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Normalized Excess Measures in Characteristic Function Games

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Abstract

In this paper we develop a general framework which allows us to treat most classical solution concepts of characteristic function games as special cases.

As an application we also treat the well known "airport cost game" in this general setting.

1. Introduction

During the last years several solution concepts of characteristic function games have been proposed which extend the classical solution concepts of the core and the nucleolus. Examples are the concepts of the least core, the weak least core and the weak nucleolus, the proportional least core and the proportional nucleolus, the disruption nucleolus, etc. For a survey of most of these concepts, see. H.P. Young at al. (1980).

The purpose of the present paper is to develop a general framework which allows us to treat all these solution concepts as special cases.

This general framework is developed in section 2. It is based on the notion of the excess of a coalition in a given imputation, relative to some normalizing function f. In section 3 we discuss some important special cases by using specific forms for the normalizing function f.
Finally, in section 4 we discuss a case where the characteristic function has a special structure (the "airport cost game"). This special structure will allow us to derive several solution concepts explicitly.

2. The Normalized Case and the Normalized Nucleolus of a Game

Consider a characteristic function game $(\mathcal{N}, v)$, consisting of a set of N players, $\mathcal{N} = \{1,2,\ldots,N\}$, and a characteristic function v which associates with each coalition $S \subset \mathcal{N}$ a real number $v(S)$, with $v(\emptyset) = o$.
Let X represent the set of all imputations, i.e.,

$$X = \{x = (x_1,\ldots, x_N) \in R^N \mid x_i \geq v(\{i\}),\ i:1,\ldots, N \text{ and } x(\mathcal{N}) = v(\mathcal{N})\}$$

The notation $x(S)$ is used to denote $\sum_{i \in S} x_i$.

For any imputation $x \in X$ and for any coalition $S \subset \mathcal{N}$, one can define the (absolute) excess of S in x as the difference $x(S) - v(S)$. It represents the excess of what coalition S as a whole receives, $x(S)$, over its worth, $v(S)$.
In order to define the normalized excess of S in x, we first define a normalizing function $f : 2^{\mathcal{N}} \to R$. The only restrictions we impose on such a function are : first, that $f(S) \geq o$ for all $S \subset \mathcal{N}$, and, secondly, that the set $\mathcal{T} = \{S \subset \mathcal{N} \mid f(S) > o\}$ be nonempty. The normalized excess of S in x is then defined as

$$d(x,S) = \frac{x(S) - v(S)}{f(S)} \qquad (2.1)$$

where $x \in X$ and $S \in \mathcal{T}$. For example, one could define f by $f(S) = |S|$. In this case, $d(x,S)$ represents the excess of S in x per number of S.
Another possibility is to put $f(S) = v(S)$, so that $d(x,S)$ represents the excess of S in x, measured as a percentage of $v(S)$. Other reasonable specifications of f will be discussed in section 3.

For any $x \in X$, we can define a function D as

$$D(x) = \underset{S \in \mathcal{T}}{\text{Min}}\ d(x,S) \qquad (2.2)$$

The value $D(x)$ then gives, for any imputation x, the minimal normalized excess over all $S \in \mathcal{T}$. As the function d is linear in x, it is clear that D is continuous and concave in X.
Using this function D, we now consider two maximization problems which will generate, for any given specification of f, various solution concepts of the game $(\mathcal{N}, v)$.

The first problem is

$$\underline{\text{Problem (p)}} : \underset{x}{\text{Max}}\ D(x) \quad \text{s.t. } x \in X$$

When solving problem (p), we are looking for the imputation(s) which maximize the minimal normalized excess. It is clear that the motivation of problem (p) is similar to the motivation of the nucleolus of a game. See. D. Schmeidler (1969), and M. Maschler et al. (1979). As X is a compact set and D is continuous in X, (p) always has a solution. Let $\overset{\circ}{X}$ denote the set of solutions of (p), and let t° denote the maximal value of D.

Our first proposition provides an easy way to determine an upper bound for t°, or even the value of t° itself.

Proposition 2.1.

(i) For any $S \in \mathcal{T}$ such that also $\mathcal{N}\setminus S \in \mathcal{T}$,

$$t^\circ \leq \frac{v(\mathcal{N}) - v(\mathcal{N}\setminus S) - v(S)}{f(S) + f(\mathcal{N}\setminus S)}$$

(ii) Suppose that

$$\forall\, x \in X, \sum_{S \in \mathcal{T}} x(S) = K, \text{ K is a constant} \quad (2.3)$$

then

$$t^\circ \leq \frac{K - \sum_{S\in\mathcal{T}} v(S)}{\sum_{S\in\mathcal{T}} f(S)}$$

(iii) If (2.3) holds and if for same imputation $y \in X$ and some real number t

$$d(y,S) = t\ ,\ \forall\, S \in \mathcal{T} \quad (2.4)$$

then $y \in X^\circ$, and

$$t = t^\circ = D(y) = \frac{K - \sum_{S\in\mathcal{T}} v(S)}{\sum_{S\in\mathcal{T}} f(S)}$$

