STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

COMPARING THE ALLEN AND THE DIRECT DEFINITION OF THE ELASTICITY OF SUBSTITUTION

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A B S T R A C T

When in a production process there are more than two inputs, the elasticity of substitution between any two outputs can be defined in two ways. One has the "Allen-definition", and the "direct" definition. The purpose of this note is to derive an exact relationship between the two definitions, and to show that knowledge of cost shares and of price elasticities of factor demands is sufficient to calculate both substitution elasticities.

When in a production process there are more than two inputs, the elasticity of substitution between any two inputs can be defined in two different ways. First, one has the "Allen-definition" which is based on price elasticity of factor demands. Secondly, one has the "direct" definition based on the production function. See, e.g., /1/, and the references given there. The purpose of this note is to derive an exact relationship between the two definitions, and to show that knowledge of cost shares and of price elasticities of factor demands is sufficient to calculate both substitution elasticities.

Let $x \in R^n_+$ be a vector of inputs where R^n_+ denotes the nonnegative orthant of R^n. Let us write the production function as $y=f(x)$, where $y \in R_+$ denotes output. We will assume that f is twice continuously differentiable and strongly quasi-concave in R^n_+. Consider then the problem

$$\begin{aligned} \min_x \quad & w'x \\ \text{s.t.} \quad & f(x) \geq y \\ & x \geq 0 \end{aligned} \qquad (1)$$

where $w \in R^n_+$ is the vector of factor prices. Let us denote the solution of this problem as $x^\circ = x(w,y)$, so that the cost function is given by $C(w,y) = w'x(w,y)$. Define then the elasticities

$$\eta_{ij} = \frac{w_j}{x_i} \frac{\partial x_i(w,y)}{\partial w_j}, \qquad i,j=1,\ldots,n$$

and the shares

$$S_j = \frac{w_j x_j(w,y)}{C(w,y)}, \qquad j=1,\ldots,n$$

The <u>Allen-definition</u> of the elasticity of substitution between factor i and factor j is then given by

$$\sigma^A_{ij} = \frac{\eta_{ij}}{S_j} \qquad (2)$$

On the other hand, the direct definition is given by

$$\sigma^D_{ij} = \left. \frac{d(\frac{x_j}{x_i})(\frac{f_i}{f_j})}{d(\frac{f_i}{f_j})(\frac{x_j}{x_i})} \right|_{\substack{dy = 0 \\ dx_k = 0,\ k \neq i,j}} \tag{3}$$

where $f_i = \frac{\partial f(x)}{\partial x_i}$. From the optimality condition of problem (1) we know that

$$\frac{f_i}{f_j} = \frac{w_i}{w_j}$$

so that (3) can also be written as

$$\sigma^D_{ij} \quad \left. \frac{\frac{dx_j}{x_j} - \frac{dx_i}{x_i}}{\frac{dw_i}{w_i} - \frac{dw_j}{w_j}} \right|_{\substack{dy = 0 \\ dx_k = 0,\ k \neq i,j}} \tag{4}$$

Note that, in both definitions, w_i/w_j is varied and y is kept constant. In the case of σ^A_{ij}, factor prices different from w_i and w_j are kept constant, while the quantities of all inputs are allowed to vary. In the case of σ^D_{ij}, quantities of inputs different from x_i and x_j are kept constant, while the prices of all inputs are allowed to vary.

We now want to express σ^D_{ij} in terms of factor price elasticities so that σ^A_{ij} and σ^D_{ij} can easily be compared. To fix ideas, we will calculate σ^D_{12}.

Let us partition the matrix of factor price elasticities as follows:

$$\left[\begin{array}{cc|ccc} \eta_{11} & \eta_{12} & \eta_{13} & \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \eta_{23} & \cdots & \eta_{2n} \\ \hline \eta_{31} & \eta_{32} & \eta_{33} & \cdots & \eta_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \eta_{n1} & \eta_{n2} & \eta_{n3} & \cdots & \eta_{nn} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

We first derive two preliminary results. First, the inverse of A_{22} exists. We have that

$$A_{22} = \begin{bmatrix} \eta_{33} & .. & \eta_{3n} \\ .. & \cdots & .. \\ \eta_{n3} & .. & \eta_{nn} \end{bmatrix} = \begin{bmatrix} \frac{1}{x_3} & 0 & .. & 0 \\ 0 & \frac{1}{x_4} & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & \frac{1}{x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_3}{\partial w_3} & .. & \frac{\partial x_3}{\partial w_n} \\ \cdots & .. & \cdots \\ \frac{\partial x_n}{\partial w_3} & .. & \frac{\partial x_n}{\partial w_n} \end{bmatrix} \begin{bmatrix} w_3 & 0 & .. & 0 \\ 0 & w_4 & .. & 0 \\ .. & .. & .. & .. \\ .. & .. & .. & w_n \end{bmatrix}$$

If then A_{22} would be singular, the system

$$\begin{bmatrix} \frac{\partial x_3}{\partial w_3} & \cdots & \frac{\partial x_3}{\partial w_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial w_3} & \cdots & \frac{\partial x_n}{\partial w_n} \end{bmatrix} z = 0$$

would have a nonzero solution $z \varepsilon R^{n-2}$. It would then also follow that

$$\begin{bmatrix} 0, & 0, & z' \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \cdots & \frac{\partial x_1}{\partial w_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial w_1} & \cdots & \frac{\partial x_n}{\partial w_n} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = 0$$

This is not possible as this quadratic form can only be zero for vectors proportional to w. This is well known from traditional microeconomic theory.

