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An Econometric Quantity Rationing Model for the Labour Market

by

Hilde MEERSMAN, University of Antwerp (UFSIA)
Joseph PLASMANS, University of Tilburg and Antwerp (UFSIA)*

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Universitaire Faculteiten St.-Ignatius
Prinsstraat 13 - 2000 Antwerpen
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Abstract

In section 1 of this paper Walrasian and effective demands and supplies are derived, based on a Johansen-type utility function and a competitive short run C.E.S.-production technology in two labour inputs: the number of employed people and the number of (average) working hours per employed person. Since we have 3 markets, 8 quantity rationing regimes occur. The resulting spill-over effects of these quantity rationing regimes are evaluated.

In the second section, an econometric estimation procedure for a quantity rationing model with 3 markets is derived, based on the 2-market estimation method of Gourieroux-Laffont-Monfort (1980 a). It is proved that, in general, the model is not always statistically identified. This problem is solved by taking one of the dependent variables as exogenously determined (in our case: the labour force participation or the supply of workers).

In the last section, it is assumed that the agents know in theory the constraints perceived on the other markets, as was the case with the derivation of the effective demands and supplies in section 1. Then, the joint likelihood function of the 3-market quantity rationing model, derived in section 2, is evaluated by means of the spill-over effects, derived in section 1. The result is a <u>complete and statistically identified</u> quantity-rationing model for 3 markets.

Introduction

Under rigid and exogenous prices, a quantity rationing model (QRM) is explicitly derived when the labour market has been split into two submarkets: one market for the number of employed people and one market for the (average) number of working hours per employed person. The purpose of this splitting is, among others, to study the impact of a shorter working time and of a growing unemployment in a non-Walrasian economy.

In this paper we consider a closed economy without any endogenous treatment of the government. The decisions are taken by consumers and producers. The (representative) producer supplies the commodities and demands a number of workers and also an average number of hours of work per worker. The (representative) consumer demands commodities, decides whether or not he will enter the labour market and supplies a number of hours of work. The representative producer is supposed to maximise expected profits under a short run C.E.S.-production technology in both labour inputs. The representative consumer is supposed to maximise a strongly separable Johansen-type utility function in commodities, leisure and real cash balances subject to a budget constraint. The supply of the number of workers can be based on a labour force participation model.

Since it is assumed that prices and wages are not sufficiently flexible to bring the commodity and labour markets into equilibrium, these may exist excess demand and excess supply in the different markets. Hence, we have to consider <u>effective</u> demand and supply functions. Following Clower (1965), Barro and Grossman (1976), Malinvaud (1977) and many others we define the "effective" demand (supply) in a specific market as the demand (supply) taking into account an economic agent's quantity constraints ("rationings") in all the <u>other</u> markets(*). If there is no rationing in the other markets, the effective demand (supply) will coincide with the Walrasian or "notional" demand (supply).

It is supposed that producers and consumers can be rationed in the commodity market, in the market for the number of workers and in the market for the average number of hours of work per worker. Hence, eight disequilibrium on quantity rationing regimes can be derived (**).

In section 1 of this paper, the effective demands and supplies for the resulting rationing regimes are discussed, together with a functional evaluation of the occurring spill-overs, measuring the effects of a rationing in an agent's particular market on the demand and supply in the agent's other markets.

In section 2, an econometric estimation procedure for a QRM with three markets will be outlined, based on the well-known Gourieroux-Laffont-Monfort-paper for 2 markets in Econometrica (1980 a).
As was also the case for the 2-market model, the implied 3-market QRM is not statistically identified. Therefore, the labour force participation (supply of workers) is assumed to be determined exogenously in the system: this asymetry entails statistical identifiability, as will be shown in section 2.

Finally, in section 3, the results of the 2 previous sections will be combined. Assuming that each agent knows in theory the quantity constraints perceived on the markets, as is implied by the effective demands and supplies in section 1, the joint sample likelihood function,

* The effective quantities are sometimes called "Clower effective demands (and supplies)", in contrast to "Drêze effective demands (and supplies"), where the quantity constraint on the agent's market considered is also taken account of (see Drêze (1975)).

** As also remarked by Sneessens (1981) p. 2, the term "disequilibrium" can be quite misleading since the Walrasian equilibrium is only one very special case of equilibrium (with perfectly flexible prices), but equilibria can also be defined under fixed prices (as was, in fact, done in our previous paper (1980)).

derived in section 2, will be evaluated with the help of the spill-over effects of section 1. A complete and statistically identifiable QRM for 3 markets results.

SYMBOLS

y quantities in the commodity market
x_1 number of workers
x_2 average hours of work per worker

superscript D or S denotes Walrasian demand or supply
superscript D' or S' denotes effective demand or supply

w nominal wage rate
s average coefficient to calculate the producer's contribution to the Social Security system.
p wholesale price index
p_c consumer price index
q average coefficient to calculate the consumer's contribution to the Social Security system.
v average personal income tax rate
z average cost per worker, being independent of the hours of work
M nominal money stock

Realised transactions are indicated by the superscript $\bar{.}$

1. Quantity Rationing Regimes, Notional and Effective Quantities and Spill-over effects.

Considering three not necessarily clearing markets: a commodity market with volume y, a market for workers with quantity x_1 and a market for average hours of work per worker with quantity x_2, we will advocate the principle of voluntary trade, i.e., in our case, the realised transactions are taken to be the minimum of effective demand and effective supply.

1.1. The different quantity rationing regimes

As we have three markets, there will exist eight possible quantity rationing (or disequilibrium) regimes which are assembled in the table below.

I	II	III	IV
$\bar{y} = y^{D'} < y^{S'}$ $\bar{x}_1 = x_1^{D'} < x_1^{S'}$ $\bar{x}_2 = x_2^{D'} < x_2^{S'}$	$\bar{y} = y^{D'} < y^{S'}$ $\bar{x}_1 = x_1^{D'} < x_1^{S'}$ $x_2^{D'} > x_2^{S'} = \bar{x}_2$	$\bar{y} = y^{D'} < y^{S'}$ $x_1^{D'} > x_1^{S'} = \bar{x}_1$ $\bar{x}_2 = x_2^{D'} < x_2^{S'}$	$\bar{y} = y^{D'} < y^{S'}$ $x_1^{D'} > x_1^{S'} = \bar{x}_1$ $x^{D'} > x_2^{S'} = \bar{x}_2$
V	VI	VII	VIII
$y^{D'} > y^{S'} = \bar{y}$ $\bar{x}_1 = x_1^{D'} < x_1^{S'}$ $\bar{x}_2 = x_2^{D'} < x_2^{S'}$	$y^{D'} > y^{S'} = \bar{y}$ $\bar{x}_1 = x_1^{D'} < x_1^{S'}$ $x_2^{D'} > x_2^{S'} = \bar{x}_2$	$y^{D'} > y^{S'} = \bar{y}$ $x_1^{D'} > x_1^{S'} = \bar{x}_1$ $\bar{x}_2 = x_2^{D'} < x_2^{S'}$	$y^{D'} > y^{S'} = \bar{y}$ $x_1^{D'} > x_1^{S'} = \bar{x}_1$ $x_2^{D'} > x_2^{S'} = \bar{x}_2$

The above eight quantity rationing regimes, describing the economy when all the three markets are in disequilibrium simultaneously (otherwise, 27 disequilibrium situations) can also be characterised respectively as:

case I : PCC, case II : PCP, case III : PPC, case IV : PPP,
case V : CCC, case VI : CCP, case VII : CPC and case VIII : CPP.

Here, the first letter indicates the (representative) agent, i.e., consumer or producer, who is rationed in the commodity market, the second letter that agent being rationed in the labour force market and the third letter the agent being rationed in the working hours market. Hence, the first quantity rationing regime PCC, denoting an excess supply in all markets, implies that the producer is only rationed in the commodity market and that the consumer is being rationed in both labour type markets. It is the situation of unemployment and stock formation, i.e., a Keynesian underutilisation, which can be restored by extra expenditures.

If there is now any rationing in a market, this will influence the demand and supply in the other markets. This influence is called the spill-over effect (see Patinkin (1949, 1956)). Hence, the concept of a "spill-over" refers to a situation where an economic agent is forced to revise his notional or desired level of trans-

actions in one market, once he meets at least one constraint on the level of his transactions in another market.
An example: the producer's spill over effects from the commodity market to the labour markets amount to $\frac{\partial x_1^{D'}}{\partial \bar{y}}$ and $\frac{\partial x_2^{D'}}{\partial \bar{y}}$ respectively.
When the producer is confronted with a quantity constraint in the commodity market, this means that he cannot realise his Walrasian supply. As we have assumed absence of stock formation, the producer will have to shift his production below the Walrasian output. Therefore, he will need less labour input. The effective demand for workers will be smaller than the Walrasian demand. The same holds for the effective demand for average hours of work per worker.

Now, a specific QRM will be elaborated and Walrasian and effective demands and supplies will be derived.

1.2. Notional and Effective Demands and Supplies

For the (representative) producer, we will consider a stochastic short run production technology in both labour inputs and for the (representative) consumer a Johansen-type expected utility indicator will be utilised.

1.2.1. The producer's side of the model

We consider a representative producer (or body of producers) who produces each period t ($t=0,\ldots,\infty$) according to a production function in the number of workers and in the average hours of work per worker. We assume that the periods between sample points follow each other fast enough to hold the capital stock fixed for each period.
We use a CES-production function with Hicks-neutrality:

$$y_t = Ae^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \tag{1.1}$$

with δ a distribution parameter, ρ a substitution parameter and μ a returns to scale parameter. We assume that the production is subject to stochastic influences caused by weather components, deficiencies, breakdowns, etc. If the stochastic component is additive we have

$$\underline{y}_t = y_t + \varepsilon_t \tag{1.2}$$

where $E(\varepsilon_t)$ is assumed to be zero so that $E(\underline{y}_t) = y_t$

The producer is going to maximise the expected present value of his net profit over an infinite time horizon. The after-tax profit in period t is the result of:

i) - The revenue $p_t\underline{y}_t$ from selling the output $\underline{y}_t$ at the price p_t.

ii)- The labour cost.

The producer has to pay wages $w_t x_{1t} x_{2t}$, where w_t is the gross hourly wage rate. He also has to pay payroll taxes. These contributions are a fraction of the wages. They are calculated according to coefficients determined by law.

We assume that on average the producer has to pay a percentage s_t of the total wage bill $w_t x_{1t} x_{2t}$.

