SESO UNIVERSITAIRE FACULTEITEN ST.-IGNATIUS TE ANTWERPEN

STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

THE OPTIMAL SUPPLY OF QUALITIES UNDER INCREASING RETURNS TO SCALE

W. Pauwels*

Rapport 81/120
October 1981

*This paper was written while the author was on leave at the Center for Operations Research and Econometrics, Université Catholique de Louvain. The author wants to thank professor K.J. Lancaster of Columbia University with whom he discussed parts of this paper. He also wants to thank professor M. Weverbergh for computational assistance.

Universitaire Faculteiten St.-Ignatius
Prinsstraat 13 - 2000 Antwerpen

D/1981/1169/18

ABSTRACT

Suppose a particular commodity can be produced in an infinite number of different qualities, and that, for any given quality, production occurs under increasing returns to scale.
If then consumers differ in their tastes for quality, two conflicting forces are at work: on the one hand, there is a need for variety of products; on the other hand, there is a cost advantage from concentrating production on a limited number of products. The following questions then arise. How many, and which, qualities should be produced, and how much of each quality should be produced?
These questions are investigated in the case of a monopoly, and from a welfare point of view.

1. INTRODUCTION

In an interesting article, M. Mussa and S. ROSEN /7/ analysed the optimal supply of qualities by a monopolist, and compared this optimum with the welfare optimum. They assumed constant returns to scale in production so that the focus of their analysis is on the role of demand conditions and on substitution by consumers among the varieties available. They show that, in both the monopoly and the welfare case, a continuum of qualities will be produced.

In this paper we extend their analyses to the case where production occurs under increasing returns to scale. This is an important extension. Indeed, in this case there are two conflicting forces at work: on the one hand, consumers demand variety; on the other hand, producers have a cost advantage from concentrating production on a finite number of products. See, e.g. J.F. Meade /6/, and K.J. Lancaster /4/, /5/. In contrast to the constant returns to scale case, only a finite number of qualities will be produced, in both the monopoly and the welfare case. The following questions then arise: How many, and which, varieties should be produced? How much of each variety should be produced? And, in the monopoly case, how should these varieties be priced? These are the questions we want to answer.

The paper is organized as follows. In the second section we study the demand conditions of the model. These conditions are derived from those used in M. Mussa and S. Rosen. Special attention is given to the case where only a finite number of different qualities is offered. In the third and fourth section we derive the monopoly solution and the welfare optimum. A numerical example is given in the last section.

2. DEMAND CONDITIONS

Consider a commodity which can be produced in different qualities. Let quality be measured by a real number q, $q \geq 0$, higher qualities being represented by higher values of q. Buyers purchase only one unit of the good, but they have a choice among the various qualities offered. The utility function for all consumers has the form

$$U(x,q;\theta) = x + \theta q$$

where x is a composite commodity, representing all other commodities. θ indexes consumer types. It measures the willingness to pay, in terms of x, for an increase in q. Consumers differ only in θ. Preferences of all potential consumers are described by a density function $f(\theta)$, defined over an interval $[\underline{\theta},\bar{\theta}]$. We assume f is continuously differentiable and positive over this interval.

Consider now a monopolist who announces a price for every quality offered. Let this price schedule be given by a function P(q). A given quality can therefore be sold only at one price, which is the same for all consumers.

A consumer of type θ then faces the following problem

$$\max_{x,q} \quad x + \theta q$$

$$\text{s.t. } P(q) + x \leq y$$
$$q \geq 0, \; y \geq 0$$

y represents income, and commodity x is used as numéraire. If the price schedule P(q) is continuously differentiable, and if the optimal values $\hat{x}$ and $\hat{q}$ are positive, one obtains the optimality condition