Proof :

(i) Let $x^\circ \in X^\circ$ so that $D(x^\circ) = t^\circ$. It follows that

$$t^\circ \leq d(x^\circ, S),\ \forall\, S \in \mathcal{T}$$

or, equivalently,

$$t^\circ f(S) \leq x^\circ(S) - v(S),\ \forall\, S \in \mathcal{T} \quad (2.5)$$

If now $S \in \mathcal{T}$ is such that also $\mathcal{N} \setminus S \in \mathcal{T}$, one also has

$$t^\circ f(\mathcal{N} \setminus S) \leq x^\circ(\mathcal{N} \setminus S) - v(\mathcal{N} \setminus S) = v(\mathcal{N}) - x^\circ(S) - v(\mathcal{N} \setminus S)$$

Adding the last two inequalilities gives result (i).

(ii) This follows from summing (2.5) over all $S \in \mathcal{T}$, and using (2.3).

(iii) First observe that, since $D(y) = t$ and $y \in X$, we must have $t^\circ \geq t$.

Suppose that $t^\circ > t$. Then for some imputation $x^\circ \in X^\circ$ we have $t = d(y,S) < d(x^\circ,S)$, $\forall S \in \mathcal{T}$

or, equivalently,

$$y(S) < x^\circ(S), \forall S \in \mathcal{T}$$

Summing this inequality over all $S \in \mathcal{T}$ gives $K < K$, a contradiction.

Hence, $t = t^\circ$, and

$$t^\circ f(S) = y(S) - v(S), \forall S \in \mathcal{T}.$$

Summing this equality over all $S \in \mathcal{T}$, and using (2.3), concludes the proof of (iii).

Part (i) of the proposition allows us to determine upper bounds for t° quite easily. If one defines the complement, $\bar{v}$, of a game v by

$$\bar{v}(S) = v(\mathcal{N}) - v(\mathcal{N} \setminus S), \forall S \subset \mathcal{N} \qquad (2.6.)$$

(see A. Charnes et al. - 1978 -), the upper bounds given by (i) can also be written as

$$t^\circ \leq \frac{\bar{v}(S) - v(S)}{f(S) + f(\mathcal{N} \setminus S)} \qquad (2.7)$$

It is clear that, if v is superadditive, $\bar{v}(S) \geq v(S)$ for all $S \subset \mathcal{N}$. If there is one coalition S for which $\bar{v}(S) = v(S)$, then $t^\circ \leq 0$.

Part (ii) of proposition 2.1. also allows us to determine an upper bound for t° if condition (2.3) is satisfied. As will be seen in section 3 condition (2.3) will in fact be satisfied for many reasonable specifications of f.

Finally, if again condition (2.3) holds, and if we can find an imputation which equalizes the normalized excesses over all $S \in \mathcal{T}$, then by part (iii) of proposition 2.1, y must solve (p). Equalizing these normalized excesses involves the solution of the system of linear equations (2.4). Of course, a solution of such a system need not always exist.

Example 2.2

Suppose v is given in normalized o-1 form, and assume $f(\{i\}) = a_i > o$, $i:1, \ldots, N$, and $f(S) = o$ for all other coalitions. Then $\mathcal{T} = (\{1\}, \{2\}, \ldots, \{N\})$, and $D(x) = \text{Min}\ \{\frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_N}{a_N}\}$. In this case proposition 2.1.(iii) applies with $K = 1$. This results in the optimal values,

$$t^\circ = \frac{1}{a_1 + \ldots + a_N}$$

$$x^\circ = t^\circ(a_1, \ldots, a_N)$$

Hence, x_i° is proportional to $f(\{i\})$, the normalizing factor of player i.

Example 2.3

If $f(S) > o$ for all $S \subset \mathcal{N}$ with $|S| = k$, and $f(S) = o$ for all other coalitions, then $\mathcal{T} = \{S \subset \mathcal{N} \mid |S| = k\}$ and
$K = \sum_{S \in \mathcal{T}} x(S) = \binom{N-1}{k-1} v(\mathcal{N})$.
In this case proposition 2.1 (ii) gives

$$t^\circ \leq \frac{\binom{N-1}{k-1} v(\mathcal{N}) - \sum_{|S| = k} v(S)}{\sum_{|S| = k} f(S)} = u_k$$

Example 2.4

If $f(S) > o$ for all conditions S, proposition 2.1 (ii) and example 2.3 give

$$t^\circ \leq \text{Min}\ \{u_1, u_2, \ldots, u_{N-1}\}$$

One should remark that in problem (p) and in proposition 2.1. the coalitions $S \in \mathcal{T}^c$ (the complement of $\mathcal{T}$) are "left out of the game". This implies that the solution set $X°$ and the value $t°$ do not depend on the values $v(S)$ for $S \in \mathcal{T}^c$. To remove this restriction, we now consider the following maximization problem.