The second preliminary result states that

$$A_{11} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = E \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (5)$$

where E is defined as

$$E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = A_{12} \; A_{22}^{-1} \; A_{21}$$

As factor demands are homogeneous of degree zero in w, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

so that we also have

$$\begin{bmatrix} I_2 & -A_{12} \; A_{22}^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} A_{11} - A_{12} \; A_{22}^{-1} \; A_{21} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

(5) then easily follows.

Let us now return to the calculation of σ_{12}^{D}.

By taking the total differential of the factor demands, keeping dy=0, we obtain

$$\begin{bmatrix} \frac{dx_1}{x_1} \\ \\ \frac{dx_2}{x_2} \end{bmatrix} = A_{11} \begin{bmatrix} \frac{dw_1}{w_1} \\ \\ \frac{dw_2}{w_2} \end{bmatrix} + A_{12} \begin{bmatrix} \frac{dw_3}{w_3} \\ \\ \frac{dw_n}{w_n} \end{bmatrix} \qquad (6)$$

$$\begin{bmatrix} \frac{dx_3}{x_3} \\ \cdots \\ \frac{dx_n}{x_n} \end{bmatrix} = A_{21} \begin{bmatrix} \frac{dw_1}{w_1} \\ \\ \frac{dw_2}{w_2} \end{bmatrix} + A_{22} \begin{bmatrix} \frac{dw_3}{w_3} \\ \cdots \\ \frac{dw_n}{w_n} \end{bmatrix} \qquad (7)$$

When calculating σ^D_{12}, one requires $\frac{dx_3}{x_3}=\ldots=\frac{dx_n}{x_n}=0$. From (7) we then obtain

$$\begin{bmatrix} \frac{dw_3}{w_3} \\ \cdots \\ \frac{dw_n}{w_n} \end{bmatrix} = - A_{22}^{-1} A_{21} \begin{bmatrix} \frac{dw_1}{w_1} \\ \\ \frac{dw_2}{w_2} \end{bmatrix} \qquad (8)$$

Substituting (8) into (6) gives

$$\begin{bmatrix} \frac{dx_1}{x_1} \\ \\ \frac{dx_2}{x_2} \end{bmatrix} = \begin{bmatrix} A_{11} - E \end{bmatrix} \begin{bmatrix} \frac{dw_1}{w_1} \\ \\ \frac{dw_2}{w_2} \end{bmatrix}$$

so that

$$\frac{dx_2}{x_2} - \frac{dx_1}{x_1} = (-1,1)\begin{bmatrix}\frac{dx_1}{x_1}\\ \frac{dx_2}{x_2}\end{bmatrix} = (-1,1)\ \begin{bmatrix}A_{11}-E\end{bmatrix}\begin{bmatrix}\frac{dw_1}{w_1}\\ \frac{dw_2}{w_2}\end{bmatrix}$$

$$=(\eta_{21}-\eta_{11}-e_{21}+e_{11})\frac{dw_1}{w_1} + (\eta_{22}-\eta_{12}-e_{22}+e_{12})\frac{dw_2}{w_2} \qquad (9)$$

From (5) we know that

$$\eta_{11} + \eta_{12} = e_{11} + e_{12}$$

$$e_{21} + e_{22} = \eta_{21} + \eta_{22}$$

Adding the left hand sides and the right hand sides of these equations we obtain

$$\eta_{21} - e_{21} - \eta_{11} + e_{11} = -(\eta_{22} - e_{22} - \eta_{12} + e_{12})$$

We can therefore write (9) as

$$\frac{dx_2}{x_2} - \frac{dx_1}{x_1} = (\ \eta_{21} - \eta_{11} - e_{21} + e_{11})(\frac{dw_1}{w_1} - \frac{dw_2}{w_2})$$

We then finally obtain

$$\sigma^D_{12} = \frac{\frac{dx_2}{x_2} - \frac{dx_1}{x_1}}{\frac{dw_1}{w_1} - \frac{dw_2}{w_2}} = \eta_{21} - \eta_{11} - e_{21} + e_{11} \qquad (10)$$

This expresses σ^D_{12} only in terms of price elasticities of factor demands. Knowledge of these price elasticities is, therefore, sufficient to calculate σ^D_{12}. From (10) and (2) we also see that

$$\sigma^D_{12} = S_1\ \sigma^A_{12} - \eta_{11} - e_{21} + e_{11}$$

which gives the relationship between σ^A_{12} and σ^D_{12}.

Finally, let us consider the two-factor case. It is clear that, in this case, the E matrix vanishes. We then have from (10)

$$\sigma_{12}^{D} = \eta_{21} - \eta_{11}$$

From the homogeneity condition we know that $\eta_{11}+\eta_{12}=0$, so that

$$\sigma_{12}^{D} = \eta_{21} + \eta_{12} = \frac{S_1}{S_2}\eta_{12} + \eta_{12} = \frac{\eta_{12}}{S_2} = \sigma_{12}^{A}$$

Hence $\sigma_{12}^{D} = \sigma_{12}^{A}$. For the n-input case this equality need not hold. For an example, see /1/, pp.291-295.

REFERENCES

/1/ DIXON P.B., BOWLES S. and KENDRICK D., Notes and Problems in Microeconomic Theory, North Holland Publishing Company, Amsterdam, 1980.