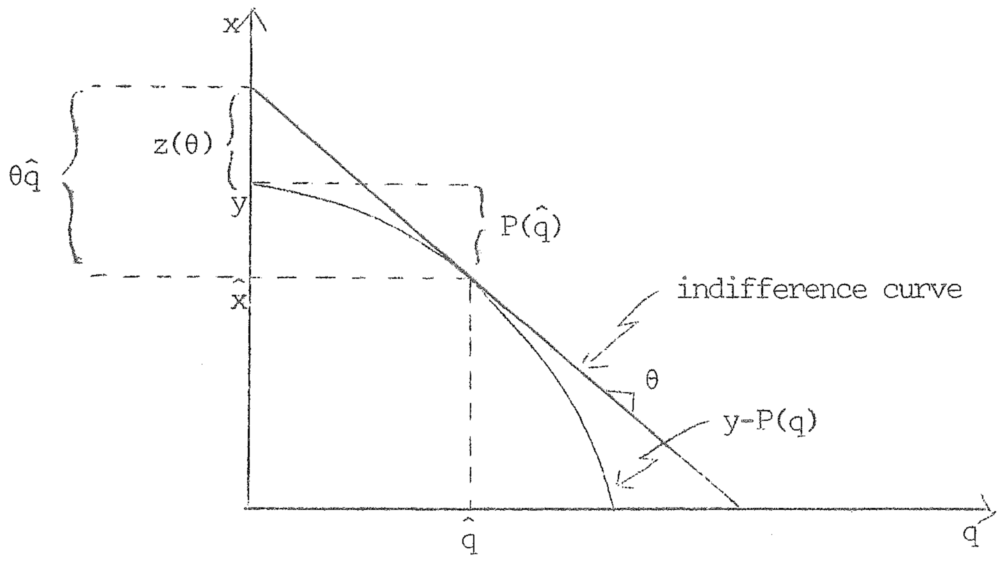
$$P'(\hat{q}) = \theta \qquad (1)$$

This equation gives the demand for q as a function of θ (independent of y). The second order condition requires that

$$P''(\bar{q}) > 0 \qquad (2)$$

(1) and (2) imply that consumers with a higher value of θ demand a higher quality. Consumer equilibrium is represented on figure 1.

Figure 1.



Up to now, we have described the monopolist's behavior in terms of the price schedule P(q). This behavior, however, can also equivalently be described in terms of an assignment function $q(\theta), \theta \varepsilon [\underline{\theta}, \bar{\theta}]$, whereby the monopolist assigns qualities to all consumer types. Consumer behavior requires this function to be non decreasing. The relationship between $q(\theta)$ and P(q) is derived as follows. Denote consumer surplus enjoyed by a consumer of type θ by $z(\theta)$. Then, by definition,

$$z(\theta) = \theta q(\theta) - P\{q(\theta)\} \qquad (3)$$

If $q(\theta)$ and P(q) are continuously differentiable, one can take derivatives, so that

$$\dot{z}(\theta) = q(\theta) + \theta \dot{q}(\theta) - P'\{q(\theta)\}\dot{q}(\theta)$$

For $q(\theta)$ to be optimal for a consumer of type θ, (1) has to satisfied, so that

$$\dot{z}(\theta) = q(\theta) \qquad (4)$$

Define, now the moponolist's effective supply interval $[t_o, \bar{\theta}]$, $t_o \geq \underline{\theta}$, by

$$\forall \theta \varepsilon [\underline{\theta}, t_o), \ q(\theta) = 0$$
$$\forall \theta \varepsilon (t_o, \bar{\theta}], \ q(\theta) > 0$$

If then follows from (4) that, if the monopolist determines t_o, $z(t_o)$ and $q(\theta)$ for $\theta \varepsilon [t_o, \bar{\theta}]$, $z(\theta)$ is also determined over $[t_o, \bar{\theta}]$. Prices are then given from (3) by

$$P\{q(\theta)\} = p(\theta) = \theta q(\theta) - z(\theta) \qquad (5)$$

The monopolist's total revenue is given by

$$\int_{t_o}^{\bar{\theta}} \{\theta q(\theta) - z(\theta)\} f(\theta) d\theta \qquad (6)$$

We will now rewrite (6) in a simpler form. It is clear that the monopolist will never grant a consumer of type t_o a positive consumer surplus, so that $z(t_o) = 0$. The solution of (4) is then

$$z(\theta) = \int_{t_o}^{\theta} q(\sigma) d\sigma$$

Using this solution in (6), and changing the order of integration, gives

$$\int_{t_o}^{\bar{\theta}} \theta q(\theta) f(\theta) d\theta - \int_{t_o}^{\bar{\theta}} \int_{t_o}^{\theta} q(\sigma) d\sigma f(\theta) d\theta =$$

$$\int_{t_o}^{\bar\theta} \theta q(\theta)f(\theta)d\theta - \int_{t_o}^{\bar\theta} \int_{\sigma}^{\bar\theta} f(\theta)d\theta q(\sigma)d\sigma =$$

$$\int_{t_o}^{\bar\theta} \theta q(\theta)f(\theta)d\theta - \int_{t_o}^{\bar\theta} q(\sigma) \{1-F(\sigma)\}d\sigma =$$

$$\int_{t_o}^{\bar\theta} q(\theta)\{\theta f(\theta) - [1-F(\theta)]\}d\theta$$

where $F(\theta) = \int_{\underline{\theta}}^{\theta} f(\sigma)d\sigma$.