<u>Problem (P)</u> :
$$\max_x D(x)$$
$$\text{s.t. } x(s) \geq v(s), \forall S \in \mathcal{T}^c$$
$$x \in X$$

The difference with the previous problem (p) is that in (P) we not only require that $x \in X$, but also that, for all $S \in \mathcal{T}^c$, the absolute excess in x be non-negative. It is clear that problems (p) and (P) are the same if $\mathcal{T}^c = \emptyset$. In what follows we will assume that the feasible set of problem (P) is non-empty, and hence compact. In this case, (P) must have an optimal solution. Let X^* be the solution set of (P), and let t^* denote the maximal value of D. We will call X^* the <u>normalized core of v</u>.

The next result in the analogue of proposition 2.1. for (P).

<u>Proposition 2.5</u>

(i) For any $S \in \mathcal{T}$, $t^* \leq \dfrac{v(\mathcal{N}) - v(\mathcal{N} \setminus S) - v(S)}{f(S) + f(\mathcal{N} \setminus S)}$

(ii) If, for same imputation $y \in X$ and same real number t the following set of linear equations holds

$$d(y,S) = t, \forall S \in \mathcal{T}$$
$$y(S) = v(S), \forall S \in \mathcal{T}^c \qquad (2.8)$$

then $y \in X^*$, and

$$t = t^* = D(y) = \frac{v(N)2^{N-1} - \sum_{S \subset \mathcal{N}} v(S)}{\sum_{S \in \mathcal{T}} f(S)} \qquad (2.9)$$

$$y_i = \frac{\sum_{S \ni i}(v(S) - v(S \setminus \{i\})) + t^* \sum_{S \ni i}(f(S) - f(S \setminus \{i\}))}{2^{N-1}} \qquad (2.10)$$

Proof :

(i) As in the proof of proposition 2.1(i) we have

$$t^* f(S) \leq x^*(S) - v(S)$$

Whether $f(\mathcal{N}\setminus S)$ is zero or non-zero we also have

$$t^* f(\mathcal{N}\setminus S) \leq x^*(\mathcal{N}\setminus S) - v(\mathcal{N}\setminus S)$$
$$= v(\mathcal{N}) - x^*(S) - v(\mathcal{N}\setminus S)$$

Combining these inequalities gives result (i).

(ii) First note that if an imputation $y \in X$ and a real number t solves (2.8), then y is feasible in (P). Hence, $t = D(y) \leq t^*$. Suppose now $t < t^*$. Then for some $x^* \in X^*$ we must have $t = d(y,S) < d(x^*, S)$, $\forall\ S \in T$

or, equivalently,

$$y(S) < x^*(S),\ \forall\ S \in T$$

For all $S \in T^c$ we have

$$y(S) = v(S) \leq x^*(S)$$

Hence,

$$\sum_{S \subset \mathcal{N}} y(S) < \sum_{S \subset \mathcal{N}} x^*(S)$$

which cannot hold. Therefore $t = t^*$.

The value of t^*, given by (2.9), easily follows from (2.8) after summation. To prove (2.10), note that

$$y_i = y(S) - y(S\setminus\{i\})$$

for all coalitions $S \ni i$. As also

$$y(S) = v(S) + t^* f(S),\ \forall\ S \subset \mathcal{N}$$

it follows that

$$y_i = (v(S) - v(S\setminus\{i\})) + t^*(f(S) - f(S\setminus\{i\}))$$

Summing this expression over all $S \ni i$ gives result (2.10).

It is well known that the core of a game $(\mathcal{N}, v)$, denoted by $C(v)$, can be defined as the set of imputations for which the absolute excesses, $x(S) - v(S)$, are non-negative for all $S \subset \mathcal{N}$. Considering then the definition of D and the constraints of problem (P), one can expect a close connection between the sets C(v) and X^*. This leads to the following proposition, which also gives us a lower bound for t^* in case $C(v) \neq \emptyset$.

Proposition 2.6

$C(v) \neq \emptyset$ if and only if $t^* \geq o$. In this case $X^* \subset C(v)$. If the strict core of the game v is non empty, then $t^* > o$.

Proof :

(a) Suppose $C(v) \neq \emptyset$, and let $x \in C(v)$. Then $x(S) \geq v(S)$ for all $S \subset \mathcal{N}$, and hence $d(x,S) \geq o$ for all $S \in T$. But then $t^* \geq o$.

(b) Conversely, if $x^* \in X^*$ and $D(x^*) = t^* \geq o$, then

$$d(x^*, S) = \frac{x^*(S) - v(S)}{f(S)} \geq t^* \geq o, \ \forall S \in T$$

Since also $x^*(S) \geq v(S)$ for $S \in T^c$, it follows that $x^* \in C(v)$.

(c) If the strict core of the game is non empty, there exists an imputation $x \in X$ such that for all $S \subset \mathcal{N}$, $S \neq \mathcal{N}$, $x(S) > v(S)$. But then $d(x,S) > o$ for all $S \in T$. Hence, $t^* > o$.