There are also employer's contributions which are independent of the hours of work, but vary with the number of workers (e.g., education within the firm, holiday-allowances). We assume that their average cost per worker is z_t.

The total labour cost is

$$(1+s_t)w_t x_{1t} x_{2t} + z_t x_{1t} \tag{1.3}$$

iii)- Other costs, like capital (maintenance) costs, net depreciation, etc., are represented by c_t;

iv)- Taxes to be paid over the gross profit $p_t\underline{y}_t - w_t(1+s_t)x_{1t}x_{2t} - z_t x_{1t} - c_t$. We assume that the producer pays taxes according to an average (company) tax rate u_t.

The aftertax profit in period t is given by:

$$(1-u_t)\{p_t\underline{y}_t - w_t(1+s_t)x_{1t}x_{2t} - z_t x_{1t} - c_t\} \tag{1.4}$$

Let k_t be the discount factor for period t defined as:

$$k_t := 1 \quad \text{for } t = 0$$

$$:= \prod_{\Theta=0}^{t} \frac{1}{1+r_\Theta} \quad \text{for } t > 0,$$

with r_Θ being the discount rate at the end of period Θ.

Then, the present value of the net profit amounts to:

$$\sum_{t=0}^{\infty} k_t(1-u_t)\{p_t\underline{y}_t - w_t(1+s_t)x_{1t}x_{2t} - z_t x_{2t} - c_t\} \quad (1.5)$$

and the expected value of this present value of net profits can be written as:(*)

$$E\{\sum_{t=0}^{\infty} k_t(1-u_t)\{p_t\underline{y}_t - w_t(1+s_t)x_{1t}x_{2t} - z_t x_{1t} - c_t)\}\}$$

$$= \sum_{t=0}^{\infty} k_t(1-u_t)\{p_t E(\underline{y}_t) - w_t(1+s_t)x_{1t}x_{2t} - z_t x_{1t} - c_t\} \quad (1.6)$$

$$= \sum_{t=0}^{\infty} k_t(1-u_t)\{p_t y_t - w_t(1+s_t)x_{1t}x_{2t} - z_t x_{1t} - c_t\}$$

To obtain the <u>Walrasian</u> supply and demand functions of the producer, we have to solve the following programme:

$$\max.\ \pi = \sum_{t=0}^{\infty} k_t(1-u_t)\{p_t y_t - w_t(1+s_t)x_{1t}x_{2t} - z_t x_{1t} - c_t\}$$

(*) The order of the expectation and summation operators E and Σ may be changed because of the following general mathematical properties:
- if $\underline{x}_1,\ldots,\underline{x}_t$ are summable in quadratic mean, then $E\{\sum_{t=0}^{\infty}\underline{x}_t\} = \sum_{t=0}^{\infty} E\{\underline{x}_t\}$;
- a sufficient (but not necessary!) conditions such that $\underline{x}_1,\ldots,\underline{x}_t$ are summable in quadratic mean is that $\sum_{t=0}^{\infty}\sqrt{E\{\underline{x}_t^2\}} < \infty$.

If, e.g., the error terms ε_t are independently identically distributed as $\varepsilon_t \overset{i.i.d.}{\sim} N(o,\sigma_\varepsilon^2)$ and if we define:

$c_t' := c_t + w_t(1+s_t)x_{1t}x_{2t} - z_t x_{1t}$

$\underline{x}_t := k_t(1-u_t)(p_t\underline{y}_t - c_t')$,

so that, taking account of (1.2):

$\underline{x}_t^2 = k_t^2(1-u_t)^2(p_t^2\underline{y}_t^2 - 2p_t\underline{y}_t c_t' + c_t'^2)$ and

$E\{\underline{x}_t^2\} = k_t^2(1-u_t)^2\{(p_t y_t - c_t')^2 + p_t^2\sigma_\varepsilon^2\}$. Hence

$\sum_{t=0}^{\infty}\sqrt{E\{\underline{x}_t^2\}} = \sum_{t=0}^{\infty} k_t(1-u_t)(c_t''^2 + p_t^2\sigma_\varepsilon^2)^{1/2}$,

so that, using d'Alembert's criterium, this sum is finite: $\lim_{t\to\infty} k_t^{1/t}(1-u_t)^{\frac{1}{t}}(c_t''^2+p_t^2\sigma_\varepsilon^2)^{\frac{1}{2t}} < 1$ (because $k_t<1$ for $t>0$), so that the above properties are valid.

$$\text{s.t. } y_t = Ae^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \tag{1.7}$$

The first-order conditions of the above programming problem are:

$$\mu\delta A^{-\frac{\rho}{\mu}} p_t y_t^{\frac{\rho+\mu}{\mu}} e^{-\frac{\lambda\rho t}{\mu}} x_{1t}^{-(\rho+1)} = w_t(1+s_t)x_{2t} + z_t \tag{1.8}$$

$$\mu(1-\delta)A^{-\frac{\rho}{\mu}} p_t y_t^{\frac{\rho+\mu}{\mu}} e^{-\frac{\lambda\rho t}{\mu}} x_{2t}^{-(\rho+1)} = w_t(1+s_t)x_{1t} \tag{1.9}$$

$$y_t = Ae^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta)\, x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}}$$

Since (1.8) involves a time-dependent additive term in a multiplicative expression and we are mainly interested in linearised models for the further analysis of QRM's, we divide both sides of (1.8) by the variable costs $w_t(1+s_t)x_{2t}$ and approximate $\ln\left(\frac{z_t}{w_t(1+s_t)x_{2t}} + 1\right)$ by a Taylor series expansion around a zero fixed cost-variable cost ratio per worker*), or

$$\ln\left(\frac{z_t}{w_t(1+s_t)x_{2t}} + 1\right) \cong \frac{z_t}{w_t(1+s_t)x_{2t}} \tag{1.10}$$

Solving (1.7-1.9) using (1.10), would still entail highly nonlinear equations however, so that it will be assumed henceforth, that fixed labour costs per worker are so small with respect to variable labour costs per worker that they can be neglected without too great imprecision.
The Walrasian quantities in natural logarithms are then derived as:

$$\ln y_t^{SW} = \frac{2}{2-\mu}\ln A + \frac{\mu}{2-\mu}\ln\mu - \frac{\mu}{\rho(2-\mu)}\ln\delta - \frac{\mu}{\rho(2-\mu)}\ln(1-\delta)$$
$$- \frac{\mu(\rho+2)}{\rho(2-\mu)}\ln 2 + \frac{2\lambda}{2-\mu}t - \frac{\mu}{2-\mu}\ln w_t + \frac{\mu}{2-\mu}\ln p_t - \frac{\mu}{2-\mu}\ln(1+s_t) \tag{1.11}$$

$$\ln x_{1t}^{DW} = \frac{1}{2-\mu}\ln A + \frac{1}{2-\mu}\ln\mu + \frac{1-\mu}{\rho(2-\mu)}\ln\delta - \frac{1}{\rho(2-\mu)}\ln(1-\delta)$$

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*) Fixed labour costs during period t, z_t, can be assumed to be small relative to the total variable labour costs $w_t(1+s_t)x_{2t}$ per employed person.

$$-\frac{\mu+\rho}{\rho(2-\mu)}\ \ln 2 + \frac{\lambda}{2-\mu}\ t - \frac{1}{2-\mu}\ \ln w_t + \frac{1}{2-\mu}\ \ln p_t - \frac{1}{2-\mu}\ln(1+s_t) \qquad (1.12)$$

$$\ln x_{2t}^{DW} = \frac{1}{2-\mu}\ \ln A + \frac{1}{2-\mu}\ \ln \mu + \frac{1-\mu}{\rho(2-\mu)}\ \ln(1-\delta) - \frac{1}{\rho(2-\mu)}\ \ln \delta$$
$$-\frac{\mu+\rho}{\rho(2-\mu)}\ \ln 2 + \frac{\lambda}{2-\mu}\ t - \frac{1}{2-\mu}\ \ln w_t + \frac{1}{2-\mu}\ \ln p_t - \frac{1}{2-\mu}\ \ln(1+s_t) \qquad (1.13)$$

from which it follows that the Walrasian commodity supply is(generally)an increasing function of the commodity price and a decreasing function of variable labour costs and the demand for workers increases when the variable labour costs decrease. Finally, the demand for working hours will decrease when wages increase.

To obtain the effective demand and supply functions we have to consider the rationing schemes separately.

i) The producer is rationed in the commodity market (only). Then, the effective commodity supply coincides with the Walrasian supply and the effective demand for workers and for hours can be found by solving the following programming problem, assuming that the producer knows his restrictions on the commodity market (then, he will produce without inventory)*).

$$\underset{\{x_{1t},x_{2t}\}}{\text{Max.}}\ \pi = \sum_{t=0}^{\infty} k_t(1-u_t)\{p_t y_t - w_t(1+s_t)x_{1t}x_{2t} - c_t'\}^{**}$$
$$\text{s.t. } y_t = Ae^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta)\ x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \qquad (1.14)$$
$$y_t = \bar{y}_t$$

This yields as effective factor demand equations:

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*) In section 2 of this paper we will also assume that inventories are allowed (e.g. when the producer does not exactly know his restrictions on the commodity market). See also the approach followed by Gourieroux, Laffont and Monfort (1980 a).