Defining now

$$R'(\theta) = \theta f(\theta) - \{1 - F(\theta)\} \qquad (7)$$

the monopolist's total revenue is given by the simple expression

$$\int_{t_o}^{\bar\theta} q(\theta) R'(\theta)d\theta \qquad (8)$$

Up to now we have assumed that $q(\theta)$ is a continuously differentiable non-decreasing function. Suppose now that the monopolist chooses for $q(\theta)$ a non-decreasing step function, with jumps occurring at the points $t_1, t_2 \ldots, t_{N-1}$ where

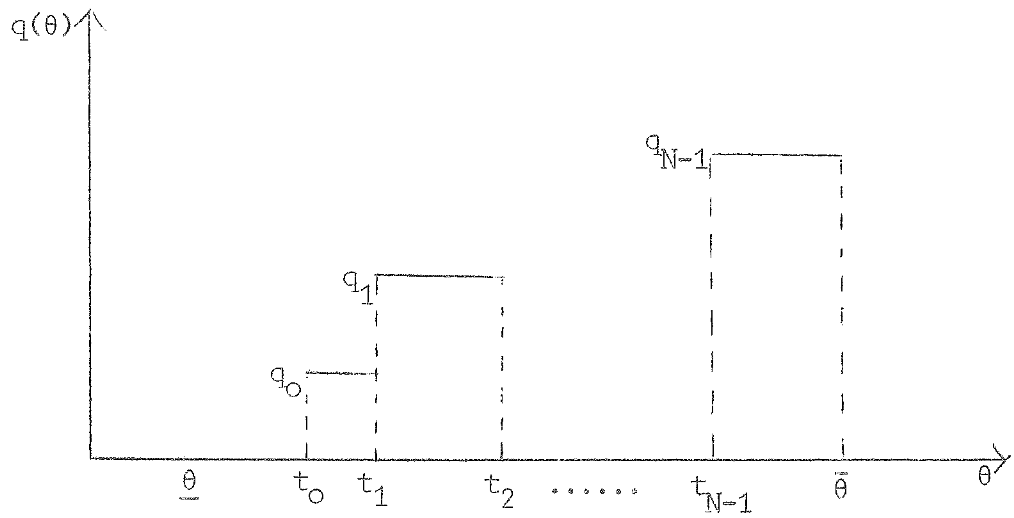
$$\underline{\theta} \leqq t_o < t_1 < t_2 \ldots < t_{N-1} < t_N = \bar\theta \qquad (9)$$

and with corresponding values

$$0 < q_o < q_1 < q_2 \ldots < q_{N-1} \qquad (10)$$

Such a function is drawn on figure 2. It implies that the monopolist bunches consumers of different tastes on to the same quality.

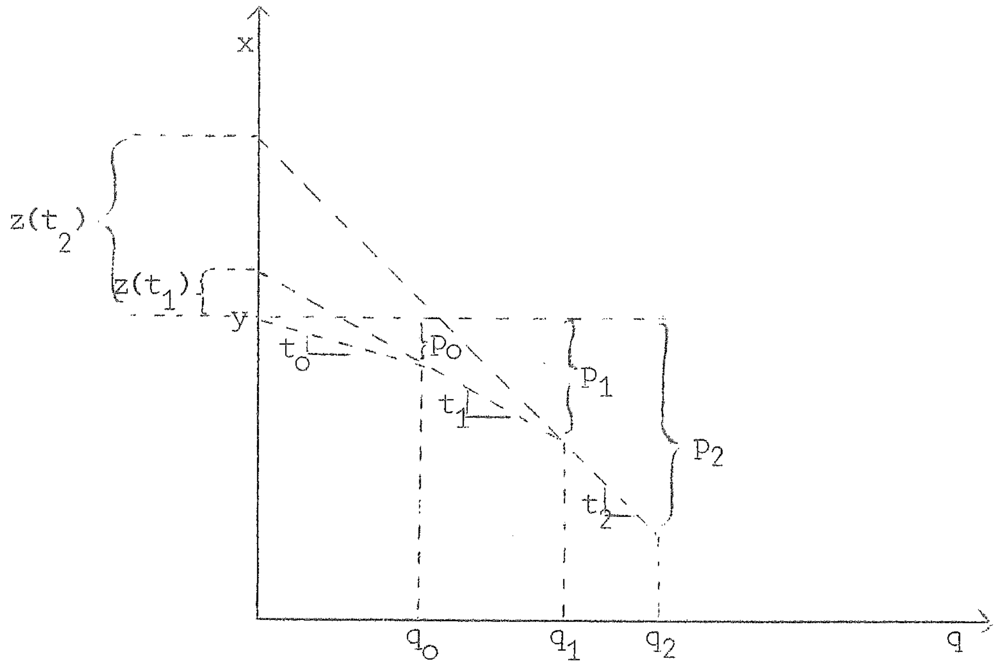
Figure 2.



In terms of figure 1, the P(q) schedule should now be replaced by N discrete points $p_o=P(q_o)$, $p_1=P(q_1)\ldots,p_{N-1}=P(q_{N-1})$. How can these prices be determined from a given step function $q(\theta)$?

