A DISEQUILIBRIUM ANALYSIS OF
THE COMmodity MARKET, EMPLOYMENT,
AND THE HOURS OF WORK

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The purpose of this paper is to construct a model for markets in disequilibrium when prices are exogeneous. We consider the commodity and the labour market. In order to analyse rations on employment and on the average hours of work, the labour market is divided into submarkets. The number of workers and the average hours of work per worker are separately treated so that we have three markets.

We consider a closed economy without government. The decisions are taken by consumers and producers. The producer supplies the commodities and demands a number of workers and also an average number of hours of work per worker. As we do not allow stock formation, the production and the supply of commodities will coincide. The consumer demands commodities, decides whether or not he will enter the labour market and supplies a number of hours of work. For the producer, the Walrasian quantities are determined by profit maximization subject to a production function. For the consumer, they are determined by utility maximization subject to a budget constraint. The Walrasian functions are represented as follows:

\[ y^S \]  commodity supply
\[ y^D \]  commodity demand
\[ x_1^S \]  labour force participation
\[ x_1^D \]  number of workers demanded
\[ x_2^S \]  average hours of work supplied
\[ x_2^D \]  average hours of work per worker demanded

In a first part the spill-over between the three markets are analysed in a rather general way. In a second part the effective demand and supply functions are determined, starting from specific functions. For the producer we use a Cobb-Douglas production function and for the consumer a specific form of the Johansen utility function. The purpose of this part is to construct functions which can be used for estimation. Where possible the spill-over coefficients are determined and discussed.
1. The different disequilibrium regimes and the spill-over effects

When we assume that prices and wages are not sufficiently flexible to bring the labour and commodity market into equilibrium, we cannot use the Walrasian demand and supply functions any longer. As there may exist excess demand or excess supply in the different markets, we have to consider the effective demand and supply functions. Following Clower (1965) and Benassy (1975, 1976, 1977, 1978) we define the effective demand (supply) in a specific market as the demand (supply) taking into account the quantity constraints in all the other markets (1). If there is no rationing in the other markets the effective demand (supply) will coincide with the Walrasian or notional demand (supply).

1.1. The different disequilibrium regimes

As we have three markets, there will exist eight possible disequilibrium regimes which are given in table I. The realised transactions \( \bar{y}, \bar{x}_1 \) and \( \bar{x}_2 \) are assumed to be the minimum of effective demand and effective supply (2). If there is any rationing in a market, this will influence the demand and the supply on the other markets. This influence is called the spill-over effect (Patinkin (1949, 1956)).

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(2) It can also be assumed that the realised transactions are the minimum of the Walrasian demand and supply (cfr. Clower (1965)).
1.2. The spill-over effects

In an economy with \( n \) markets we can define the effective demand on market \( i \), as the demand taking into account all the rationings in the other markets. Generally, we can write the effective demand (\( D'_i \)) as a function of the rationings in the other markets (\( Q \)) and of other factors (\( P \))

\[
D'_i = f(Q,P)
\]

where \( Q = (Q_1, Q_2, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n) \)

The spill-over effect from a rationing in market \( j \neq i \) on the effective demand on market \( i \), is then defined as

\[
\frac{\partial D'_i}{\partial Q_j}
\]

In our model we can consider several spill-over effects.

1.2.1. The spill-over effects for the producer

1.2.1.1. The spill-over from the commodity market to the labour market, \( \frac{\partial x'_i}{\partial y} \) and \( \frac{\partial x'_2}{\partial y} \)

When the producer is confronted with a quantity constraint in the commodity market, this means that he cannot realise his Walrasian supply. As we assumed no stock formation, the producer will have to bring his production below the Walrasian production. Therefore he will need less labour input. The effective demand for workers will be smaller than the Walrasian demand. The same holds for the effective demand for average hours of work per worker.
1.2.1.2. The spill-over effect from a rationing on the number of workers in the commodity market and on the demand for hours
\[ \frac{\partial y^S}{\partial x_1} \text{ and } \frac{\partial y^D}{\partial x_2} \]

When the producer is rationed on the number of workers, this can have two consequences. If he continues with the Walrasian average hours per worker, his effective production will fall below his Walrasian production and so will his effective supply. On the other hand, the producer can try to meet his Walrasian supply, by compensating the rationed number of workers by an increase of the average hours of work per worker. This effective demand for hours will be greater than the Walrasian one. In this case the producer will select that procedure which results in his highest profit.