**) The term c_t' stands for $c_t + z_t x_{1t}$ in problem (1.7).

$\ln x_{1t}^{D'} = \frac{1}{\mu} \ln \bar{y}_t - \frac{1}{\mu} \ln A + \frac{1}{\rho} \ln 2 + \frac{1}{\rho} \ln \delta - \frac{\lambda}{\mu} t$	(1.15)
$\ln x_{2t}^{D'} = \frac{1}{\mu} \ln \bar{y}_t - \frac{1}{\mu} \ln A + \frac{1}{\rho} \ln 2 + \frac{1}{\rho} \ln (1-\delta) - \frac{\lambda}{\mu} t$	(1.16)
$\ln y_t^{s'} = \ln y_t^{s}$	

It is found that spill-over elasticities are equal for both labour demands:

$$\frac{\partial \ln x_{1t}^{D'}}{\partial \ln \bar{y}_t} = \frac{1}{\mu} = \frac{\partial \ln x_{2t}^{D'}}{\partial \ln \bar{y}_t} \tag{1.17}$$

ii) The producer is rationed on the number of workers. Then the effective demand for workers is equal to the Walrasian demand and the effective demand for average hours of work per worker and the effective commodity supply are found as:

$$\max_{\{y_t, x_{2t}\}} \Pi = \sum_{t=0}^{\infty} k_t(1-u_t) \{p_t y_t - w_t(1+s_t)x_{1t}\, x_{2t} - c_t')\}$$

$$\text{s.t. } y_t = A\, e^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta) x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \tag{1.18}$$

$$x_{1t} = \bar{x}_{1t}$$

Using Lagrange multipliers σ_1 and σ_2 we get as first order conditions:

$$\left[\begin{array}{l}
p_t = \sigma_1 \\
w_t(1+s_t)x_{2t} = \sigma_1 A e^{\lambda t} \mu\delta \; \{\delta\, x_{1t}^{-\rho} + (1-\delta)\; x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}} \; x_{1t}^{-\rho-1} \; -\sigma_2 \\
w_t(1+s_t)x_{1t} = \sigma_1\; A\; e^{\lambda t}\; \mu(1-\delta)\; \{\delta\; x_{1t}^{-\rho} + (1-\delta)\; x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}} \; x_{2t}^{-\rho-1} \\
y_t = A e^{\lambda t}\{\delta x_{1t}^{-\rho} + (1-\delta) x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \qquad (1.19) \\
x_{1t} = \bar{x}_{1t}
\end{array}\right.$$

<=>

$$\begin{cases} w_t(1+s_t)x_{2t} = p_t A e^{\lambda t}\mu\delta\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}}\bar{x}_{1t}^{-\rho-1} - \sigma_2 \\ w_t(1+s_t)\bar{x}_{1t} = p_t A e^{\lambda t}\mu(1-\delta)\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}} x_{2t}^{-\rho-1} \\ y_t = A e^{\lambda t}\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \end{cases} \quad (1.20)$$

We will use the last two equations to find $y_t^{S'}$ and $x_{2t}^{D'}$.

We will have to use first order approximations for $\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}}$ and for $\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}}$

(1) <u>Approximation for $\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}}$</u>

$$-\frac{\mu+\rho}{\rho}\ln\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\} = \mu+\rho \ln x_{2t} - \frac{\mu+\rho}{\rho}\ln\{\delta(\frac{\bar{x}_{1t}}{x_{2t}})^{-\rho} + 1-\delta\}$$

Let

$$f(\ln\frac{x_1}{x_2}) = -\frac{\mu+\rho}{\rho}\ln\{\delta(\frac{\bar{x}_{1t}}{x_{2t}})^{-\rho} + 1-\delta\}$$

A first order Taylor approximation around $\bar{x}_{1t} = x_{2t}$ (Kmenta-approximation) gives

$$f(\ln\frac{x_1}{x_2}) = \delta(\mu+\rho)\ln\frac{\bar{x}_{1t}}{x_{2t}}$$

This yields

$$\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu+\rho}{\rho}} \cong \bar{x}_{1t}^{\delta(\mu+\rho)}\, x_{2t}^{(1-\delta)(\mu+\rho)} \quad (1.21)$$

(ii) <u>Approximation for $\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}}$</u>

$$-\frac{\mu}{\rho}\ln\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\} = \mu \ln x_{2t} - \frac{\mu}{\rho}\ln\{\delta(\frac{\bar{x}_{1t}}{x_{2t}})^{-\rho} + 1-\delta\}$$

$$g(\ln\frac{\bar{x}_{1t}}{x_{2t}}) = -\frac{\mu}{\rho}\ln\{\delta(\frac{\bar{x}_{1t}}{x_{2t}})^{-\rho} + 1 - \delta\}$$

A first order Taylor approximation for $\bar{x}_{1t} = x_{2t}$ gives

$$g(\ln\frac{\bar{x}_{1t}}{x_{2t}}) = \delta\mu \ln\frac{\bar{x}_{1t}}{x_{2t}}$$

Hence, we have:

$$\{\delta\bar{x}_{1t}^{-\rho} + (1-\delta)x_{2t}^{-\rho}\}^{-\frac{\mu}{\rho}} \cong \bar{x}_{1t}^{\mu\delta}x_{2t}^{\mu(1-\delta)} \qquad (1.22)$$

Assuming equalities, we have to solve the following two equations:

$$w_t(1+s_t)\bar{x}_{1t} = p_t A\mu(1-\delta)\bar{x}_{1t}^{\delta(\mu+\rho)}x_{2t}^{-\rho-1+(1-\delta)(\mu+\rho)}$$
$$y_t = Ae^{\lambda t}\bar{x}_{1t}^{\mu\delta}x_{2t}^{\mu(1-\delta)} \qquad (1.23)$$

This yields:

$$\ln x_{2t}^{D'} = \frac{1}{a}\ln w_t + \frac{1}{a}\ln(1+s_t) - \frac{1}{a}\ln p_t - \frac{1}{a}\ln A - \frac{1}{a}\ln\mu - \frac{\lambda}{a}t + \frac{1-\delta(\mu+\rho)}{a}\ln\bar{x}_{1t} \qquad (1.24)$$

$$\ln y_t^{S'} = -\frac{1+\rho\delta}{a}\ln A - \frac{(1-\delta)\mu}{a}\ln\mu + \frac{(1-\delta)\mu}{a}\ln w_t + \frac{(1-\delta)\mu}{a}\ln(1+s_t) - \frac{(1-\delta)\mu}{a}\ln p_t - \lambda\frac{1+\rho\delta}{a}t + \frac{\mu\{1-\delta(\rho+2)\}}{a}\ln\bar{x}_{1t} \qquad (1.25)$$

with $\boxed{a := (\mu+\rho)(1-\delta) - \rho - 1}$

Hence, the spill-over elasticities for the effective demand for hours per worker and for the effective commodity supply are respectively:

$$\frac{\partial \ln x_{2t}^{D'}}{\partial \ln\bar{x}_{1t}} = \frac{1-\delta(\mu+\rho)}{(\mu+\rho)(1-\delta)-\rho-1} \quad \text{and}$$

$$\frac{\partial \ln y_t^{S'}}{\partial \ln\bar{x}_{1t}} = \frac{\mu\{1-\delta(\rho+2)\}}{(\mu+\rho)(1-\delta)-\rho-1} \qquad (1.26)$$

iii) The producer is rationed on the average hours of work per worker.
Using the same approximations as under ii) we find:

$$\ln x_{1t}^{D'} = -\frac{1}{b}\ln A - \frac{1}{b}\ln\mu - \frac{1}{b}\ln\delta - \frac{\lambda}{b}t + \frac{1}{b}\ln w_t + \frac{1}{b}\ln(1+s_t) - \frac{1}{b}\ln p_t + \frac{1-(1-\delta)(\mu+\rho)}{b}\ln\bar{x}_{2t} \tag{1.27}$$

$$\ln y_t^{S'} = \frac{\rho(\delta-1)-1}{b}\ln A - \frac{\delta\mu}{b}\ln\mu - \frac{\delta\mu}{b}\ln\delta + \lambda'\frac{\rho(\delta-1)-1}{b}t + \frac{\mu\delta}{b}\ln w_t + \frac{\mu\delta}{b}\ln(1+s_t) - \frac{\mu\delta}{b}\ln p_t + \frac{\mu(\delta\rho+2\delta-\rho-1)}{b}\ln\bar{x}_{2t} \tag{1.28}$$

with

$$\boxed{b := \delta(\mu+\rho) - \rho - 1}$$

so that the spill-over elasticities are respectively:

$$\frac{\partial\ln x_{1t}^{D'}}{\partial\ln\bar{x}_{2t}} = \frac{1-(1-\delta)(\mu+\rho)}{\delta(\mu+\rho)-\rho-1}$$ and

$$\frac{\partial\ln y_t^{S'}}{\partial\ln\bar{x}_{2t}} = \frac{\mu(\delta\rho+2\delta-\rho-1)}{\delta(\mu+\rho)-\rho-1}$$

iv) The producer is rationed on the commodity market and on the number of workers.
Using the Kmenta-approximation we have

$$\ln\bar{y}_t = \ln A + \lambda t + \mu\delta\ln\bar{x}_{1t} + \mu(1-\delta)\ln x_{2t}$$

$\Leftrightarrow$

$$\ln x_{2t}^{D'} = -\frac{1}{\mu(1-\delta)}\ln A - \frac{\lambda}{\mu(1-\delta)}t + \frac{1}{\mu(1-\delta)}\ln\bar{y}_t - \frac{\delta}{1-\delta}\ln\bar{x}_{1t} \tag{1.29}$$

with two spill-over effects involved.

v) The producer is rationed on the commodity market and on the average hours of work per worker.
Then, the effective demand for workers is also directly derived from the CES-production function also using Kmenta's approximation:

$$\ln \bar{x}^{D'}_{1t} = -\frac{1}{\mu\delta} \ln A - \frac{\lambda}{\mu\delta} t + \frac{1}{\mu\delta} \ln \bar{y}_t - \frac{1-\delta}{\delta} \ln \bar{x}_{2t} \qquad (1.30)$$

also involving two spill-over elasticities.

vi) The producer is rationed on the number of workers and on the average hours of work per worker.

$$\ln y^{S'}_t = \ln A + \lambda t + \mu\delta \ln \bar{x}_{1t} + \mu(1-\delta) \ln \bar{x}_{2t} \qquad (1.31)$$

where the spill-over elasticities are simply the (linearised) output elasticities with respect to the number of workers and the average hours of work per worker respectively.

1.2.2. The consumer's side of the model

We consider a representative consumer (or a body of consumers) who has for every period $t=0,\ldots,\infty$ an instantaneous utility function

$$U_t = \frac{\beta_1}{\alpha_1} \left(\frac{y_t}{\beta_1}\right)^{\alpha_1} - \frac{\beta_2}{\alpha_2} \left(\frac{x_{2t}}{\beta_2}\right)^{\alpha_2} + \frac{\beta_3}{\alpha_3} \left(\frac{M_t/p_{c,t}}{\beta_3}\right)^{\alpha_3} \qquad (1.32)$$

where $\alpha_1, \alpha_2, \alpha_3 < 1$

$\beta_1, \beta_2, \beta_3 > 0$

The budget restriction is given by

$$p_{c,t}y_t + M_t = \{w_t(1-q_t)x_{2t} + N_t\}(1-v_t) + M_{t-1} \qquad (1.33)$$

with $p_{c,t}$ the consumer price index

M_t the nominal money stock at the end of period t

q_t the average coefficient for calculating the employee's share of the payroll taxes.

w_t the nominal wage rate per hour of work

N_t non-labour income

v_t average personal income tax rate

The utility function for the consumer over the period $0,\ldots\ldots,\infty$ is given by

$$U = \sum_{t=0}^{\infty} l_t U_t$$

The factor l_t is a discount factor which gives less importance to future utilities.