It may, of course, happen that the normalized core X^* of v contains more than one imputation. In this case one may want to select a single "best" imputation out of X^*. In order to indicate a procedure to obtain a single best solution, we first reformulate (P) as a linear programming problem. Consider then

Problem (LP) :
$$\max_{x,t} \ t$$
$$\text{s.t.} \quad x(S) \geq v(S) + tf(S), \ S \subsetneq \mathcal{N}$$
$$x \in X$$

The following proposition proves the equivalence of problems (P) and (LP).

Proposition 2.7

x^* solves (P) with $D(x^*) = t^*$ if and only if (x^*, t^*) solves (LP).

Proof :

The proof is divided into several parts.

(a) Let x^* solve (P) with $D(x^*) = t^*$. Then

$$x^*(S) \geq v(S), \forall S \in T^c$$

$$d(x^*,S) = \frac{x^*(S) - v(S)}{f(S)} \geq t^*, \qquad \forall S \in T$$

and $\quad d(x^*, S^1) = t^*$ for some $S^1 \in T$

But then

$$x^*(S) \geq v(S) + t^*f(S) \text{ for all } S \underset{\neq}{\subset} \mathcal{N}.$$

It follows that (x^*, t^*) is feasible in (LP), so that, if $(\bar{x}, t_1)$ is a solution of (LP), we must have $t^* \leq t_1$.

(b) Suppose next that $(\bar{x}, t_1)$ solves (LP). Then

$$\bar{x}(S) \geq v(S), \forall S \in T^c$$
$$\bar{x}(S) \geq v(S) + t_1 f(S), \forall S \in T$$

and

$$\bar{x}(\bar{S}) = v(\bar{S}) + t_1 f(\bar{S}) \text{ for some } \bar{S} \in T.$$

Hence,

$$\min_{S \in T} d(\bar{x}, S) = D(\bar{x}) = t_1$$

so that

$$t_1 = D(\bar{x}) \leq D(x^*) = t^*.$$

(c) To conclude the proof, note that (a) and (b) imply that $t_1 = t^*$ and $t^* = D(\bar{x}) = D(x^*)$.

Suppose now that X^* contains several imputations. Then a second stage linear programming problem can be formulated as follows. Let t_1 be the optimal value of t in (LP), and let A_1 denote the collection of all coalitions for which the constraint in (LP) holds with equality for all imputations in X^*. Then form the second stage problem.

$$\begin{aligned} \underset{x,\ t}{\text{Max}} \quad & t \\ \text{s.t.}\quad & x(S) = v(S) + t_1 f(S),\ \forall\ S \in A_1 \\ & x(S) \geq v(S) + tf(S),\ \forall\ S \notin A_1,\ S \subsetneq \mathcal{N} \\ & x \in X \end{aligned} \qquad (2.11)$$

If the optimal solution is still not unique, form a third stage problem, etc. At each stage, at least one additional linear equality is imposed so that at most N - 1 linear programs will be required in order to obtain a unique solution. This unique solution will be called the normalized nucleolus of v.

Before considering some special cases, it should be noted that one can consider more general excess measures d(x,S) such as

$$d(x,S) = \frac{x(S) - g(S)\ v(S)}{f(S)}$$

or

$$d(x,S) = \frac{x(S) - g_1(S)v(S) - g_2(S)v(\mathcal{N}\setminus S)}{f(S)}$$

for suitable normalizing functions f, g, g_1 and g_2.

With d(x,S) as defined in (2.1), note that

$$\begin{aligned} d(x,\mathcal{N}\setminus S) &= \frac{x(\mathcal{N}\setminus S) - v(\mathcal{N}\setminus S)}{f(\mathcal{N}\setminus S)} \\ &= \frac{v(\mathcal{N}) - v(\mathcal{N}\setminus S) - x(S)}{f(\mathcal{N}\setminus S)} \\ &= \frac{\bar{v}(S) - x(S)}{f(\mathcal{N}\setminus S)} \end{aligned}$$

where $\bar{v}$ is the complement of v. If $f(S) = f(\mathcal{N}\setminus S)$, it follows that

$$d(x,S) + d(x,\mathcal{N}\setminus S) = \frac{\bar{v}(S) - v(S)}{f(S)}$$

Finally, one could also define normalized excess measures of the form

$$\lambda \, d_1(x,S) + \mu d_2(x,S)$$

or

$$\lambda \, d_1(x,S) + \mu d_2(x,\mathcal{N}\setminus S)$$

Where d_1 and d_2 are excess measures as before, and where λ and μ are real numbers. Special cases of this form have been shown useful in A. Charnes et al. (1978). Note that if d_1 has f as normalizing function while d_2 has g as normalizing function, then $\lambda d_1 + \mu d_2$ has $(fg) / (\lambda g + \mu f)$ as normalizing function.

3. Some Important Special Cases

By using specific forms for the normalizing function f, we can use the results of the previous section to derive several classical solution concepts.