1.2.1.3. The spill-over effect from a rationing on the average hours of work per worker on the commodity market and on the demand for workers,
\[ \frac{\partial y^S}{\partial x_2} \text{ and } \frac{\partial y^D}{\partial x_2} \]

For this situation we can follow the same reasoning as in 1.2.1.2. but starting from a rationing on the average hours of work.

1.2.2. The spill-over effects for the consumer

1.2.2.1. The spill-over effect from the commodity market to the labour market, \[ \frac{\partial x^S_1}{\partial y} \text{ and } \frac{\partial x^S_2}{\partial y} \]

If the consumer is rationed in the commodity market, that is if he cannot buy what he wants, some consumers will not find it necessary to work any longer and they will leave the labour market. The effective labour participation will be smaller than the Walrasian one. Others will supply less hours and so the effective supply of hours will also be smaller than the Walrasian one. We can, however,
state that the spill-over effects will be rather small because, when
rationed on the commodity market, the consumer can save the part
of his income which he cannot spend.

1.2.2.2. The spill-over effect from a rationing on the number of
workers on the commodity market and on the supply of hours
of work, \( \frac{\partial y^D}{\partial x_1} \) and \( \frac{\partial x^S}{\partial x_1} \)

If there is a rationing on the number of workers this means that
there is unemployment. In a household where someone who wants to
work but is unable to find any work or where someone becomes un-
employed, the reactions on the commodity demand and the supply of
hours will depend upon the unemployment insurance. If high, then
the effective commodity demand and the effective supply of hours
will not differ very much from the Walrasian ones. If the insurance
is low or even zero, the effective demand will be smaller than the
Walrasian demand. How much smaller will depend upon the lost in-
come. It is possible that other members of the household want to
compensate this loss by supplying more hours than the Walrasian
quantity.

1.2.2.3. The spill-over effect from a rationing on the average hours
of work per worker on the commodity market and on the labour
force participation, \( \frac{\partial y^D}{\partial x_2} \) and \( \frac{\partial x^S}{\partial x_2} \)

We can follow the same reasoning as in 1.2.2.2, but starting from a
rationing on the average hours of work per worker.

2. Disequilibrium analysis for a specific model

We will concentrate our disequilibrium analysis and the study of the
spill-over effects on a specific model. For the producer we will
consider a Cobb-Douglas production function and for the consumer
a special form of the Johansen utility function.
2.1. The producer side of the model

We consider a representative producer (or a representative body of producers) who has for each period $t$ ($t=0,\ldots,\infty$) a production function in the number of workers and in the average hours of work per worker. We assume that the periods between sample points follow each other fast enough to hold the capital stock fixed for each period. We use a Cobb-Douglas production function with Hicks-neutrality

$$y_t = A e^{\lambda t} x_1^a x_2^b$$

We assume that the production is subject to stochastic influences caused by weather changes, deficiencies, breakdowns, etc. If the stochastic component is additive, we have

$$y_t = y_t + \epsilon_t$$

where $E(\epsilon_t)$ is assumed to be zero so that $E(y_t) = y_t$.

The producer is going to maximize the expected present value of his net profit over an infinite time horizon. We assume that the net profit in period $t$ is the result of:
- the revenue $p_t y_t$ from selling the output $y_t$ at the price $p_t$;
- the labour cost.

The producer has to pay wages $w_t x_1^t x_2^t$, where $w_t$ is the gross hourly wage rate. He also has to pay contributions to the social security. These contributions are a fraction of the wages. They are calculated according to legally given coefficients (1). We assume that the producer has to pay an average percentage $s_t$ of the total wage bill $w_t x_1^t x_2^t$.