The labour force participation (LFP) is provisionally assumed to be exogenous.

x_{1t} = LFP_t (see also Meersman and Plasmans (1980), pp. 20-32 and section 2 of this paper).

In order to determine the Walrasian commodity demand and the supply of average hours of work, we have to solve the following programming problem:

$$\text{Max } U = \sum_{t=0}^{\infty} l_t \left\{ \frac{\beta_1}{\alpha_1}\left(\frac{y_t}{\beta_1}\right)^{\alpha_1} - \frac{\beta_2}{\alpha_2}\left(\frac{x_{2t}}{\beta_2}\right)^{\alpha_2} + \frac{\beta_3}{\alpha_3}\left(\frac{M_t/p_{c,t}}{\beta_3}\right)^{\alpha_3} \right\} \tag{1.34}$$

$$\text{s.t. } p_{c,t}y_t + M_t = \{w_t(1-q_t)x_{2t} + N_t\}(1-v_t) + M_{t-1}$$

This yields the following functions:*)

$$\left[\begin{array}{l} \ln y_t^D = \ln \beta_1 - \dfrac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \dfrac{1-\alpha_3}{1-\alpha_1} \ln\left(\dfrac{M_t}{p_{c,t}}\right) \\[2ex] \ln x_{2t}^S = \ln \beta_2 - \dfrac{1-\alpha_3}{1-\alpha_2} \ln \beta_3 - \dfrac{1}{1-\alpha_2} \ln\left\{\dfrac{w_t}{p_{c,t}}(1-q_t)(1-v_t)\right\} \\[2ex] \qquad + \dfrac{1-\alpha_3}{1-\alpha_2} \ln \dfrac{M_t}{p_{c,t}} \\[2ex] \ln x_{1t}^S = \ln LFP_t \end{array}\right. \tag{1.35}$$

The effective demand and supply functions are derived under the various consumer's rationing schemes.

i) The consumer is rationed in the commodity market. Then, the

*)Under the assumption that the (Walrasian) money stock is known. See also footnote *) on p. 19.

effective commodity demand is equal to the Walrasian demand*).

$$\ln y_t^{D'} = \ln y_t^D = \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_1} \ln \left(\frac{M_t}{p_{c,t}}\right)^W \qquad (1.36)$$

In order to derive the effective supply of hours of work we make the following assumption.
The rationing on the commodity market is reflected in the money stock and we assume :

$$\ln \frac{M_t}{p_{c,t}} = \ln \left(\frac{M_t}{p_{c,t}}\right)^W + \gamma_1(\ln y_t^D - \ln \bar{y}_t)$$

$$= \ln \left(\frac{M_t}{p_{c,t}}\right)^W + \gamma_1 \ln \beta_1 - \gamma_1 \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3$$

$$+ \gamma_1 \frac{1-\alpha_3}{1-\alpha_1} \ln \left(\frac{M_t}{p_{c,t}}\right)^W - \gamma_1 \ln \bar{y}_t \qquad (1.37)$$

This expression wil now be substituted into the effective supply of average hours of work:

$$\ln x_{2t}^{s'} = \ln \beta_2 + \gamma_1 \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_1 - \left\{\frac{1-\alpha_3}{1-\alpha_2} + \gamma_1 \frac{(1-\alpha_3)^2}{(1-\alpha_1)(1-\alpha_2)}\right\} \ln \beta_3$$

$$- \frac{1}{1-\alpha_2} \ln \left\{\frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\right\} + \frac{1-\alpha_3}{1-\alpha_2} \left\{1 + \gamma_1 \frac{1-\alpha_3}{1-\alpha_1}\right\} \ln \left(\frac{M_t}{p_{c,t}}\right)^W$$

$$- \gamma_1 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{y}_t \quad {}^{**)} \qquad (1.38)$$

Also

$$\ln x_{1t}^{s'} = \ln LFP_t$$

---

*) The demand for money is in this case always equal to the Walrasian quantity $\left(\frac{M_t}{p_{c,t}}\right)^W$.

**) Assuming that the real Walrasian money demand can be approximated by an adaptive expectations model , we can write:

$$\left(\frac{M_t}{p_{c,t}}\right)^W \cong A \prod_{i=0}^{n} \left(\frac{M_{t-i}}{p_{c,t-i}}\right)^{W_i}, \quad w_i \geq 0, \quad \sum_{i=0}^{n} w_i = 1 \qquad (1.39)$$

The spill-over elasticity is $-\gamma_1 \dfrac{1-\alpha_3}{1-\alpha_2}$.

ii) The consumer is rationed by his labour force participation.

Assuming that the number of unemployed people influences the money stock in the following way:

$$\ln\left(\frac{M_t}{p_{c,t}}\right) = \ln\left(\frac{M_t}{p_{c,t}}\right)^W + \gamma_2(\ln LFP_t - \ln\bar{x}_{1t}) \qquad (1.40)$$

where an increase in unemployment leads to a decrease in the money stock, $\gamma_2 < 0$. The resulting effective commodity demand and effective supply of working hours are:

$$\left[\begin{array}{l} \ln y_t^{D'} = \ln \beta_1 - \dfrac{1-\alpha_3}{1-\alpha_1}\ln \beta_3 + \dfrac{1-\alpha_3}{1-\alpha_1}\ln\left(\dfrac{M_t}{p_{c,t}}\right)^W + \gamma_2 \dfrac{1-\alpha_3}{1-\alpha_1}\ln LFP_t \\ \qquad - \gamma_2 \dfrac{1-\alpha_3}{1-\alpha_1}\ln \bar{x}_{1t} \\ \ln x_{2t}^{S'} = \ln \beta_2 - \dfrac{1-\alpha_3}{1-\alpha_2}\ln \beta_3 - \dfrac{1}{1-\alpha_2}\ln\left\{\dfrac{w_t}{p_{c,t}}(1-q_t)(1-v_t)\right\} \\ \qquad + \dfrac{1-\alpha_3}{1-\alpha_2}\ln\left(\dfrac{M_t}{p_{c,t}}\right)^W + \gamma_2\dfrac{1-\alpha_3}{1-\alpha_2}\ln LFP_t - \gamma_2\dfrac{1-\alpha_3}{1-\alpha_2}\ln\bar{x}_{1t} \end{array}\right. \qquad (1.41)$$

and also:

$$\ln x_{1t}^{S'} = \ln x_{1t}^{S} = \ln LFP_t$$

If γ_2 is negative, the spill-over elasticities are both positive.

iii) The consumer is rationed by the average hours of work.

The effective supply of hours is the Walrasian one. To find the effective commodity demand we will make an assumption that is analogue to (1.40):

$$\ln\left(\frac{M_t}{p_{c,t}}\right) = \ln\left(\frac{M_t}{p_{c,t}}\right)^W + \gamma_3(\ln x_{2t}^S - \ln \bar{x}_{2t}) \qquad (1.42)$$

Then, the effective commodity demand satisfies:

$$\ln y_t^{D'} = \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_1} \ln \left(\frac{M_t}{p_{c,t}}\right)^W + \gamma_3 \frac{1-\alpha_3}{1-\alpha_1} \ln x_{2t}^s$$

$$- \gamma_3 \frac{1-\alpha_3}{1-\alpha_1} \ln \bar{x}_{2t}$$

$$= \ln \beta_1 + \gamma_3 \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_2 - \left\{\frac{1-\alpha_3}{1-\alpha_1} + \gamma_3 \frac{(1-\alpha_3)^2}{(1-\alpha_1)(1-\alpha_2)}\right\} \ln \beta_3$$

$$- \gamma_3 \frac{1-\alpha_3}{(1-\alpha_1)(1-\alpha_2)} \ln\left\{\frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\right\} + \frac{1-\alpha_3}{1-\alpha_1} \left\{1 + \gamma_3 \frac{1-\alpha_3}{1-\alpha_2}\right\} \ln\left(\frac{M_t}{p_{c,t}}\right)^W$$

$$- \gamma_3 \frac{1-\alpha_3}{1-\alpha_1} \ln \bar{x}_{2t} \qquad (1.43)$$

so that the spill over elasticity is $-\gamma_3 \frac{1-\alpha_3}{1-\alpha_1}$, which is generally positive.

Also

$$\ln x_{1t}^{s'} = \ln LFP_t$$

$$\ln x_{2t}^{s'} = \ln x_{2t}^s$$

iv). The consumer is rationed in the commodity market and by the labour force participation rate.

The effective commodity demand is the same as in (1.41). For the effective supply of working hours, we make the following assumption:

$$\ln \left(\frac{M_t}{p_{c,t}}\right) = \ln\left(\frac{M_t}{p_{c,t}}\right)^W + \gamma_4(\ln y_t^D - \ln \bar{y}_t) + \gamma_5(\ln LFP_t - \ln \bar{x}_{1t}) \qquad (1.44)$$

Then, the effective supply of hours of work becomes:

$$\ln x^{s'}_{2t} = \ln \beta_2 - \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_3 - \frac{1}{1-\alpha_2} \ln \{\frac{w_t}{p_{c,t}} (1-q_t)(1-v_t) + \frac{1-\alpha_3}{1-\alpha_2} \ln (\frac{M_t}{p_{c,t}})^W$$

$$+ \gamma_4 \frac{1-\alpha_3}{1-\alpha_2} \ln y^D_t + \gamma_5 \frac{1-\alpha_3}{1-\alpha_2} \ln LFP_t - \gamma_4 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{y}_t - \gamma_5 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{x}_{1t}$$

$$= \ln \beta_2 - \{\frac{1-\alpha_3}{1-\alpha_2} + \gamma_4 \frac{(1-\alpha_3)^2}{(1-\alpha_1)(1-\alpha_2)}\} \ln \beta_3 + \gamma_4 \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_1$$

$$- \frac{1}{1-\alpha_2} \ln \{ \frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\} + \frac{1-\alpha_3}{1-\alpha_2} \{1+\gamma_4 \frac{1-\alpha_3}{1-\alpha_1}\} \ln (\frac{M_t}{p_{c,t}})^W$$

$$+ \gamma_5 \frac{1-\alpha_3}{1-\alpha_2} \ln LFP_t - \gamma_4 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{y}_t - \gamma_5 \frac{1-\alpha_3}{1-\alpha_2} \ln \bar{x}_{1t} \qquad (1.45)$$

We have two spill-over elasticities now:

$$\frac{\partial \ln x^{s'}_{2t}}{\partial \ln \bar{y}_t} = - \gamma_4 \frac{1-\alpha_3}{1-\alpha_2} \text{ and } \frac{\partial \ln x^{s'}_{2t}}{\partial \ln \bar{x}_{1t}} = - \gamma_5 \frac{1-\alpha_3}{1-\alpha_2} \qquad (1.46)$$

v) The consumer is rationed in the commodity market and on the average hours of work.