3.1. The Least Core and Nucleolus

Define f by putting $f(S) = 1$ for all $S \subset \mathcal{N}$. In this case $d(x,S)$ measures the absolute excess of an x, and problems (p), (P) and (LP) are all equivalent. The normalized core X^* is then known as the least core. See L.S. Shapley and M. Shubik (1973). From (2.7) we also know that in this case

$$t^\circ = t^* \leq \frac{1}{2} (\bar{v}(S) - v(S)), \; \forall \, S \subsetneq \mathcal{N}$$

If the least core X^* contains more than one imputation, one can solve the sequence of (LP)-problems described in section 2 until a unique solution is obtained. This single imputation is the nucleolus of v. See D. Schmeidler (1969) and M. Maschler et al. (1979). The method of determining the nucleolus of a game by a sequence of (LP)-problems is well known. See A. Kopelowitz (1967).

3.2. The Generalized Least Core and Generalized Nucleolus

If the normalizing function f is specified as $f(S) = \sum_{i \in S} a_i$ where $a_i > o$, $i:1, \ldots, N$, we can call X^* the generalized least core, while the unique solution of the sequence of (LP)-problems can be called the generalized nucleolus.

3.3. The weak least Core and Weak Nucleolus

If in 3.2. one takes $a_i = 1$, $i:1, \ldots, N$ then $f(S) = |S|$ and $d(x,S)$ represents the excess of S in x per member of S. X^* is then called the weak least core, while the unique solution of the sequence of (LP)-problems is called the weak nucleolus. See H.P.Young et al. (1980), and J.H. Grotte (1970) and (1972).

3.4. The Proportional Least Core and Proportional Nucleolus

If we define f by $f(S) = v(S)$, $d(x,S)$, represents the proportional excess of S in x. The set X^* is then known as the proportional least core, while the unique solution of the sequence of (LP)-problems is called the proportional nucleolus. See H.P. Young et al. (1980), M. Maschler et al. (1979), and J.P. Heaney (1979).

3.5. The Disruption Core and Disruption Nucleolus

Let the normalizing function f be specified as

$$\begin{aligned} f(S) &= v(\mathcal{N}) - v(\mathcal{N} \setminus S) - v(S) \\ &= \bar{v}(S) - v(S), \ \forall \ S \subset \mathcal{N} \end{aligned} \qquad (3.1)$$

If v is superadditive and non constant sum, $f(S) \geq o$ for all S, and $f(S) > o$ for some S. If v has a non-empty strict core $f(S)$ is strictly positive for all S. Note that, with this definition of f,

$$f(S) = f(\mathcal{N} \setminus S) \text{ for all } S, \qquad (3.2)$$

and that

$$d(x,S) + d(x, \mathcal{N} \setminus S) = 1, \text{ for all } S$$

$f(S)$ can be interpreted as the total gain that can be realized if coalition S cooperates with coalition $\mathcal{N} \setminus S$. $d(x,S)$ then expresses the excess of S in x as a fraction of this total gain $f(S)$.

Alternatively, $d(x,S)$ can also be written as

$$d(x,S) = \frac{x(S) - v(S)}{v(\mathcal{N}) - v(\mathcal{N}\setminus S) - v(S)} = \frac{1}{1 + \hat{d}(x,S)}, \quad S \in T \tag{3.3}$$

where

$$\hat{d}(x,S) = \frac{x(\mathcal{N}\setminus S) - v(\mathcal{N}\setminus S)}{x(S) - v(S)}$$

$\hat{d}(x,S)$ is known as the propensity of coalition S to disrupt imputation x. See D. Gately (1974), and S.C. Littlechild and K.G. Vaidya (1976).

For any $x \in X$, we can then define a function $\hat{D}$ as

$$\hat{D}(x) = \underset{S \in T}{\text{Max}} \ \hat{d}(x,S) \tag{3.4}$$

$\hat{D}(x)$ then gives, for any imputation $x \in X$, the maximal propensity to disrupt over all $S \in T$. Comparing (2.2) and (3.4), and using (3.3) it follows that

$$\hat{D}(x) = \frac{1}{D(x)} - 1 \tag{3.5}$$

It is then clear that, in this case, the solution set X^* of problem(P) will be equal to the solution set of the problem

$$\begin{aligned} &\underset{x}{\text{Min}} \ \hat{D}(x) \\ &\quad \text{s.t. } x(S) \geq v(S), \ \forall S \in T^c \\ &\quad\quad x \in X \end{aligned} \tag{3.6}$$

Hence, an imputation $x^* \in X^*$ will minimize the maximal propensity to disrupt, subject to the constraints given in (3.6). This is why we call X^* the <u>disruption core</u> of v.

From proposition 2.5 (i), using (3.2), we can conclude that $t^* \leq 1/2$. From proposition 2.6 we also know that C(v) is non empty if and only if $t^* \geq o$. In this case, we therefore have $o \leq t^* \leq 1/2$.