Their are also contributions which are independent of the hours of work, but vary with the number of workers. We assume that the average cost per worker is $z_t$.

The total labour cost is

$$(1+s_t)w_t x_1^t x_2^t + z_t x_1^t$$

- other costs as capital costs, net depreciation, etc., are represented by $c_t$;

(1) The Social Security contributions are calculated on different wage levels. The coefficients differ for the several branches of the Social Security. These are, however, plans for changing this system in 1981. The contributions will then be calculated on the total wages or on total sales according to one coefficient.
- taxes to be paid over the gross profit \( p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \). We assume that the producer pays taxes according to an average tax rate \( u_t \).

The net profit in period \( t \) is given by

\[
(1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right]
\]

let \( k_t \) be the discount factor for period \( t \) and

\[
k_t = \begin{cases} 
1 & \text{for } t=0 \\
\frac{1}{(1+r_\theta)^t} & \text{for } t > 0 \ (r_\theta = \text{discount rate at the end of period } \theta)
\end{cases}
\]

The present value of the net profit is

\[
\sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right]
\]

The expected value is

\[
E\left\{ \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right] \right\}
\]

\[
= \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t E(y_t) - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right]
\]

\[
= \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right]
\]

2.1.1. The Walrasian supply and demand functions

To obtain the Walrasian supply and demand functions of the producer, we have to solve following programme
\[
\max_{y_t, x_{1t}, x_{2t}} \sum_{t=0}^{\infty} k_t (1-u_t) [p_t v_t - w_t (1+s_t) x_{2t} - z_t x_{1t} - e_t]
\]
\[
s.t. \quad y_t = A^t e^{\lambda t} x_{1t}^a x_{2t}^b \quad \psi_t = 0, \ldots, \infty
\]

Assuming an interior solution, we have to impose some additional conditions on the parameters \(a\) and \(b\) (1), viz.

\[
a + b < 1 \quad 0 < a < 1 \quad a > b \quad 0 < b < 1
\]

The first-order conditions are

\[
p_t a \frac{y_t}{x_{1t}} = w_t (1+s_t) x_{2t} + x_t \quad /1/
\]
\[
p_t b \frac{y_t}{x_{2t}} = w_t (1+s_t) x_{1t} \quad /2/
\]
\[
y_t = A e^{\lambda t} x_{1t}^a x_{2t}^b \quad /3/
\]

Solving /2/ for \(x_{1t}\) and substituting in /1/ gives

\[
\begin{align*}
\frac{a}{b} x_{2t} w_t (1+s_t) &= w_t (1+s_t) x_{2t} + z_t \\
x_{1t} &= b \frac{p_t y_t}{w_t (1+s_t) x_{2t}} \\
y_t &= A e^{\lambda t} x_{1t}^a x_{2t}^b
\end{align*}
\]

\[
<=>
\]
\[
\begin{align*}
x_{2t}^D &= \frac{b}{a-b} z_t w_t^{-1} (1+s_t)^{-1} - 1 \\
x_{1t} &= b \frac{p_t e^{\lambda t} x_{1t}^a x_{2t}^b}{w_t (1+s_t) x_{2t}} \\
y_t &= A e^{\lambda t} x_{1t}^a x_{2t}^b
\end{align*}
\]

(1) See Appendix 1.
Substituting /4/ into /5/ and solving for $x_{1t}$, gives

$$x_{1t}^D = \frac{1}{1-a} \frac{1-b}{(a-b)^{1-a}} \frac{b}{1-a} \frac{1}{p_t} \frac{\lambda t}{1-a} \frac{1}{w_t} \frac{b}{1-a} \frac{1-b}{(1+s_t)^{1-a}} z_t$$

/6/

The Walrasian commodity supply is found by substituting /4/ and /6/ in /3/

$$y_t^s = A \frac{1-a}{1-a} \frac{1-a}{(a-b)^{1-a}} \frac{b}{1-a} \frac{1}{p_t} \frac{\lambda t}{1-a} \frac{1}{w_t} \frac{b}{1-a} \frac{1-b}{(1+s_t)^{1-a}} z_t$$