No influence on LFP_t, since it is assumed exogenous.

vi) The consumer is rationed on the LFP and on the average hours of work. Assuming that;

$$\ln (\frac{M_t}{p_{c,t}}) = \ln (\frac{M_t}{p_{c,t}})^W + \gamma_6(\ln LFP_t - \ln \bar{x}_{1t}) + \gamma_7(\ln x^s_{2t} - \ln \bar{x}_{2t}) \qquad (1.47$$

the effective commodity demand can be written as:

$$\ln y_t^{D'} = \ln \beta_1 + \gamma_7 \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_2 - \left\{\frac{1-\alpha_3}{1-\alpha_1} + \gamma_7 \frac{(1-\alpha_3)^2}{(1-\alpha_1)(1-\alpha_2)}\right\} \ln \beta_3$$
$$- \gamma_7 \frac{1-\alpha_3}{(1-\alpha_1)(1-\alpha_2)} \ln \left\{\frac{w_t}{p_{c,t}} (1-q_t)(1-v_t)\right\} + \frac{1-\alpha_3}{1-\alpha_1} \left\{1 + \gamma_7 \frac{1-\alpha_3}{1-\alpha_2}\right\} \ln \left(\frac{M_t}{p_{c,t}}\right)^W$$
$$+ \gamma_6 \frac{1-\alpha_3}{1-\alpha_1} \ln LFP_t - \gamma_6 \frac{1-\alpha_3}{1-\alpha_1} \ln \bar{x}_{1t} - \gamma_7 \frac{1-\alpha_3}{1-\alpha_1} \ln \bar{x}_{2t} \qquad (1.48)$$

In the next section, an econometric estimation procedure for an identified QRM with three markets will be outlined, based on Gourieroux-Laffont-Monfort (1980a).

2. An Econometric Estimation Procedure for an identified QRM with 3 markets.

Consider again three not necessarily clearing markets with volumes y, x_1 and x_2 and assume that the realised transactions are equal to the minimum of effective demand and effective supply. Then, the estimation procedure runs as follows.

2.1. General outline of the estimation procedure

In a first step, the effective demand and supply functions are specified as functions of the realised transactions according to the different quantity rationing regimes.*)
So, for our 3-market case, we have for each regime i (i = 1,2 ...,8):

*) Notice that, in general, the supply of workers (LFP) is assumed to be endogenous.

$$\begin{bmatrix} y_{it}^{D'} \\ y_{it}^{S'} \\ x_{i1t}^{D'} \\ x_{i1t}^{S'} \\ x_{i2t}^{D'} \\ x_{i2t}^{S'} \end{bmatrix} = F_i(\bar{y}_t, \bar{x}_{1t}, \bar{x}_{2t}) + \begin{bmatrix} \delta_{oit} \\ \lambda_{oit} \\ \delta_{1it} \\ \lambda_{1it} \\ \delta_{2it} \\ \lambda_{2it} \end{bmatrix} \qquad (2.1)$$

with $\bar{y}_t$, $\bar{x}_{1t}$ and $\bar{x}_{2t}$ the quantities transacted of commodities, workers and working hours respectively, and δ_{jit} and λ_{jit} (j = 0, 1, 2; i = 1, 2,..... 8; t = 1, 2,...,T) functions of prices (as, e.g., the nominal wage rate w_t, the wholesale price index p_t, the consumer price index $p_{c,t}$, the producer's and consumer's relative contributions to social security s_t and q_t, the average personal income tax rate v_t, the average cost per worker being independent of the hours of work z_t) and other exogenous variables (as, e.g., income, weather indicators, instrumental variables, etc...).

Linearising the functions F_i, δ_{ji} and λ_{ji} (j = 0, 1, 2; i = 1, 2,...,8) around the corresponding Walrasian quantities and the exogenously given prices and variables respectively, the right-hand side vector of (2.1) can be written in simplifying notation as:

$$\begin{bmatrix} \underline{\delta}_{jit} \\ \underline{\lambda}_{jit} \end{bmatrix}_{j = 0, 1, 2} = B_i X_{it} + u_{it} \quad , \quad (2.2) \quad *)$$

where B_i is a matrix of unknown parameters, X_{it} is a vector of

*) Note that 0 stands for the commodity market, 1 for the workers' market and 2 for the number of working hours' market.

(known) exogenous variables including prices and u_{it} is a 6-dimensional random vector which density g.

In a <u>second step</u> the realised transactions are derived from the effective demands and supplies through the application of the minimum-conditions. The whole set of equations is solved for $(\underline{\delta}_{jit}, \underline{\lambda}_{jit})'$ (j = 0, 1, 2):

$$\begin{bmatrix} \underline{\delta}_{jit} \\ \underline{\lambda}_{jit} \end{bmatrix}_{j=0,1,2} = f_i \left(\begin{bmatrix} \underline{y}^{D'}_{it} \\ \underline{y}^{S'}_{it} \\ \underline{x}^{D'}_{i1t} \\ \underline{x}^{S'}_{i1t} \\ \underline{x}^{D'}_{i2t} \\ \underline{x}^{S'}_{i2t} \end{bmatrix} \right) = B_i X_{it} + \underline{u}_{it} \tag{2.3}$$

<u>Third</u>, starting from the density of $\underline{u}_{it}$, the density ϕ_i of $(\underline{y}^{D'}_{it}, \underline{y}^{S'}_{it}, \underline{x}^{D'}_{1it}, \underline{x}^{S'}_{1it}, \underline{x}^{D'}_{2it}, \underline{x}^{S'}_{2it})'$ is derived. Let $|J_i|$ be the Jacobian matrix which corresponds to the underlying transformation, then

$$\phi_i = g(f_i(y^{D'}_{it}, \ldots, x^{S'}_{2it})' - B_i X_{it}) |J_i| \qquad (i=1,2,\ldots,8) \tag{2.4}$$

The joint density of the transactions $(\bar{y}, \bar{x}_1, \bar{x}_2)$ is obtained by numerical integration over the excess quantities of all 8 regimes:

$$\begin{aligned} h_t(y,\bar{x}_1,\bar{x}_2) = \int_y^\infty \int_{X_1}^\infty \int_{X_2}^\infty \{ & g(f_1(\bar{y},Y,\bar{x}_1,X_1,\bar{x}_2,X_2)' - B_1X_{1t}) \, |J_1| \\ & + g(f_2(\bar{y},Y,\bar{x}_1,X_1,X_2,\bar{x}_2)' - B_2X_{2t}) \, |J_2| \\ & + g(f_3(\bar{y},Y,X_1,\bar{x}_1,\bar{x}_2,X_2)' - B_3X_{3t}) \, |J_3| \qquad (2.5) \\ & + g(f_4(\bar{y},Y,X_1,\bar{x}_1,X_2,\bar{x}_2)' - B_4X_{4t}) \, |J_4| \\ & + g(f_5(Y,\bar{y},\bar{x}_1,X_1,\bar{x}_2,X_2)' - B_5X_{5t}) \, |J_5| \end{aligned}$$

$$+ g(f_6(Y,\bar{y},\bar{x}_1,X_1,X_2,\bar{x}_2)' - B_6X_{6t}) \, |J_6|$$
$$+ g(f_7(Y,\bar{y},X_1,\bar{x}_1,\bar{x}_2,X_2)' - B_7X_{7t}) \, |J_7|$$
$$+ g(f_8(Y,\bar{y},X_1,\bar{x}_1,X_2,\bar{x}_2)' - B_8X_{8t}) \, |J_8| \}$$
$$dY dX_1 dX_2$$

Assuming that $(\bar{y}_t,\bar{x}_{1t},\bar{x}_{2t})$, $t = 1,\ldots,T$, are independently distributed for all t, we can construct the likelihood function L

$$L = \prod_{t=1}^{T} h_t(\bar{y}, \bar{x}_1, \bar{x}_2) \qquad (2.6)$$

Finally, this function has to be maximised in order to find the ML-estimates of the parameters.

Now, we have to specify the f_i-functions $(i=1,2,\ldots,8)$ for each of the eight regimes. We will only carry out this exercise for the first regime . For the other regimes the same procedure should be followed, which can be found in Meersman and Plasmans (1982).

2.2. A detailed specification of the 1st quantity rationing regime (general excess supply: $y^{D'} < y^{S'}$, $x_1^{D'} < x_1^{S'}$, $x_2^{D'} < x_2^{S'}$)

In section 1 of this paper, examples of effective demand and supply functions which are linear in natural logarithms of the realised transactions can be found (under the hypothesis that the agents know the constraints existing in the various markets). Of course, alternative specifications (e.g., if inventories are allowed in the model) may be considered here. Therefore, we will use a more general notation. As the consumer is rationed in this regime on the number of workers and on the average hours of work, and as the producer is rationed in the commodity market, we can write according to (2.1-2.3):

$$\begin{aligned}
y^{D'} &= a_{11}\bar{x}_1 + a_{21}\bar{x}_2 + \delta_{01} \\
y^{S'} &= y^S \\
x_1^{D'} &= c_{01}\bar{y} + \delta_{11} \\
x_1^{S'} &= d_{21}\bar{x}_2 + \lambda_{11} \\
x_2^{D'} &= e_{01}\bar{y} + \delta_{21} \\
x_2^{S'} &= f_{11}\bar{x}_1 + \lambda_{21}
\end{aligned} \tag{2.7}$$

with $\bar{x}_1 := x_1^{D'}$; $\bar{x}_2 := x_2^{D'}$ and $\bar{y} := y^{D'}$.

The Walrasian demands and supplies are found by replacing the realised transactions by the Walrasian quantities in the effective demand (and supply) functions (fundamental identity property).