With f as defined in (3.1) we can define the function $w_t(S)$ as

$$w_t(S) = t\bar{v}(S) + (1-t)v(S) = v(S) + tf(S)$$

so that problem (LP) can be written as

$$\begin{aligned} &\max_{x\,t} \quad t \\ &\quad \text{s.t. } x(S) \geq w_t(S) \ :\ \forall\, S \subsetneq \mathcal{N} \\ &\qquad x \in X. \end{aligned} \tag{3.7}$$

If then $o \leq t^* \leq 1$, as will certainly be the case when $C(v) \neq \emptyset$, the function $w_{t^*}(S)$ is known as a constant mollifier of v. See A. Charnes et al. (1978). Problem (3.7) can then be interpreted as follows : we want to find an imputation $x \in X$ such that x belongs to the core of the constant mollifier $w_t(S)$ for as high a value of t as possible.

If the disruption core of v is not a singleton, one can solve a sequence of (LP)-problems until a unique solution is obstained. This solution is known as the disruption nucleolus of v. See S.C. Littlechild and K.G. Vaidya (1976).

Finally, let f be defined as

$$f(S) = v(\mathcal{N}) - v(\mathcal{N} \setminus S) - v(S) \text{ for } |S| = k$$
$$o \text{ for } |S| \neq k.$$

Assuming that f is strictly positive for all S with $|S| = k$, then in this case attention is limited to the propensities to disrupt of coalitions of size k. Recalling then (3.5) and using proposition 2.1 (iii), we know that, if there exists an imputation which equalizes the propensity to disrupt over all coalitions of size k, then this imputation also minimizes, over X, the maximal propensity to disrupt over all coalitions of size k.

4. The Airport Cost Game

The airport cost game (S.C. Littlechild (1974), S.C. Littlechild and G.F. Thompson (1977), S.C. Littlechild and G. Owen (1976), S.C. Littlechild and K.G. Vaidya (1976)) exemplifies a class of games for which the characteristic function has a certain special structure. We will now give some simple expressions for the normalized nucleolus for this kind of game, for a large class of normalizing functions f.

In the airport cost game the set of players consists of movements (take-offs and landings) by different types of aircraft. The characteristic function is given by the negative of the airport runway construction cost function. This cost function has the following special form. Let there be n types of players (aircraft types), and let N_i denote the set of players of type i. Let c, denote the cost to serve a player of type i alone. Suppose the types of players are numbered such that

$$o < c_1 < c_2 < \dots < c_n$$

The cost of serving a set S of players is then defined by

$$c(S) = \max \{c_i \mid S \text{ contains a player of type } i\}$$

or :

$$C(S) = c_{i(S)}$$

where

$$i(S) = \max \{i \mid S \text{ contains a player of type } i\}.$$

The characteristic function of the game is then defined by

$$v(S) = - C(S)$$

Pay-off vectors x for this game will have negative components, representing the contribution of the various players for using the airport. As it is more convenient to work with positive numbers, we will only work with the cost function c(S) and a vector z of charges, where z = -x. Imputations are then characterized by

$$z_i \leq C(\{i\}),\ i \in \mathcal{N}$$
$$z(\mathcal{N}) = C(\mathcal{N})$$

The normalized excess of S in z is then given by

$$d(z, S) = \frac{C(S) - z(S)}{f(S)}$$

Problem (LP) is reformulated as

$$\begin{aligned} \max_{z,t}\quad & t \\ \text{s.t. } & z(S) + tf(S) \leq C(S),\ S \subsetneq \mathcal{N} \\ & z \in Z \end{aligned} \qquad (4.1)$$

where Z is the set of imputations.

In the next proposition we will assume that the normalizing function is positive for all $S \subset \mathcal{N}$, and that it is monotone, i.e.

$$\forall\ S_1, S_2 \subset \mathcal{N},\ S_1 \subset S_2 \rightarrow f(S_1) \leq f(S_2).$$

Proposition 4.1.

Let (z^*, t^*) be an optimal solution to problem (4.1). Then

$$t^* = \mathrm{Min}\{ \frac{c_i}{\sum_{j \in M_i} f(\mathcal{N}\backslash\{j\}) + f(M_i)},\ i=1, \ldots, n-1,\ \frac{cn}{\sum_{i \in \mathcal{N}} f(\mathcal{N}\backslash\{i\})} \qquad (4.2)$$

where $M_i = N_1 U \ldots U N_i$, i:1, ..., n

Furthermore, if the minimum in (4.2) is attained for the first time at the k_1-th element of the set, then

$$z_i^* = f(\mathcal{N}\backslash\{i\})t^*,\ \forall\ i \in M_{k_1} \qquad (4.3)$$

Proof :

The proof is divided into several parts. In part 1 we reduce (LP) to a simpler form. In part 2 we prove (4.2); (4.3) is proved in part 3.

Part 1

We first reduce the number of constraints in (LP) using special properties of this game.