/7/

After taking logarithms of /4/, /6/ and /7/ we find loglinear relationships

$$\ln y_t^s = \frac{1}{1-a} \ln A + \frac{a-b}{1-a} \ln(a-b) + \frac{1}{1-a} \ln b + \frac{\lambda t}{1-a} + \frac{a}{1-a} \ln p_t - \frac{b}{1-a} \ln w_t$$

$$- \frac{b}{1-a} \ln(1+s_t) - \frac{a-b}{1-a} \ln z_t$$

/8/

$$\ln x_{1t}^D = \frac{1}{1-a} \ln A + \frac{1-b}{1-a} \ln(a-b) + \frac{1}{1-a} \ln b + \frac{\lambda t}{1-a} + \frac{1}{1-a} \ln p_t$$

$$- \frac{b}{1-a} \ln w_t - \frac{b}{1-a} \ln (1+s_t) - \frac{1-b}{1-a} \ln z_t$$

/9/

$$\ln x_{2t}^D = \ln b = \ln(a-b) + \ln z_t - \ln w_t - \ln (1+s_t)$$

/10/

The commodity supply is an increasing function of the price and is decreasing in labour costs. The demand for workers is a decreasing function of labour cost. The demand for hours will decrease as wages increase, but will increase as the cost per worker increases.

2.1.2. The effective demand and supply functions

After the Walrasian demand and supply functions, we will determine the effective demand and supply functions and pay some attention to the spill-over effects.
2.1.2.1. The producer is rationed in the commodity market

In this case, the effective commodity supply coincides with the Walrasian supply. The effective demand for workers and for hours can be found by solving the following programme

\[
\max_{t=0}^{\infty} \sum k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right]
\]

s.t. \[ y_t = A e^{\lambda_t x_{1t} x_{2t}} \quad t=0, ..., \infty \]

\[ y_t = \bar{y}_t \]

This results in (1):

\[
\ln x_{2t}^D = \ln b - \ln(a-b) - \ln w_t - \ln(1+s_t) + \ln z_t = \ln x_{2t}^D \quad /11/
\]

\[
\ln x_{1t}^D = -\frac{1}{a} \ln A - \frac{b}{a} \ln b + \frac{b}{a} \ln(a-b) - \frac{\lambda_t}{a} + \frac{b}{a} \ln w_t + \frac{b}{a} \ln(1+s_t) - \frac{b}{a} \ln z_t
\]

\[ + \frac{1}{a} \ln \bar{y}_t \quad /12/\]

We can remark that the Walrasian demand for average hours of work per worker remains unaffected by the rationing on the commodity market(2). This is not the case for the demand for workers. The spill-over elasticity is given by

\[
\frac{\delta \ln x_{1t}^D}{\delta \ln \bar{y}_t} = \frac{1}{a}
\]

As we know \(0 < a < 1\) we can conclude

\[ 1 < \frac{1}{a} \]

(1) The derivation of the effective demand and supply functions is given in Appendix 2.

(2) This is caused by the specific form of the profit function. We have a cost for the men hours and a cost for the number of men. If we should introduce a cost for the hours of work, independent of the number of men, the effective demand for hours would differ from the Walrasian demand.
In order to have a better insight in the spill-over effect, we will rewrite the effective demand for workers as follows

\[
\ln x_{1t}^{D'} = \ln x_{1t}^{D} - \ln x_{1t}^{D} + \ln x_{1t}^{D'}
\]

\[
= \ln x_{1t}^{D} - \left( 1 - \frac{a}{1-a} \right) \ln A - \left( 1 - \frac{b}{1-a} - \frac{1}{1-a} \right) \ln b - \left( \frac{1-b}{1-a} \right) \ln (a-b) - \frac{1}{1-a} \ln p_t
\]

\[-\left( \frac{\lambda}{1-a} + \frac{\lambda}{1-a} \right) t + \left( \frac{b}{1-a} + \frac{b}{1-a} \right) \ln w_t + \left( \frac{b}{1-a} - \frac{b}{1-a} \right) \ln z_t
\]

\[+ \frac{1}{a} \ln \tilde{y}_t
\]

\[= \ln x_{1t}^{D} - \frac{1}{a} (\ln y_t^S - \ln \tilde{y}_t)
\]

We can clearly see that the effective demand for workers is equal to the Walrasian demand minus \(1/a\) times the rationed quantity (Walrasian quantity - realised transactions).