For the Walrasian commodity supply we may write:

$$y^S = b_{11}x_1^D + b_{21}\,x_2^D + \lambda_{01} \tag{2.8}$$

so that we have

$$\begin{aligned}
y^D &= a_{11}x_1^S + a_{21}x_2^S + \delta_{01} \\
y^S &= b_{11}x_1^D + b_{21}x_2^D + \lambda_{01} \\
x_1^D &= c_{01}y^S + \delta_{11} \\
x_1^S &= d_{21}x_2^S + \lambda_{11} \\
x_2^D &= e_{01}y^S + \delta_{21} \\
x_2^S &= f_{11}x_1^S + \lambda_{21}
\end{aligned} \tag{2.9}$$

Solving for y^S gives

$$y^S = \frac{b_{11}\delta_{11} + b_{21}\delta_{21} + \lambda_{01}}{1 - b_{11}c_{01} - b_{21}e_{01}} \tag{2.10}$$

We finally have the following model

$$\begin{cases} y^{D'} = a_{11}x_1^{D'} + a_{21}x_2^{D'} + \delta_{01} \\ y^{S'} = \dfrac{b_{11}\delta_{11} + b_{21}\delta_{21} + \lambda_{01}}{1 - b_{11}c_{01} - b_{21}e_{01}} \\ x_1^{D'} = c_{01}y^{D'} + \delta_{11} \\ x_1^{S'} = d_{21}x_2^{D'} + \lambda_{11} \\ x_2^{D'} = e_{01}y^{D'} + \delta_{21} \\ x_2^{S'} = f_{11}x_1^{D'} + \lambda_{21} \end{cases} \tag{2.11}$$

Solving for $\begin{Bmatrix} \delta_{j1} \\ \lambda_{j1} \end{Bmatrix}$ $j=0,1,2$ gives

$$\begin{Bmatrix} \delta_{j1} \\ \lambda_{j1} \end{Bmatrix}_{j=0,1,2} = A_1 \begin{Bmatrix} y^{D'} \\ \vdots \\ x_2^{S'} \end{Bmatrix} \tag{2.12}$$

with

$$A_1 = \begin{bmatrix} 1 & 0 & -a_{11} & 0 & -a_{21} & 0 \\ (b_{11}c_{01}+b_{21}e_{01})(1-b_{11}c_{01}-b_{21}e_{01}) & & -b_{11} & 0 & -b_{21} & 0 \\ -c_{01} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -d_{21} & 0 \\ -e_{01} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -f_{11} & 0 & 0 & 1 \end{bmatrix}$$

$$|A_1| = (1 - b_{11}c_{01} - b_{21}e_{01})(1 - a_{21}e_{01} - a_{11}c_{01}) \tag{2.13}$$

In Meersman-Plasmans (1982), pp. 8-14, the evaluations of the A_i-matrices ($i = 2,3,\ldots,8$) with corresponding determinants are given. The first necessity of the resulting model is the proof of statistical identifiability. We will prove in the appendix of this paper that the simpler 2-market model of Gourieroux-Laffont-Monfort is <u>not</u> identified. This is, by analogy, also true for the above 3-market model, which is not worked out in detail, however, in order to avoid unnecessary elaborations.
Since the identification problem is caused by the similarity between some regimes, it is argued in the appendix that a sufficient condition to obtain statistical identifiability of the 3-market QRM is the exogenisation of the labour force participation rate.

In the sequel of this section, the 3-market QRM will be respecified with exogenous supply of the number of workers.

2.3. Estimation Procedure for an identified 3-market QRM

Assuming that the labour participation (LFP) is determined exogenously, the effective demand and supply functions, containing at most two spill-over components and a component being a function of prices and other exogenous variables, can be derived following the lines set forth in the first part of this section. Hence, the set of equations (2.3) can be replaced by:

$$\begin{bmatrix} \delta_{oit} \\ \lambda_{oit} \\ \delta_{1it} \\ \delta_{2it} \\ \lambda_{2it} \end{bmatrix} = C_i \begin{bmatrix} y_t^{D'} \\ y_t^{S'} \\ x_{1t}^{D'} \\ x_{2t}^{D'} \\ x_{2t}^{S'} \end{bmatrix} + k_i LFP_t = f_i \qquad (2.14)$$

$$(i = 1,2,\ldots,8)$$

with $LFP_t = x_{1t}^S = x_{1t}^{S'}$, and where

δ_{oit}, λ_{oit}, δ_{1it}, δ_{2it}, λ_{2it} are functions of prices and other variables,

C_i are (5 x 5) matrices of unknown parameters,

k_i are 5-dimensional vectors of unknown parameters belonging to the impact of the workers' supply,

$$\begin{bmatrix} \delta_{oit} \\ \lambda_{oit} \\ \delta_{1it} \\ \delta_{2it} \\ \lambda_{2it} \end{bmatrix} = \Lambda_i X_{it} + \varepsilon_{it} \qquad (2.15)$$

with Λ_i a matrix of unknown parameters

X_{it} a vector of exogenous variables, including prices

ε_{it} a 5-dimensional random vector with probability density g.

The simultaneous density of the transactions $\bar{y}_t$, $\bar{x}_{1t}$ and $\bar{x}_{2t}$ can be written for each observation period as (see (2.5)):

$$\begin{aligned} h_t(\bar{y}, \bar{x}_1, \bar{x}_2) = \int_{\bar{y}}^{\infty}\int_{\bar{x}_1}^{\infty}\int_{\bar{x}_2}^{\infty} \Big[& g\{ C_1.(\bar{y},y,\bar{x}_1,\bar{x}_2,x_2)' + k_1 LFP - \Lambda_1 X\} \, |C_1| \\ & + g\{ C_2.(\bar{y},y,\bar{x}_1,\bar{x}_2,\bar{x}_2)' + k_2 LFP - \Lambda_2 X\} \, |C_2| \\ & + g\{ C_3.(\bar{y},y,x_1,\bar{x}_2,x_2)' + k_3 LFP - \Lambda_3 X\} \, |C_3| \\ & + g\{ C_4.(\bar{y},y,x_1,x_2,\bar{x}_2)' + k_4 LFP - \Lambda_4 X\} \, |C_4| \\ & + g\{ C_5.(y,\bar{y},\bar{x}_1,\bar{x}_2,x_2)' + k_5 LFP - \Lambda_5 X\} \, |C_5| \\ & + g\{ C_6(y,\bar{y},\bar{x}_1,x_2,\bar{x}_2)' + k_6 LFP - \Lambda_6 X\} \, |C_6| \\ & + g\{ C_7(y,\bar{y},x_1,\bar{x}_2,x_2)' + k_7 LFP - \Lambda_7 X\} \, |C_7| \\ & + g\{ C_8(y,\bar{y},x_1,x_2,\bar{x}_2)' + k_8 LFP - \Lambda_8 X\} \, |C_8| \Big] \\ & dy \; dx_1 \, dx_2 \qquad (2.16) \end{aligned}$$

Again assuming that $(\bar{y}_t, \bar{x}_{1t}, \bar{x}_{2t})$ $(t = 1,\ldots,T)$ are serially independent, the joint-likelihood function of the sample becomes:

$$L = \prod_{t=1}^{T} h_t(\bar{y}, \bar{x}_1, \bar{x}_2) \qquad (2.17)$$

Repeating the procedure of the second part of this section and taking into account the property that $x_{1t}^{S} = x_{1t}^{S'} = LFP_t$, so that the workers' supply equations are omitted, we can specify the various C_i-matrices and corresponding determinants, together with the unknown k_i-vectors in (2.16) for each regime i=1,2,...,8 as follows:

$$C_1 = \begin{bmatrix} 1 & 0 & -a_{11} & -a_{21} & 0 \\ b_{11}c_{01}+b_{21}e_{01} & (1-b_{11}c_{01}-b_{21}e_{01}) & -b_{11} & -b_{21} & 0 \\ -c_{01} & 0 & 1 & 0 & 0 \\ -e_{01} & 0 & 0 & 1 & 0 \\ 0 & 0 & -f_{11} & 0 & 1 \end{bmatrix}, \; k_1=\{0\} \qquad (2.18)$$

with

$|C_1| = (1 - b_{11}c_{01} - b_{21}e_{01})\,(1 - a_{21}e_{01} - a_{11}c_{01}),$

$$C_2 = \begin{bmatrix} 1 & 0 & -a_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 & -b_{22} \\ -c_{02} & 0 & 1 & 0 & -c_{22} \\ -e_{01} & 0 & 0 & 1 & 0 \\ 0 & 0 & -f_{11} & 0 & 1 \end{bmatrix}, k_2=\{0\} \qquad (2.19)$$

with

$|C_2| = 1 - f_{11}c_{22} - a_{12}c_{02},$

$$C_3=\begin{bmatrix} 1 & 0 & 0 & -a_{23} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -c_{01} & 0 & 1 & 0 & 0 \\ -e_{03} & 0 & 0 & 1 & 0 \\ -f_{03} & 0 & 0 & f_{03}a_{23} & 1-f_{03}a_{23} \end{bmatrix},\ k_3=\begin{bmatrix} 0 \\ -b_{13} \\ 0 \\ -e_{23} \\ -f_{13} \end{bmatrix} \qquad (2.20)$$

with

$|C_3| = (1 - a_{23}f_{03})\ (1 - a_{23}e_{03})$,

$$C_4=\begin{bmatrix} 1 & 0 & 0 & 0 & -a_{24} \\ 0 & 1 & 0 & 0 & -b_{24} \\ -c_{02} & 0 & 1 & 0 & -c_{22} \\ -e_{03} & 0 & 0 & 1 & 0 \\ -f_{03} & 0 & 0 & 0 & 1 \end{bmatrix},\ k_4=\begin{bmatrix} -a_{14} \\ -b_{14} \\ 0 \\ -e_{13} \\ -f_{13} \end{bmatrix} \qquad (2.21)$$

with

$|C_4| = 1 - f_{03}a_{24}$,

$$C_5=\begin{bmatrix} 1 & 0 & -a_{11} & -a_{21} & 0 \\ 0 & 1 & -b_{11} & -b_{21} & 0 \\ 0 & -c_{05} & 1 & -c_{25} & 0 \\ 0 & -e_{05} & -e_{15} & 1 & 0 \\ 0 & -f_{05} & -f_{15} & 0 & 1 \end{bmatrix}, \qquad k_5=\{0\} \qquad (2.22)$$

with

$|C_5| = 1 - b_{11}c_{25}e_{05} - c_{05}e_{25}b_{21} - b_{21}e_{05} - b_{11}c_{05} - c_{25}e_{15}$,

$$C_6 = \begin{bmatrix} 1 & 0 & -a_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 & -b_{22} \\ 0 & 0 & 1 & 0 & -c_{26} \\ 0 & -e_{05} & -e_{15} & (1-e_{05}b_{22}-e_{15}c_{26}) & (e_{05}b_{22}+e_{15}c_{26}) \\ 0 & -f_{05} & -f_{15} & 0 & 1 \end{bmatrix} \quad (2.23)$$

$k_6 = \{0\}$, with $|C_6| = (1 - e_{05}b_{22} - e_{15}c_{26})\ (1 - b_{22}f_{05} - c_{26}\ f_{15})$,

$$C_7 = \begin{bmatrix} 1 & 0 & 0 & -a_{23} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -c_{05} & (1-c_{05}b_{13}-c_{25}e_{17}) & -c_{25} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -f_{07} & 0 & 0 & 1 \end{bmatrix}, k_7 = \begin{bmatrix} 0 \\ -b_{13} \\ c_{05}b_{13}+c_{25}e_{17} \\ -e_{17} \\ 0 \end{bmatrix} \quad (2.24)$$

with

$|C_7| = 1 - c_{05}b_{13} - c_{25}e_{17}$

$$C_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & -a_{24} \\ 0 & 1 & 0 & 0 & -b_{24} \\ 0 & 0 & 1 & 0 & -c_{26} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -f_{07} & 0 & 0 & 1 \end{bmatrix}, \ k_8 = \begin{bmatrix} -a_{14} \\ -b_{14} \\ 0 \\ -e_{17} \\ 0 \end{bmatrix} \quad (2.25)$$

with

$|C_8| = 1 - f_{07}b_{24}$

If we choose the parameter values such that all the above determinants are different from zero, we say that the resulting system of equations is complete (since, in general, the coherency conditions are fulfilled, see Gourieroux-Laffont-Monfort (1980b)). We already know from the appendix that the above system is statistically identifiable in that case.