The geometry of the answer is given in figure 3, which has to be compared with figure 1. According to figure 2, a consumer with θ equal to t_o should be made indifferent between not buying at all, or buying a quality q_o. Given the values of q_o and t_o, this fixes p_o at $q_o t_o$. A consumer with θ equal to t_1 should be made indifferent between buying quality q_o and buying quality q_1. This fixes p_1 at $q_1 t_1 - q_o(t_1 - t_o)$. In this way, the whole price schedule p_o, p_1, ..., p_{N-1} is traced out.

Figure 3.



One then finds

$$P_o = q_o t_o$$
$$P_1 = q_1 t_1 - q_o(t_1 - t_o)$$
$$P_2 = q_2 t_2 - q_o(t_1 - t_o) - q_1(t_2 - t_1)$$

or

$$P_i = q_i t_i - \sum_{j=0}^{i-1} q_j(t_{j+1} - t_j), \quad i:1,2,\ldots,N-1 \tag{11}$$

This result can also be obtained from (4) and (5). Start with solving (4) over the interval (t_o, t_1), taking $z(t_o)=0$. If this solution has the value $z(t_1)$ at t_1, consider (4) over the interval (t_1, t_2), using $z(t_1)$ as initial value. Proceeding in this way, we obtain a continuous solution $z(\theta)$ over $[t_o, \bar{\theta}]$. This solution is given by

$$z(\theta) = q_o(\theta - t_o) \quad \text{for} \quad \theta\varepsilon [t_o, t_1]$$
$$z(\theta) = q_o(t_1 - t_o) + q_1(\theta - t_1) \quad \text{for} \quad \theta\varepsilon [t_1, t_2]$$
$$\ldots$$
$$z(\theta) = \sum_{i=0}^{N-2} q_i(t_{i+1} - t_i) + q_{N-1}(\theta - t_{N-1}) \text{ for } \theta\varepsilon [t_{N-1}, \bar{\theta}]$$

Using this solution in (5), we obtain the prices given by (11).

It then also follows that, in case $q(\theta)$ is a step function, (8) is still valid, so that total revenue can be written as

$$q_o \int_{t_o}^{t_1} R'(\theta)d\theta + q_1 \int_{t_1}^{t_2} R'(\theta)d\theta + \ldots + q_{N-1} \int_{t_{N-1}}^{\bar{\theta}} R'(\theta)d\theta,$$

or as

$$q_o\{R(t_1) - R(t_o)\} + q_1\{R(t_2)-R(t_1)\} +\ldots+q_{N-1}\{R(\bar{\theta})-R(t_{N-1})\} \qquad (12)$$

where

$$R(\theta) = \theta F(\theta) - \theta \qquad (13)$$

Finally, total revenue can also be written as

$$p_oQ_o + p_1Q_1 + \ldots + p_{N-1} Q_{N-1} \qquad (14)$$

where

$$Q_i=F(t_{i+1}) - F(t_i), \ i=0,1,\ldots,N-1 \qquad (15)$$

Using (11), one can easily check that (12) and (14) are, indeed, identical.

It is important to understand the full content of the information captured by the function $R(\theta)$ defined in (13). Let us first consider the effect on total revenue of an increase q_i. Using (12), the marginal revenue of an increase in q_i is given by

$$R(t_{i+1}) - R(t_i) = \int_{t_i}^{t_{i+1}} R'(\theta)d\theta \qquad (16)$$

The meaning of this expression is the following. An increase in q_i, affects the price p_i the monopolist can charge for q_i. Using (11) we have

$$\frac{\partial}{\partial q_i}(p_i Q_i) = Q_i \frac{\partial p_i}{\partial q_i} = Q_i t_i \qquad (17)$$

This is a direct positive effect of q_i on p_i. However, an increase in q_i also (negatively) affects the prices the monopolist can charge on the markets $q_{i+1}, \ldots, q_{N-1}$. From (11), this effect is given by

$$\frac{\partial}{\partial q_i} \sum_{j=i+1}^{N-1} p_j Q_j = \sum_{j=i+1}^{N-1} Q_j \frac{\partial p_j}{\partial q_i} =$$

$$- (t_{i+1} - t_i) \sum_{j=i+1}^{N-1} Q_j \qquad (18)$$

which is negative. The intuitive explanation of this effect is that serving consumers with relatively low values of θ limits the monopolist's possibilities for capturing consumer surplus from consumers with relatively high values of θ. For example, if the monopolist decides to supply q_o at a price $q_o t_o$, consumer surplus of consumers with $\theta \geq t_1$ cannot be lowered below $q_o(t_1 - t_o)$. It follows that, for quality q_1, the monopolist cannot charge a price $q_1 t_1$, because the consumers with $\theta = t_1$ will switch to q_o and enjoy a consumer surplus of $q_o(t_1 - t_o)$. This is why p_1 is not equal to $q_1 t_1$, but to $q_1 t_1 - q_o(t_1 - t_o)$. This same effect is studied in J.J. Gabszewicz /2/.