2.1.2.2. The producer is rationed on the number of workers

The effective demand for workers is equal to the Walrasian demand.

\[
\ln x_{1t}^{D'} = \ln x_{1t}^{D}
\]

By maximizing the profit function subject to the production function and to \(x_{1t}^D = \tilde{x}_{1t}\), we find the effective demand for average hours of work per worker and the effective commodity supply

\[
\ln x_{2t}^{D'} = \frac{1}{1-b} \ln A + \frac{1}{1-b} \ln b + \frac{\lambda}{1-b} t + \frac{1}{1-b} \ln p_t - \frac{1}{1-b} \ln w_t - \frac{1}{1-b} \ln (1+s_t)
\]

\[-\frac{1-a}{1-b} \ln \tilde{x}_{1t}
\]

\[
\ln y_t^{S'} = \frac{1}{1-b} \ln A + \frac{b}{1-b} \ln b + \frac{\lambda}{1-b} t + \frac{b}{1-b} \ln p_t - \frac{b}{1-b} \ln w_t - \frac{b}{1-b} \ln (1+s_t)
\]

\[+ \frac{a-b}{1-b} \ln \tilde{x}_{1t}
\]

/13/

/14/
These functions can be rewritten as

\[
\ln x_{1t}^D = \ln x_{2t}^D + \frac{1-a}{1-b} (\ln x_{1t}^D - \ln x_{1t}^-)
\]

\[
\ln y_t^S = \ln y_t^S - \frac{a-b}{1-b} (\ln x_{1t}^D - \ln x_{1t}^-)
\]

The spill-over elasticity for the effective demand for hours is given by

\[
\frac{\partial \ln x_{2t}^D}{\partial \ln x_{1t}^-} = -\frac{1-a}{1-b}
\]

As we know that 0 < a < 1, 0 < b < 1 and a > b, we have

\[-1 < -\frac{1-a}{1-b} < 0\]

The effective demand for average hours of work per worker will be greater than the Walrasian demand.

The difference between the effective demand for hours and the Walrasian demand will be less than proportionate to the rationed quantity of the number of workers. The spill-over elasticity for the effective commodity supply is

\[
\frac{\partial \ln y_t^S}{\partial \ln x_{1t}^-} = \frac{a-b}{1-b}
\]

with \(0 < \frac{a-b}{1-b} < 1\)

Although the effective demand for hours is greater than the Walrasian demand, the effective commodity supply will be less than the Walrasian supply. The producer tries to compensate the lower number of workers by demanding more hours, but this compensation is only partly.
2.1.2.3. The producer is rationed on the average hours of work per worker.

The effective demand for the average hours of work per worker coincides with the Walrasian demand

\[ \ln x_{2t}^{D'} = \ln x_{2t}^{D} \]

The effective demand for workers and the effective supply of commodities are found by maximizing profit subject to the production function and to \( x_{2t} = \bar{x}_{2t} \).

\[ \ln x_{1t}^{D'} = \frac{1}{1-a} \ln A + \frac{1}{1-a} \ln a + \frac{\xi}{1-a} t + \frac{1}{1-a} p_t + \frac{b}{1-a} \ln \bar{x}_{2t} - \frac{1}{1-a} \ln w_t(1+s_t)\bar{x}_{2t}^2+z_t \]

\[ \ln y_t^{S'} = \frac{1}{1-a} \ln A + \frac{a}{1-a} \ln a + \frac{\xi}{1-a} t + \frac{a}{1-a} p_t + \frac{b}{1-a} \ln \bar{x}_{2t} - \frac{a}{1-a} \ln w_t(1+s_t)\bar{x}_{2t}^2+z_t \]

