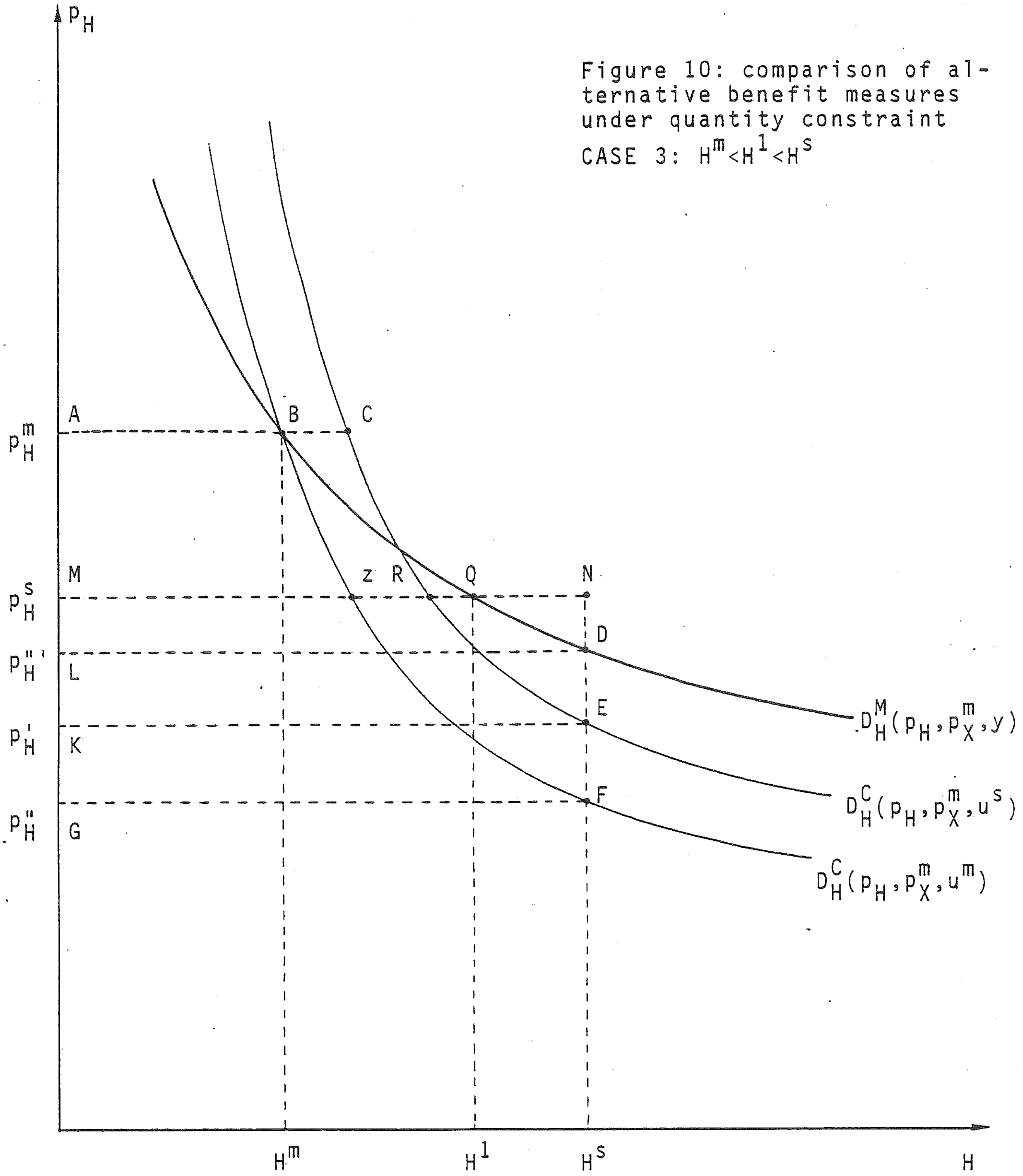


Figure 10: comparison of alternative benefit measures under quantity constraint
CASE 3: $H^m < H^1 < H^s$
p_H
p_H^m
p_H^s
$p_H^{\prime\prime\prime}$
$p_H^{\prime}$
$p_H^{\prime\prime}$
A
B
C
M
z
R
Q
N
D
L
E
K
F
G
$D_H^M(p_H, p_X^m, y)$
$D_H^C(p_H, p_X^m, u^s)$
$D_H^C(p_H, p_X^m, u^m)$
H^m
H^1
H^s
H

did find MB>CV - which implies that it will always be the case that Marshallian benefits are at least as great as the compensating variation - but the relative magnitude of EV could not be determined a priori[1].