Adding the direct price effect (17) to the indirect price effects (18) gives exactly expression (16).

In a similar way, the effects on revenue of an increase in t_i can be analysed. From (12), the marginal revenue of an increase in t_i is given by

$$-(q_i - q_{i-1}) R'(t_i) \qquad (19)$$

The interpretation of this expression is as follows. An increase in t_i has a (positive) quantity effect on the q_{i-1}-market, given by

$$p_{i-1} \frac{\partial Q_{i-1}}{\partial t_i} = p_{i-1} f(t_i) \qquad (20)$$

and a (negative) quantity effect on the q_i-market

$$p_i \frac{\partial Q_i}{\partial t_i} = - p_i f(t_i) \qquad (21)$$

It also has a (positive) price effect on p_i,

$$Q_i \frac{\partial p_i}{\partial t_i} = (q_i - q_{i-1}) Q_i \qquad (22)$$

and a (positive) effect on all prices $p_{i+1}, \ldots, p_{N-1}$

$$\sum_{j=i+1}^{N-1} Q_j \frac{\partial p_j}{\partial t_i} = (q_i - q_{i-1}) \sum_{j=i+1}^{N-1} Q_j \qquad (23)$$

Adding (20), (21), (22) and (23) given exactly (19).

3. INCREASING RETURNS TO SCALE: THE MONOPOLY SOLUTION

If we now introduce a cost function, we have a complete model for the monopoly case. The cost function used by M. Mussa and S. Rosen is of the form

$$c(q)Q, \text{ with } c'(q) > 0,\ c''(q) > 0 \qquad (24)$$

where q represents a particular quality, and where Q is the total quantity produced of that quality. This cost function implies that, for a given quality, there are constant returns to scale (constant average cost of production), while marginal costs for higher qualities are increasing.

Combining (8) and (24) the monopolist's profits are given by

$$\int_{t_o}^{\bar{\theta}} \{q(\theta)R'(\theta) - C(q(\theta))f(\theta)\}\, d\theta \qquad (25)$$

This leads to the following control problem. We want to find an arc

$$q(.),\ u(.),\ t_o,\ t_1,\ \ldots,\ t_{N-1}$$

satisfying

$$t_o < t_1 < t_2 \ldots < t_{N-1}$$

$$\dot{q}(\theta) = u(\theta) \text{ for } t_{i-1}<\theta<t_i,\ i=1,\ldots,N$$

$$u(\theta) \geqq \text{ for } t_{i-1}<\theta<t_i,\ i=1,\ldots,N$$

$$\underline{\theta} - t_o \leqq 0$$

$$q(t_i^-) - q(t_i^+) \leqq 0,\ i=1,\ldots,N \qquad (26)$$

which maximizes (25). Inequalities (26) allow for the possibility of jumps in $q(.)$. Together with $u(\theta)\geqq 0$, they imply that $q(.)$ is non-decreasing.

This problem can be solved using methods of control theory. See, e.g., K.J. Arrow and M. Kurz /1, pp.51-57/, or W.M. Getz and D.H. Martin /3/. M. Mussa and S. Rosen solve it using variational methods. Some of their main results are:

(a) there are no jumps in the optimal $q(.)$;

(b) the optimal $q(.)$ consists of connected segments of two types:

(1) segments where $\dot{q}(\theta)>0$. On these segments

$$R'(\theta) = C'\{q(\theta)\}\ f(\theta) \qquad (27)$$

holds

(2) segments where $\dot{q}(\theta)=0$ so that $q(\theta)=\bar{q}=\text{constant} >0$. Let such a segment occur over the interval (t_l, t_u). Then M. Mussa and S. Rosen show that $t_l \geqq \underline{\theta}$, $t_u<\bar{\theta}$, and that

$$R(t_u) - R(t_l) = c'(\bar{q})\{F(t_u) - F(t_l)\} \qquad (28)$$

$$\frac{R'(t_l)}{f(t_l)} = \frac{R'(t_u)}{f(t_u)} = c'(\bar{q}) \qquad (29)$$

We now want to introduce increasing returns to scale into the model. This can be done by replacing (24) by a function

$$C(q,Q) \qquad (30)$$

where $C_q > 0$, $C_{qq} > 0$

$C_Q > 0$, $\frac{C(q,Q)}{Q}$ is decreasing in Q.

It is clear that, given the existence of increasing returns to scale for the production of any quality, the monopolist will no longer produce a continuum of qualities as in M. Mussa and S. Rosen, but will exploit the economies of scale by concentrating production to a finite number of qualities. His problem is then to determine which qualities to produce and how to bunch different consumers onto these qualities.