In the last section of this paper, we will evaluate the C_i- matrices (i=1,2,...,8) and the corresponding determinants according to the effective supply and demand functions, derived in section 1. It is well understood that this evaluation is valid only in the case that both producers and consumers know the restrictions perceived in the various markets. If they do not know their quantity constraints and, hence, all kinds of stocks will generally occur, a similar evaluation can still be performed straightforwardly. Of course, the various specifications would then not be derived from programming problems as in, e.g., in (1.14), but could be characterised by a more ad hoc specification in such a case.

3. A Functional Evaluation of the Sample Likelihood

In this section we evaluate the likelihood function (2.17), where the parameters involved in the effective supply and demand functions, derived in section 1, are used for the C_i-matrices (2.18-25).
As stated previously, the basic assumption is that the agents know the restrictions perceived on the (other) markets, then. Since we have used Taylor approximations in the expressions for the producer's effective supply and demands (see (1.21) and (1.22)), the fundamental identity property cannot straightforwardly be applied by writing the Walrasian quantities as effective functions where the realised transactions are substituted directly by the Walrasian quantities.
In order to have Walrasian demand and supply functions which contain Walrasian quantities, we adopt the following procedure. Deriving $\ln p_t$ from the Walrasian commodity supply (1.11), $\ln w_t$ from the Walrasian demand for working hours (1.13) and substituting these expressions in the Walrasian demand for workers (1.12), we get:

$$\ln x_{1t}^D = \frac{1}{\mu} \ln y_t^S + \ln x_{2t}^D - \frac{2}{\mu(2-\mu)} \ln A - \frac{1}{2-\mu} \ln \mu + \frac{\mu - 1}{\rho(2-\mu)} \ln (1-\delta)$$
$$+ \frac{3 - \mu}{\rho(2-\mu)} \ln \delta + \frac{\rho + 2}{\rho(2-\mu)} \ln 2 + \frac{1}{2-\mu} \ln w_t - \frac{1}{2-\mu} \ln p_t$$
$$- \frac{2\lambda}{\mu(2-\mu)} t + \frac{1}{2-\mu} \ln (I + s_t) \qquad (3.1)$$

Similarly, deriving $\ln w_t$ from $\ln x_{1t}^D$ (1.12) and $\ln p_t$ from $\ln y_t^S$ (1.11), we get the following expression for the Walrasian demand for working hours.

$$\ln x_{2t}^D = \frac{1}{\mu} \ln y_t^S + \ln x_{1t}^D - \frac{2}{\mu(2-\mu)} \ln A - \frac{1}{2-\mu} \ln \mu$$
$$+ \frac{2}{\rho(2-\mu)} \ln \delta + \frac{\rho + 2}{\rho(2-\mu)} \ln 2 + \frac{1}{2-\mu} \ln w_t$$
$$- \frac{1}{2-\mu} \ln p_t + \frac{1}{2-\mu} \ln (1+s_t) - \frac{2\lambda}{\mu(2-\mu)} t \qquad (3.2)$$

Finally, deriving an expression for $\ln p_t$ from $\ln x_{1t}^D$ and for $\ln w_t$ from $\ln x_{2t}^D$, we get

$$\ln y_t^S = u \ln x_{1t}^D + \mu \ln x_{2t}^D + \frac{2(1-\mu)}{2-\mu} \ln A - \frac{\mu}{2-\mu} \ln \mu -$$
$$- \frac{\mu(1-\mu)}{\rho(2-\mu)} \ln \delta - \frac{\mu(1-\mu)}{\rho(2-\mu)} \ln (1-\delta) + \frac{\mu(2\mu+\rho-2)}{\rho(2-\mu)} \ln 2 + \frac{\mu}{2-\mu} \ln w_t$$
$$- \frac{\mu}{2-\mu} \ln p_t + \frac{\mu}{2-\mu} \ln (1+s_t) + \frac{2(1-\mu)}{2-\mu} \lambda t \qquad (3.3)$$

If the agents know the restrictions, perceived on the other markets, we can replace the general functions of the QRM of section 2, by the corresponding effective supply and demand functions of consumers and producers of section 1 (and (3.1-3)). The determinants $|C_i|(i=1,2,...,8)$ can also be computed with the help of these functions.

Regime 1 was characterised by the following simultaneous equations system (see (2.11) and (2.18)):

$$\left[\begin{array}{l} y^{D'} = a_{11}x_1^{D'} + a_{21}x_2^{D'} + \delta_{01} \\ y^S = b_{11}x_1^D + b_{21}x_2^D + \lambda_{01} \\ x_1^{D'} = c_{01}y^{D'} + \delta_{11} \\ x_2^{D'} = e_{01}y^{D'} + \delta_{21} \\ x_2^{S'} = f_{11}x_1^{D'} + \lambda_{21} \end{array}\right. \qquad (3.4)$$

After replacing these equations by the corresponding log-linear functions derived previously, we have

$$\left[\begin{array}{ll} \ln y^{D'} = -\gamma_6 \frac{1-\alpha_3}{1-\alpha_1} \ln x_1^{D'} - \gamma_7 \frac{1-\alpha_3}{1-\alpha_1} \ln x_2^{D'} + \delta_{01} & \text{(from (1.48))} \\ \ln y^S = \mu \ln x_1^D + \mu \ln x_2^D + \lambda_{01} & \text{(from (3.3))} \\ \ln x_1^{D'} = \frac{1}{\mu} \ln y^{D'} + \delta_{11} & \text{(from (1.15))} \\ \ln x_2^{D'} = \frac{1}{\mu} \ln y^{D'} + \delta_{21} & \text{(from (1.16))} \\ \ln x_2^{S'} = -\gamma_2(\frac{1-\alpha_3}{1-\alpha_2}) \ln x_1^{D'} + \lambda_{21} & \text{(from (1.41))} \end{array}\right. \qquad (3.5)$$

The determinant $|C_1|$ was found to be equal to $(1-b_{11}c_{01}-b_{21}e_{01})$ $(1-a_{21}e_{01}-a_{11}c_{01})$ (see (2.18)).

Substituting the parameters involved in (3.5), we get:

$$|C_1|=-\{1+\frac{1}{\mu}\frac{1-\alpha_3}{1-\alpha_1}(\gamma_7+\gamma_6)\}, \tag{3.6}$$

which is always different from zero*)

Proceeding in a similar way for the other regimes, we get respectively

$$|C_2| = 1 - \frac{1-\delta}{\delta}\frac{1-\alpha_3}{1-\alpha_2}\gamma_2 + \frac{1}{\mu\delta}\frac{1-\alpha_3}{1-\alpha_1}\gamma_2 \tag{3.7}$$

$$|C_3| = \{1 + \frac{1-\alpha_3}{1-\alpha_1}\gamma_1\frac{1}{\mu(1-\delta)}\}\{1 - \frac{1-\alpha_3}{1-\alpha_2}\gamma_1\frac{1-\alpha_3}{1-\alpha_2}\gamma_4\} \tag{3.8}$$

$$|C_4| = 1 + \frac{1-\alpha_3}{1-\alpha_1}\frac{\gamma_7}{\mu(1-\delta)} \tag{3.9}$$

$$|C_5| = -4 \tag{3.10}$$

$$|C_6| = 1 - \frac{2(\delta\rho+\delta-\rho) - \mu(1-\delta)}{\delta\mu+\delta\rho-\rho-1} \tag{3.11}$$

$$|C_7| = 1 + \frac{\delta(2\rho+2+\mu)}{(\mu+\rho)(1-\delta)-\rho-1} \tag{3.12}$$

$$|C_8| = 1 + \mu(1-\delta)\gamma_1\frac{1-\alpha_3}{1-\alpha_2} \tag{3.13}$$

All the above determinants are seen to be different from zero. Hence, we have obtained a <u>complete</u> and <u>statistically identified</u> QRM, which will be estimated empirically in a subsequent paper.

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*In Meersman and Plasmans (1982) it has been derived that the determinants of the transition matrices belonging to regimes 1,5 and 7 are zero ($|C_1|=|C_5|=|C_7|=0$) when the producer's problem is described by a Cobb-Douglas technology. In such a case the corresponding 3 systems of (linearised) equations cannot be solved uniquely and the system is said to be incomplete. It was found in the above cited paper that the assumption of a Cobb-Douglas production technology is responsible for this "degeneration".

Conclusion

In this paper we have derived a complete and statistically identifiable QRM for 3 markets based on a Johansen-type utility maximisation process for the consumer and a (short-run) C.E.S.-production technology in two different labour inputs under competitive profit maximisation for the producer. The procedure we have used is a generalisation of the estimation method for two markets in disequilibrium of Gourieroux-Laffont-Monfort (1980a). The latter method is proved to be non identifiable in general. The identification problem is solved by taking the labour force participation as an exogenous variable.

In a subsequent paper, this 3-market-model will be statistically estimated for the Belgian manufacturing sector.