We conclude this section with an illustrative example that will help to clarify the main results. Suppose the utility function is Cobb-Douglas

$$u = H^{\gamma}X^{1-\gamma}$$

It is not difficult to show that for this simple indifference map the benefit measures are as follows[2]:

$$EV = \left(\frac{p_H^m H^s}{\gamma}\right)^{\gamma}\left(\frac{p_X^m X^s}{1-\gamma}\right)^{1-\gamma} - y \qquad (16)$$

$$CV = y - \left(\frac{H^m}{H^s}\right)^{\frac{\gamma}{1-\gamma}} X^m - \frac{p_H^s H^s}{p_X^m} \qquad (17)$$

$$MB = p_H^m H^m - p_H^s H^s + \gamma y\{\ell n(\frac{H^s}{H^m})\} \qquad (18)$$

[1] Two points should be made. First, when the income elasticity of demand for H is zero then the 3 benefit measures are equal. In that case $p_H' = p_H'' = p_H'''$ and, in addition, Hicksian and Marshallian demand curves coincide. This implies MB=CV=EV, see (8),(12),(13). Second, we did not try to characterize the cut-off points where CV=EV and MB=EV. These depend in a complex way on the shadow prices, which are themselves functions of the parameters of the utility function, the price p_X^m and the level of the quantity constraint. Although the cut-off values for H^s could be determined by solving an implicit equation, it is very unlikely that they have an intuitive economic interpretation.

[2] These were calculated directly using the appropriate definitions. It is of course possible to derive (16) (17) and (18) using formulas (8) (12) and (13). It is fairly easy to show that for the Cobb-Douglas utility function the shadow prices are given by:

$$p_H' = (\gamma/(1-\gamma))\ ((y-p_H^s H^s)/H^s) = (\gamma/(1-\gamma))\ (p_X^m X^s/H^s)$$

$$p_H'' = (\gamma/(1-\gamma))\ (H^s)^{1/(\gamma-1)}\ (H^m)^{\gamma/(1-\gamma)}\ p_X^m X^m$$

$$p_H''' = \gamma y/H^s$$

Simple algebra shows that both EV and MB reach their maximum for $H^s = \frac{\gamma y}{p_H^s}$. This is precisely the amount demanded corresponding to price p_H^s. This quantity was previously denoted by H^1.

In figure 11, we have graphically represented the relation between the three benefit measures as a function of the constrained quantity H^s for given values of

$$p_H^m,\ p_X^m,\ p_H^s,\ H^m,\ X^m,\ y \text{ and } \gamma$$

The graph clearly illustrates the main findings of this paper[1]. Marshallian benefits are always larger than the compensating variation, except at the initial quantity H^m, where both benefit measures are equal. If the constrained quantity is less than H^m, we observe that MB>CV>EV (case 1). Moreover, we see that EV>MB>CV when H^s is between $\overline{H}$ and H^1, where $\overline{H}$ is the amount for which $p_H^m = p_H'$. This corresponds to the case where $H^m < H^s < H^1$ and $p_H^m > p_H'$, i.e. the case where the household is constrained to buy more H than it would have bought if it were given an unrestricted cash grant with the same market value as the program[2].

Of the three benefit measures discussed in this paper, only the equivalent variation and Marshallian consumer's surplus have been used in applied work. Our results suggest that EV may substantially underestimate MB for in-kind transfers. If price programs restrict the subsidized good at less than the initial quantity in the absence of the program or at a level that largely exceeds the quantity demanded at the subsidized price, then benefits are relatively small and MB overestimates, rather than

[1] Following parameter values were used: $p_H^m = p_X^m = 1$, $p_H^s = 0.5$, $H^m = 200$, $X^m = 800$, $y = 1{,}000$ $\gamma = 0.2$.

[2] In this example, $\overline{H} = 222.22$.