Assuming that the monopolist produces N different qualities $q_o, q_1, \ldots q_{N-1}$ for consumers with θ within, respectively, the intervals (t_o, t_1), $(t_1, t_2), \ldots, (t_{N-1}, \bar{\theta})$, his profits are given by

$$\pi(t_o, t_1, \ldots t_{N-1}; q_o, q_1, \ldots, q_{N-1}) =$$

$$\sum_{i=0}^{N-1} q_i \{R(t_{i+1}) - R(t_i)\} - \sum_{i=0}^{N-1} C\{q_i, Q_i\} \qquad (31)^1$$

This function has to maximized with respect to t_o, t_1, ..., t_{N-1}, q_o, q_1, ..., q_{N-1}, subject to (9) and (10).

If then $(\hat{t}_0, \hat{t}_1, \ldots, \hat{t}_{N-1}, \hat{q}_0, \ldots, \hat{q}_{N-1})$ satisfies (9) and (10), and maximizes (31), the following first order conditions must hold.

---

[1] It is clear that the specification of the cost functions precludes the existence of any "economies of scope". See e.g. J.C. Panzar and D. Willig /8/. Problems of sustainability of the monopoly are also neglected.

$$\left.\begin{aligned} R(\hat{t}_1) - R(\hat{t}_o) &= \frac{\partial C(\hat{q}_o,\hat{Q}_o)}{\partial q_o} \\ R(\hat{t}_2) - R(\hat{t}_1) &= \frac{\partial C(\hat{q}_1,\hat{Q}_1)}{\partial q_1} \\ &\dots \\ R(\bar{\theta}) - R(\hat{t}_{N-1}) &= \frac{\partial C(\hat{q}_{N-1},\hat{Q}_{N-1})}{\partial q_{N-1}} \end{aligned}\right\} \quad (32)$$

$$\left.\begin{aligned} &\hat{q}_o R'(\hat{t}_o) \geqq \frac{\partial C(\hat{q}_o,\hat{Q}_o)}{\partial Q_o} f(\hat{t}_o) \\ &\{\hat{q}_o R'(t_o) - \frac{\partial C(\hat{q}_o,\hat{Q}_o)}{\partial Q_o} f(\hat{t}_o)\} (\hat{t}_o - \underline{\theta}) = 0 \\ &\underline{\theta} - \hat{t}_o \leqq 0 \end{aligned}\right\} \quad (33)$$

$$\left.\begin{aligned} (\hat{q}_1-\hat{q}_o)R'(\hat{t}_1) &= \frac{\partial C(\hat{q}_1,\hat{Q}_1)}{\partial Q_1} f(\hat{t}_1) - \frac{\partial C(\hat{q}_o,\hat{Q}_o)}{\partial Q_o} f(\hat{t}_1) \\ &\dots \\ (\hat{q}_{N-1}-\hat{q}_{N-2})R'(\hat{t}_{N-1}) &= \frac{\partial C(\hat{q}_{N-1},\hat{Q}_{N-1})}{\partial Q_{N-1}} f(\hat{t}_{N-1}) - \frac{\partial C(q_{N-2},Q_{N-2})}{\partial Q_{N-2}} f(t_{N-2}) \end{aligned}\right\} \quad (34)$$

Conditions (32) state that, for every quality produced, its marginal revenue (as explained in the previous section) equals its marginal cost. If C has the form given in (24), conditions (32) are the same as condition (28). Conditions (33) state that, if there is a decrease in t_o the marginal revenue resulting must not be smaller than the increase in costs due to the increase in Q_o, and that both must be equal if $\hat{t}_o > \underline{\theta}$. Finally, conditions (34) require that the marginal revenue of an increase in t_i be equal to its marginal cost caused by the resulting increase in Q_i and decrease in Q_{i-1}.

It should be stressed the way profit function (31) is specified already implies that N different qualities will be produced. Given this number N, the only questions which are answered by maximizing (31) are the determination of these N qualities, and their distribution among the consumers. The optimal number of qualities can only be determined by comparing maximal profits for different values of N.

4. INCREASING RETURNS TO SCALE: THE WELFARE SOLUTION

When studying the welfare solution, we will assume that a central authority determines an effective supply interval $[t_o, \bar{\theta}]$ and an assignment function $q(\theta)$ defined over this interval. The aggregate willingness to pay for quality of all consumers is then

$$\int_{t_o}^{\bar{\theta}} \theta q(\theta) f(\theta) d\theta \qquad (35)$$

Introducing the function

$$H(\theta) = - \int_{\theta}^{\bar{\theta}} \sigma f(\sigma) d\sigma \qquad (36)$$

with

$$H'(\theta) = \theta f(\theta)$$

(35) can also be written in a form comparable with (8), as

$$\int_{t_o}^{\bar{\theta}} q(\theta) H'(\theta) d\theta \qquad (37)$$

If $q(\theta)$ is a step function, as defined by (9) and (10), (37) takes the form

$$q_o\{(H(t_1)-H(t_o)\}+q_1\{(H(t_2)-H(t_1)\}+\ldots+q_{N-1}\{H(\bar{\theta})-N(t_{N-1})\} \qquad (38)$$