Appendix: Statistical identifiability of the 2-market model in Gourieroux-Laffont-Monfort (1980a)

Gourieroux-Laffont-Monfort are giving the following relationship in their notation:

$$\begin{pmatrix} \delta_{1t} \\ \lambda_{1t} \\ \delta_{2t} \\ \lambda_{2t} \end{pmatrix} = \sum_{i=1}^{4} A_i \mathbb{1}_{C_i} \begin{pmatrix} D_{1t} \\ S_{1t} \\ D_{2t} \\ S_{2t} \end{pmatrix} = \Lambda X_t + u_t \quad \text{(see also (2.3))}$$

with

δ_{it}, λ_{it}	functions of prices and other variables
X_t	a vector of exogenous variables
Λ	a matrix of unknown parameters
u_t	a (4x1) random vector having a density g

$C_1: = \{D_1 > S_1,\ D_2 > S_2\}$ $\quad D_1$, S_1 effective demand and supply in market 1

$C_2: = \{D_1 > S_1,\ D_2 < S_2\}$

$C_3: = \{D_1 < S_1,\ D_2 < S_2\}$ $\quad D_2$, S_2 effective demand and supply in market 2

$C_4: = \{D_1 < S_1,\ D_2 > S_2\}$

$\mathbb{1}_{C_i}(x) = 1 \iff x \in C_i$

A_i are matrices of spill-over coefficients

$$A_1 = \begin{bmatrix} 1-\alpha_1\beta_2 & \alpha_1\beta_2 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\beta_1 \\ 0 & -\alpha_2 & 1-\alpha_2\beta_1 & \alpha_2\beta_1 \\ 0 & -\beta_2 & 0 & 1 \end{bmatrix} \quad \text{(compare with (2.13))}$$

$$A_2 = \begin{bmatrix} 1 & 0 & -\alpha_1 & 0 \\ 0 & 1 & -\beta_1 & 0 \\ 0 & -\alpha_2 & 1 & 0 \\ 0 & -\beta_2 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & -\alpha_1 & 0 \\ \alpha_2\beta_1 & 1-\alpha_2\beta_1 & -\beta_1 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\beta_2 & 0 & \alpha_1\beta_2 & 1-\alpha_1\beta_2 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\beta_1 \\ -\alpha_2 & 0 & 1 & 0 \\ -\beta_2 & 0 & 0 & 1 \end{bmatrix}$$

$Q_1 := \min \{D_1, S_1\}$
$Q_2 := \min \{D_2, S_2\}$

The density of u_t is given by $g(u_t)$
The density of (Q_1, Q_2) is

$$h_t (Q_1, Q_2) = \int_{Q_1}^{\infty} \int_{Q_2}^{\infty} \{|A_1| g(A_1\{Q_1, x, Q_2, y\}' - \Lambda X_t)$$
$$+ |A_2| g(A_2\{Q_1, x, y, Q_2\}' - \Lambda X_t)$$
$$+ |A_3| g(A_3\{x, Q_1, Q_2, y\}' - \Lambda X_t)$$
$$+ |A_4| g(A_4\{x, Q_1, y, Q_2\}' - \Lambda X_t)\} \, dxdy\}$$ (see also (2.5))

The likelihood function $L_\Theta(X)$ satisfies:

$$L_\Theta(X) = \prod_{t=1}^{T} h_t(Q_1, Q_2)$$

with

$\Theta: = (\alpha_1, \alpha_2, \beta_1, \beta_2, \Lambda, \xi)' \in \text{Ⓗ}$

$\text{Ⓗ}: = \{\theta | 1-\alpha_1\beta_2>0;\ 1-\alpha_2\beta_1>0;\ 1-\alpha_1\alpha_2>0;$
$1-\beta_1\beta_2>0;\ \alpha_1,\alpha_2,\beta_1,\beta_2>0;\ \Lambda$ is a
4xk matrix with k the number of exogenous variables;$\xi \in Z\}$,

ξ is the parameter, defining the probability density of u_t.
Let $\mathcal{P}: = \{H_\Theta : \Theta \varepsilon \text{ Ⓗ}\}$, where H_Θ is the distribution of the stochastic vector with density L_Θ (X). The class, $\mathcal{P}$ is said to be identified by Ⓗ if

$$\forall\ \theta_1, \theta_2 \in \text{Ⓗ}: \theta_1 \neq \theta_2 \rightarrow L_{\theta_1}(X) \neq L_{\theta_2}(X)$$

$$<=>$$

$$h_{\theta_1}(Q_1, Q_2) \neq h_{\theta_2}(Q_1, Q_2)$$

When we write out $h_{\theta_1}(Q_1, Q_2)$ with

$\Lambda_1 X_t \rightarrow$ first row of ΛX_t

$\Lambda_2 X_t \rightarrow$ second row of ΛX_t

and $\theta_1: = (\alpha_1,\beta_1,\alpha_2,\beta_2,\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4,\xi)$

$h_{\theta_1}(Q_1,Q_2) =$

$$\int_{Q_1}\int_{Q_2} \{ (1-\alpha_1\beta_2)(1-\beta_1\alpha_2)(1-\alpha_1\alpha_2) g \left\{ \begin{bmatrix} Q_1-\alpha_1 Q_2-\Lambda_1 X_t \\ \alpha_2\beta_1 Q_1+(1-\beta_1\alpha_2)x-\beta_1 Q_2-\Lambda_2 X_t \\ -\alpha_2 Q_1+Q_2-\Lambda_3 X_t \\ -\beta_2 Q_1+\alpha_1\beta_2 Q_2+(1-\beta_2\alpha_1)y-\Lambda_4 X_t \end{bmatrix} \right\}$$

$$+(1-\alpha_1\beta_2)g\left\{\begin{bmatrix} Q_1-\alpha_1 Q_2-\Lambda_1 X_t \\ x-\beta_1 Q_2-\Lambda_2 X_t \\ -\alpha_2 Q_1+y-\Lambda_3 X_t \\ -\beta_2 Q_1+Q_2-\Lambda_4 X_t \end{bmatrix}\right\}$$

$$+(1-\alpha_2\beta_1)g\left\{\begin{bmatrix} x-\alpha_1 Q_2-\Lambda_1 X_t \\ Q_1-\beta_1 Q_2-\Lambda_2 X_t \\ -\alpha_2 Q_1+Q_2-\Lambda_3 X_t \\ -\beta_2 Q_1+y-\Lambda_4 X_t \end{bmatrix}\right\}$$

$$+(1-\alpha_1\beta_2)(1-\beta_1\alpha_2)(1-\beta_1\beta_2)g\left\{\begin{bmatrix} (1-\alpha_1\beta_2)x+\alpha_1\beta_2 Q_1-\alpha_1 Q_2-\Lambda_1 X_t \\ Q_1-\beta_1 Q_2-\Lambda_2 X_t \\ -\alpha_2 Q_1+(1-\alpha_2\beta_1)y+\alpha_2\beta_1 Q_2-\Lambda_3 X_t \\ -\beta_2 Q_1+Q_2-\Lambda_4 X_t \end{bmatrix}\right\}\Big\}dx\,dy$$

If we consider a parameter set $\theta_2 \neq \theta_1$

$$\theta_2 := (\beta_1,\alpha_1,\beta_2,\alpha_2,\Lambda_2,\Lambda_1,\Lambda_4,\Lambda_3)$$

Then we must have, in order to obtain a statistically identified model, that $h_{\theta_1}(Q_1,Q_2) \neq h_{\theta_2}(Q_1,Q_2)$.

However, $h_{\theta_1}(Q_1,Q_2) = h_{\theta_2}(Q_1,Q_2)$ as

$$h_{\theta_2}(Q_1,Q_2) = \int_{Q_1}\int_{Q_1} \Big\{(1-\beta_1\alpha_2)(1-\alpha_1\beta_2)(1-\beta_1\beta_2)g\left\{\begin{matrix} Q_1-\beta_1 Q_2-\Lambda_2 X_t \\ \beta_2\alpha_1 Q_1+(1-\alpha_1\beta_2)x-\alpha_1 Q_2-\Lambda_1 X_t \\ -\beta_2 Q_1+Q_2-\Lambda_4 X_t \\ -\alpha_2 Q_1+\beta_1\alpha_2 Q_2+(1-\alpha_2\beta_1)y-\Lambda_3 X_t \end{matrix}\right\}$$

$$+(1-\beta_1\alpha_2)g\begin{bmatrix} Q_1-\beta_1 Q_2-\Lambda_2 X_t \\ x-\alpha_1 Q_2-\Lambda_1 X_t \\ -\beta_2 Q_1+y-\Lambda_4 X_t \\ -\alpha_2 Q_1+Q_2-\Lambda_3 X_t \end{bmatrix}$$

$$+(1-\beta_2\alpha_1)g\begin{bmatrix} x-\beta_1 Q_2-\Lambda_2 X_t \\ Q_1-\alpha_1 Q_2-\Lambda_1 X_t \\ -\beta_2 Q_1+Q_2-\Lambda_4 X_t \\ -\alpha_2 Q_1+y-\Lambda_3 X_t \end{bmatrix}$$

$$+(1-\beta_1\alpha_2)(1-\alpha_1\beta_2)(1-\alpha_1\alpha_2)g\left\{\begin{bmatrix} (1-\beta_1\alpha_2)x + \beta_1\alpha_2 Q_1 - \beta_1 Q_2 - \Lambda_2 X_t \\ Q_1-\alpha_1 Q_2-\Lambda_1 X_t \\ -\beta_2 Q_1 + (1-\beta_2\alpha_1)y + \beta_2\alpha_1 Q_2 - \Lambda_4 X_t \\ -\alpha_2 Q_1 + Q_2 - \Lambda_3 X_t \end{bmatrix}\right\} dxdy$$

which is equal to $h_{\theta_1}(Q_1, Q_2)$, if we change the parameters in the argument of the error density g by permuting the role of the first and the second component on the one hand and the role of the third and fourth component on the other hand.
The conclusion is that the 2-market-model in Gourieroux-Laffont-Monfort (1980a) is not statistically identified. Hence, exchanging the demand and supply equations on each market, we have the same density for the realisations (Q_1, Q_2). A statistically identified QRM could,however,be obtained if the set of exogenous variables is not the same in both equations. This could, e.g. (a fortiori) be reached by making either endogenous variable exogenous.

Since a similar (but more elaborate) proof can be followed for our 3-market QRM in section 2, we propose to exogenize the labour force participation rate determining the workers' supply of labour (which is already principally determined by exogenous demographic factors (see Meersman and Plasmans (80), pp. 22-24). Note, however, that exogenising the workers' supply is sufficient, but in no way necessary for statistical identification of the 3-market QRM!

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