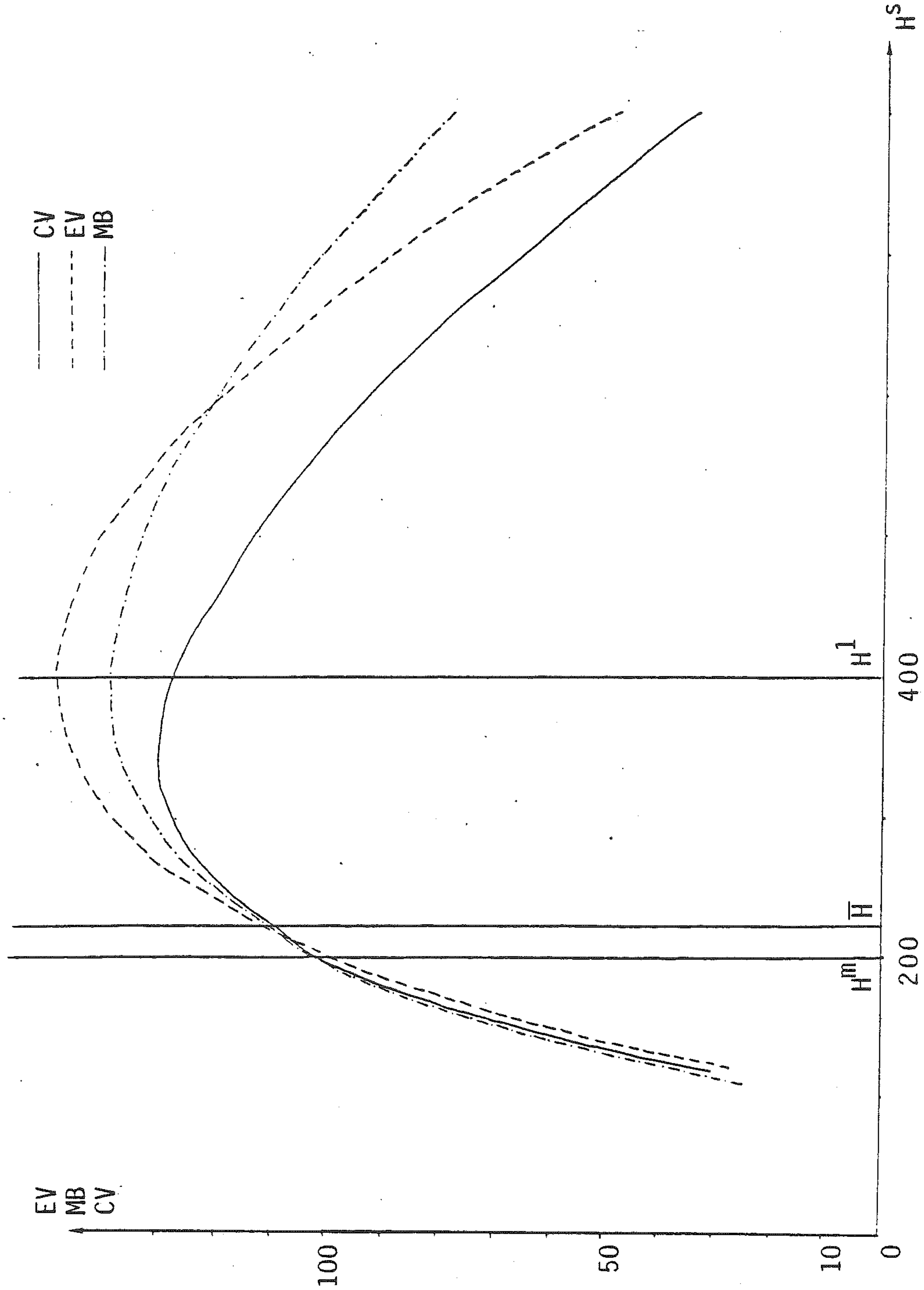


Figure 11

underestimates, the exact measure EV. This finding reinforces the arguments against the use of Marshallian benefit for the evaluation of government programs involving in-kind subsidies. It should be added to several undesirable properties of this measure that exist in the case of pure price subsidies.

4. SUMMARY AND CONCLUSION

In this paper we have investigated the relation between Hick's compensating and equivalent variations and Marshallian consumer's surplus in the case of an in-kind subsidy. By appropriately defining a set of shadow prices for the constrained good, we derived formulas for the three alternative benefit measures and analyzed their relative magnitude as a function of the amount of the constrained commodity.

We showed that Marshallian benefits will always exceed the compensating variation unless the subsidized good is restricted at the optimal amount consumed in the absence of the program, in which case both measures are equal. If the restricted amount is less than the quantity consumed without the program then we showed that both Marshallian benefit and the compensating variation exceed the equivalent variation. We also indicated a set of sufficient conditions under which the relation between the three measures is the same as for a pure price subsidy, viz. $EV \geq MB \geq CV$. Two conditions are required: first, the household should consume more of the subsidized good under the program than when given a cash grant with the same market value as the government subsidy. Moreover, the quantity offered under the program should be less than the amount that would be chosen if the observed market price equaled the subsidized price. These conditions are sufficient, though not necessary, for the previous relation to hold.

In practice, the use of Marshallian benefits may substantially overestimate or underestimate the equivalent variation in the case of a quantity-constrained price subsidy, depending upon the level of the constraint. This strongly suggests the use of the Hicksian measure for the evaluation of government programs imposing constraints on the subsidized good.

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