The marginal social benefit of an increase in q_i is then given by

$$H(t_{i+1}) - H(t_i) = \int_{t_i}^{t+1} \theta f(\theta) d\theta$$

while the marginal social benefit of a change in t_i is given by

$$-(q_i-q_{i-1})H'(t_i) = -(q_i-q_{i-1})t_i f(t_i)$$

Note that, in contrast to the monopoly case, there are no upstream interference effects. An increase of q_i over the interval (t_i,t_{i+1}) only affects the consumers within this interval. Also, given q_{i-1} and q_i, a change in t_i only affects the consumers whose equals t_i

Combining the cost function (24) with (37) gives the aggregate net surplus

$$\int_{t_o}^{\bar{\theta}} \{q(\theta)H'(\theta) - C[q(\theta)]f(\theta)\}\, d\theta \qquad (39)$$

M. Mussa and S. Rosen considered the problem of finding an arc q(.), t_o which maximizes (39) subject to $q(\theta)\geq 0$ for $\theta\varepsilon\left[t_o,\bar{\theta}\right]$. The solution is given by the equation

$$\theta = C'\{q(\theta)\} \text{ for } q(\theta) > 0 \qquad (40)$$

Hence, the optimal assignment function never has jumps, nor flat segments. Also, the form of the density function f is irrelevant for the determination of q(.).

If, however, we introduce increasing returns to scale, it will again be optimal to concentrate production on a finite number of qualities. Assuming that the qualities $q_o,q_1,\ldots,q_{N-1}$ are produced and distributed to the consumers with within respectively the intervals (t_o,t_1), $(t_1,t_2)\ldots,(t_{N-1},\bar{\theta})$; aggregate net surplus is given by

$$S(t_o,t_1,\ldots,t_{N-1};q_o,q_1\ldots,q_{N-1}) =$$

$$\sum_{i=0}^{N-1} q_i\,\{H(t_{i+1}) - H(t_i)\} - \sum_{i=0}^{N-1} C(q_i,Q_i) \qquad (41)$$

This function has to be maximized with respect to $t_o, t_1, \ldots, t_{N-1}$, $q_o, q_1, \ldots, q_{N-1}$, subject to (9) and (10).

If then $(\hat{t}_o, \hat{t}_1, \ldots, \hat{t}_{N-1}; \hat{q}_o, \hat{q}_1, \ldots, \hat{q}_{N-1})$ satisfies (9) and (10), and maximizes (41), the first order conditions are of the form given by (32), (33) and (34) where the function R should now be replaced by the function H. The interpretation of these conditions is, of course, similar to the one given in the monopoly case.

5. A NUMERICAL EXAMPLE

An important conclusion of the analysis of M. Mussa and S. Rosen is that the effective supply interval will be smaller in the monopoly solution than in the welfare solution. Also, a consumer with $\theta<\bar{\theta}$ who is served in both solutions will receive a lower quality in the monopoly solution than in the welfare solution. The basic reason is that the marginal revenue of serving a consumer of type θ with a quality $q(\theta)$ generates, in the monopoly case, a marginal revenue equal to $R'(\theta)$, while, in the welfare case, this marginal revenue is $H'(\theta)$. Because of the negative upstream interference effect in the monopoly case, we have

$$R'(\theta) = \theta f(\theta) - \{1-F(\theta)\} < \theta f(\theta) = H'(\theta)$$

for all $\theta\varepsilon\left[\underline{\theta},\bar{\theta}\right]$. For low-$\theta$ consumers $R'(\theta)$ may even be negative, while $H'(\theta)$ is always positive.

In case there are increasing returns to scale in production, this conclusion is confirmed by the following example. Let the cost function (30) be of the form

$$C(q,Q) = q^2 Q^{\beta} \quad , \quad 0 \leq \beta \leq 1$$

Assume f is uniform over the interval $[0,1]$. Then

$$R(\theta) = \theta^2 - \theta$$

$$H(\theta) = 1/2\ (\theta^2 - 1)$$

Table I gives the monopoly solution, for various values of β, in case there are one and two jumps.