It is clear that the values

$$z_i = o, \forall\ i \in \mathcal{N}_j, j \neq n$$

$$z_i = \frac{c_n}{|\mathcal{N}_n|}, i \in \mathcal{N}_n$$

$$t = o$$

are always feasible in (4.1). Hence, if t^* is the maximal value of t, we must have $t^* \geq o$. We can, therefore, assume that in (4.1) $t \geq o$.

(i) If $i(S) = k < n$, then $C(S) = c_k$, and the constraint corresponding to S becomes

$$z(S) + tf(S) \leq c_k$$

which, if f is monotone, is dominated by the constraint for coalition M_k :

$$z(M_k) + tf(M_k) \leq c_k, \ k : 1, \ldots, n-1 \qquad (4.4)$$

(ii) If $i(S) = n$, the constraint corresponding to S becomes $z(S) + tf(S) \leq c_n$

which, if f is monotone, is dominated by the constraint corresponding to coalition $\mathcal{N} \setminus \{i\}$ where $i \in M_k$ and $k = i(\mathcal{N} \setminus S)$:

$$z(\mathcal{N}\setminus\{i\}) + tf(\mathcal{N}\setminus\{i\}) \leq c_n$$

As $z(\mathcal{N}) = C(\mathcal{N}) = c_n$, we obtain

$$- z_i + t\ f(\mathcal{N}\setminus\{i\}) \leq o,\ i \in \mathcal{N} \tag{4.5}$$

We can therefore reformulate (LP) (4.1) as

$$\max_{z,t}\ t \qquad \text{s.t. } (4.4),\ (4.5),\ z \in Z \tag{4.6}$$

Part 2 : t^*, as defined in (4.2), is optimal.

Let p denote the R.H.S. of (4.2), and let t^* be optimal.

(i) $t^* \leq p$

Suppose, on the contrary, that $t^* > p$. If p attains its value for the first time at $i = k_1 < n$, then by (4.5) and $t^* > p$, we have

$$z(M_{k_1}) + t^* f(M_{k_1}) \geq (\sum_{i \in M_{k_1}} f(\mathcal{N}\setminus\{i\}) + f(M_{k_1}))t^* > c_{k_1}$$

which contradicts (4.4).

If p equals the last term of the R.H.S. of (4.2), then using (4.5) we arrive at

$$c_n = z(\mathcal{N}) \geq t^* \sum_{i \in \mathcal{N}} f(\mathcal{N}\setminus\{i\}) > p \sum_{i \in \mathcal{N}} f(\mathcal{N}\setminus\{i\}) = c_n$$

which is again a contradiction.

(ii) $p \leq t^*$

First suppose

$$p = \frac{c_n}{\sum_{i \in \mathcal{N}} f(\mathcal{N}\setminus\{i\})} \tag{4.7}$$

If we then set $z_i = f(\mathcal{N}\setminus\{i\})p$, $i \in \mathcal{N}$, (z,p) is feasible in (4.6), and hence $p \leq t^*$.

Next assume that p attains its value for the first time at

$i = k_1 < n$, i.e.,

$$p = \frac{c_{k_1}}{\sum_{i \in M_{k_1}} f(\mathcal{N}\setminus\{i\}) + f(M_{k_1})}, \quad k_1 < n \tag{4.8}$$

take then

$z_i = pf(\mathcal{N}\setminus\{i\})$, $i:1, \ldots, N-1$

$z_N = c_n - z(\mathcal{N}\setminus\{N\})$

Then again (z,p) is feasible in (4.6), so that $p \leq t^*$ also in this case.

Part 3 It remains to show that (4.3) holds.

First suppose (4.7) holds. Then

$$z^*_i = f(\mathcal{N}\setminus\{i\})p$$

should hold, since otherwise $z^*(\mathcal{N}) > c_n$.

Suppose next that (4.8) holds. If then for some $j \in M_{k_1}$, we would have

$$z_j > pf(\mathcal{N}\setminus\{j\})$$

this would imply that

$$z(M_{k_1}) > p \sum_{j \in M_{k_1}} f(\mathcal{N}\setminus\{j\})$$

or ,

$$z(M_{k_1}) + pf(M_{k_1}) > c_{k_1}$$

which contradicts (4.4).

Example 4.2

In this example we will consider various specifications of f as given in section 3. Let us use the notation $|\mathcal{N}_i| = m_i$, i:1, ..., n, so that

$$\sum_{i=1}^{n} m_i = N, \text{ and } |M_k| = \sum_{i=1}^{k} m_i .$$

(a) Nucleolus

If f(S) = 1 for all $S \subset \mathcal{N}$, we obtain

$$t^* = \text{Min}\{\frac{c_k}{\sum_{i=1}^{k} m_i + 1}, k : 1, \ldots, n-1, \frac{c_n}{N}\}$$

If this minimum is attained for the first time at $k = k_1 < n$, then

$$z^*_i = \frac{c_{k_1}}{\sum_{i=1}^{k} m_i + 1}, \forall i \in M_{k_1}$$

In the other case, the charges are the same for all players

$$z^*_i = \frac{c_n}{N}, \forall i \in \mathcal{N}.$$

See S.C. Littlechild (1974), and S.C. Littlechild and G. Owen (1976).