Table I. Monopoly solution

	one jump	two jumps
$\beta = 0$	$t_o = 1/2$	$t_o = t_1 = 1/2$
	$q_o = 1/8$	$q_o = 0,\ q_1 = 1/8$
	$\pi = 1/64$	$\pi = 1/64$
$\beta = 1/4$	$t_o = .53333$	$t_o = .5017;\ t_1 = .5256$
	$q_o = .15056$	$q_o = .0008;\ q_1 = .1502$
	$\pi = .01873$	$\pi = .01872$
$\beta = 1/2$	$t_o = .57142$	$t_o = .50811;\ t_1 = .5568$
	$q_o = .18704$	$q_o = .0071;\ q_1 = .1853$
	$\pi = .02290$	$\pi = .0228$
$\beta = 3/4$	$t_o = .61538$	$t_o = .5260;\ t_1 = .6130$
	$q_o = .24231$	$q_o = .0377;\ q_1 = .2417$
	$\pi = .0286$	$\pi = .0289$
$\beta = 1$	$t_o = 2/3$	$t_o = .6;\ t_1 = .8$
	$q_o = 1/3$	$q_o = .2;\ q_1 = .4$
	$\pi = 1/27$	$\pi = 1/25$

The corresponding results for the welfare solution are given in Table II.

Table II. Welfare solution

	one jump	two jumps
$\beta = 0$	$t_o = 0$ $q_o = 1/4$ $S = 1/16$	$t_o = t_1 = 0$ $q_o = 0,\ q_1 = 1/4$ $S = .0625$
$\beta = 1/4$	$t_o = 1/15$ $q_o = .2532$ $S = .063023$	$t_o = .0179;\ t_1 = .0672$ $q_o = .002222;\ q_1 = .25323$ $S = .063025$
$\beta = 1/2$	$t_o = 1/7$ $q_o = .26452$ $S = .06478$	$t_o = .08612;\ t_1 = .150713$ $q_o = .015047;\ q_1 = .265114$ $S = .06483$
$\beta = 3/4$	$t_o = 3/13$ $q_o = .2806$ $S = .06815$	$t_o = .2446;\ t_1 = .26507$ $q_o = .04815;\ q_1 = .29283$ $S = .06818$
$\beta = 1$	$t_o = 1/3$ $q_o = 1/3$ $S = 2/27$	$t_o = .2;\ t_1 = .6$ $q_o = .2;\ q_1 = .4$ $S = .08$

The following conclusions can be drawn from these results.

(a) The greater the degree of economies of scale (i.e. the smaller the value of β), the greater the effective supply interval. This is true in both the monopoly solution and the welfare solution, and in both the one-jump and the two-jump case.

(b) In the monopoly solution, one jump is more profitable than two jumps if the economies of scale are large ($\beta=1/4$, $\beta=1/2$), while two jumps are more profitable than one jump for small economies of scale ($\beta=3/4$, $\beta=1$). In the welfare solution, two jumps always give more aggregate net surplus than one jump (except in the special case $\beta=0$).

(c) For consumers who are served in both solutions, the quality they receive under the monopoly solution is always smaller then the quality they receive under the welfare solution (except for consumers with high values of θ if $\beta=1$).

(d) In case $\beta=1$, production of a finite number of qualities is never optimal. This is the case studied by M. Mussa and S. Rosen. They obtain, for the monopoly case (see (27))

$$q(\theta) = \theta - 1/2, \ \theta \varepsilon \left[1/2, \ 1\right], \ \pi = 1/24,$$

and for the welfare case (see (40)):

$$q(\theta) = \frac{\theta}{2}, \ \theta \varepsilon \left[0,1\right], \quad S = 1/12$$

These solutions are better than the corresponding solutions of table I and II.

REFERENCES

/1/ ARROW K.J. and M. KURZ, Public Investment, The Rate of Return, and Optimal Fiscal Policy, The Johns Hopkins Press, Baltimore and London, 1970.

/2/ GABSZEWICZ J.J., Blue and Red Cars, or Blue Cars Only? A Note on Product Variety under Monopoly, Discussion Paper 8021, Center for Operations Research and Econometrics, Université Catholique de Louvain, June 1980.

/3/ GETZ W.M. and D.H. MARTIN, Optimal Control Systems with State Variable Jump Discontinuities, Journal of Optimization Theory and Applications, Vol.31, N°2, June 1980, pp.195-205.

/4/ LANCASTER K.J., Socially Optimal Product Differentiation, The American Economic Review, vol.LXV, n°4, September 1975, pp.567-585.

/5/ LANCASTER K.J., Variety, Equity and Efficiency, Columbia University Press, New York, 1979.

/6/ MEADE J., The Optimal Balance between Economies of Scale and Variety of Products: An Illustrative Model, Economica, 41, 1974, pp.359-367.

/7/ MUSSA M. and S. ROSEN, Monopoly and Product Quality, Journal of Economic Theory, 13, 1978, pp.301-317.

/8/ PANZAR J.C. and R.D. WILLIG, Economies of Scope, The American Economic Review, Papers and Proceedings, May 1981, pp.268-272.