(b) Weak Nucleolus

If f(S) = |S| for all $S \subset \mathcal{N}$, we obtain

$$t^* = \text{Min}\{\frac{c_k}{N \sum_{i=1}^{k} m_i}, k : 1, \ldots, n-1, \frac{c_n}{N(N-1)}\}$$

If this minimum is attained for the first time at $k = k_1 < n$, then

$$z_i^* = \frac{(N-1)\, C_{k_1}}{N \sum_{i=1}^{k_1} m_i}, \ \forall\ i \in M_{k_1}$$

In the other case, $z_i^* = \frac{C_n}{N}$ for all $i \in \mathcal{N}$.

(c) Proportional Nucleolus

If $f(S) = v(S)$ for all $S \subset \mathcal{N}$, then

$$f(M_k) = c_k$$

$$f(\mathcal{N} \backslash \{j\}) = c_n \text{ for } j \in M_k,\ k : 1, \ldots, n-1.$$

For $j \in \mathcal{N}_n$, we have

$$f(\mathcal{N} \backslash \{j\}) = c_n \text{ if } m_n > 1$$

$$c_{n-1} \text{ if } m_n = 1$$

Hence,

$$t^* = \operatorname{Min}\left\{\frac{c_k}{c_n \sum_{i=1}^{k} m_i + c_k},\ k : 1, \ldots, n-1,\ \frac{c_n}{(N-1)c_n + c_n (\text{or } c_{n-1})}\right\}$$

If this minimum is attained for the first time at $k = k_1 < n$, then

$$z^* = \frac{c_n\, c_{k_1}}{c_n \sum_{i=1}^{k_1} m_i + c_{k_1}}, \ \forall\ i \in M_{k_1}$$

In the other case

$$z^*_i = \frac{c_n}{N}, \ \forall\ i \in \mathcal{N}$$

assuming that $m_n > 1$, (If $m_n = 1$, the minimum is attained for the first time at $k = k_1 < n$).

(d) Disruption Nucleolus

If f is defined as

$$f(s) = -C(\mathcal{N}) + C(S) + C(\mathcal{N}\setminus S),$$

then

$$f(M_k) = c_k \quad \text{for } k : 1, \ldots, n-1$$

and

$$f(\mathcal{N}\setminus\{j\}) = f(\{j\}) = c_k \text{ for } j \in \mathcal{N}_k, \quad k : 1, \ldots, n$$

In this case

$$t^* = \text{Min}\left\{\frac{c_k}{\sum_{j=1}^{k} m_j c_j + c_k},\ k:1, \ldots, n-1,\ \frac{c_n}{\sum_{j=1}^{n} m_j c_j}\right\}$$

If t^* is attained for the first time at $k = k_1$, then

$$z_i^* = t^* c_{k_1} \quad \text{for all } i \in M_{k_1}.$$

See S.C. Littlechild and K.G. Vaida (1976).

It may, of course, happen that the solution of (4.1), or, equivalently, of (4.6), is not unique. In the previous example, this can happen if $k_1 < n$. In order then to determine the normalized nucleoli completely, we have to solve a second stage linear programming problem of the type (2.11). In the case of the airport cost game, this problem can be simplified in a similar way as (4.1) was simplified to (4.6), and making use of (4.2) and (4.3). If we put $t^* = t_1$, this second stage problem is of the following structure

$$\max_{z,\, t_2} \; t_2$$

$$\text{s.t. } z(\mathcal{N}_{k_1+1} \cup \ldots \cup \mathcal{N}_k) + t_2 f(M_k) \leq c_k - t_1 \Sigma_{i \varepsilon M_{k_1}} f(\mathcal{N}\setminus\{i\}),\ k = k_1 + 1, \ldots, n-1$$

$$-z_i + t_2 \; f(\mathcal{N}\setminus\{i\}) \leq 0,\ i \; \varepsilon \; \mathcal{N}\setminus M_{k_1}$$

$$z \; \varepsilon \; Z.$$

This problem has exactly the same structure as (4.6), and can therefore be solved by the same method.

Example 4.3

Consider an airport cost game with n=3, m_1=4, m_2=4 and m_3=2, so that N = 10. Assume c_1 = 1, c_2 = 3 and c_3 = 10. After same calculations one obtains the following results. The Shapley value is also given for comparison.

	Nucleolus	Weak Nucleolus	Proportional Nucleolus	Disruption Nucleolus	Shapley
$z_1=z_2=z_3=z_4$	.2	.225	.244	.158	.1
$z_5=z_6=z_7=z_8$	.44	.4375	.467	.474	.433
$z_9=z_{10}$	3.72	3.675	3.578	3.737	3.934
$z(\mathcal{N})$	10	10	10	10	10

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