These functions are not log-linear in the realized transactions. The spill-over cannot be easily isolated as in the previous case. We shall consider the original functions (before logarithms).

\[ x_{1t}^{D'} = A^\frac{1}{1-a} a^\frac{1}{1-a} \xi^\frac{1}{1-a} p_t^\frac{1}{1-a} \bar{x}_{2t}^\frac{1}{1-a} [w_t(1+s_t)\bar{x}_{2t}^2+z_t]^\frac{1}{1-a} \]

\[ y_t^{S'} = A^\frac{1}{1-a} a^\frac{1}{1-a} \xi^\frac{1}{1-a} p_t^\frac{1}{1-a} \bar{x}_{2t}^\frac{1}{1-a} [w_t(1+s_t)\bar{x}_{2t}^2+z_t]^\frac{1}{1-a} \]

We can now consider the spill-over effects

\[ \frac{\partial x_{1t}^{D'}}{\partial x_{2t}} = A^\frac{1}{1-a} a^\frac{1}{1-a} \xi^\frac{1}{1-a} p_t^\frac{1}{1-a} \left( \frac{b}{1-a} \right)^{-1} [w_t(1+s_t)\bar{x}_{2t}^2+z_t]^{-1} \]

\[ + \left( -\frac{1}{1-a} \right) [w_t(1+s_t)\bar{x}_{2t}^2+z_t]^{-1} w_t(1+s_t)\bar{x}_{2t}^{-1-a} \]
\[ \frac{\partial x'_{1t}}{\partial x_{2t}} = \frac{1}{A-1-a} \frac{1}{a} \frac{\lambda t}{\lambda -1-a} \frac{1}{p_t} \frac{1}{1-a} \frac{b}{1-x_{2t}^2} - 1 \{ w_t (1+s_t) \bar{x}_{2t} + z_t \} = \frac{1}{A-1-a} \frac{1}{a} \frac{\lambda t}{\lambda -1-a} \frac{1}{p_t} \frac{1}{1-a} \frac{b}{1-x_{2t}^2} - 1 \{ w_t (1+s_t) \bar{x}_{1t} + z_t \} = \frac{1}{1-a} - 1 \{ (b-1) w_t (1+s_t) \bar{x}_{2t} + bz_t \} \]

In the same way we have
\[ \frac{\partial y'_{st}}{\partial x_{2t}} = \frac{1}{A-1-a} \frac{1}{a} \frac{\lambda t}{\lambda -1-a} \frac{1}{p_t} \frac{1}{1-a} \frac{b}{1-x_{2t}^2} - 1 \{ w_t (1+s_t) \bar{x}_{2t} + z_t \} = \frac{1}{1-a} - 1 \{ (b-a) w_t (1+s_t) \bar{x}_{2t} + bz_t \} \]

The sign of the spill-over effects will be determined by the last factor of the functions, as all the other factors are positive.

Therefore, we may write
\[ \frac{\partial x'_{1t}}{\partial x_{2t}} > 0 \]
\[ \frac{\partial y'_{st}}{\partial x_{2t}} < 0 \]
\[ b z_t - (1-b) w_t (1+s_t) \bar{x}_{2t} > 0 \]
\[ < > \]
\[ b z_t > (1-b) w_t (1+s_t) \bar{x}_{2t} \]
\[ < > \]
\[ \frac{b}{1-b} w_t (1+s_t) < \frac{z_t}{x_{2t}} \]

and
\[
\frac{\partial y_s'}{\partial x_{2t}} > 0
\]
\[
\frac{\partial}{\partial x_{2t}} < = >
\]
\[
bz_t - (a-b)w_t(1+s_t)x_{2t} > 0
\]
\[
< = >
\]
\[
bz_t < (a-b)w_t(1+s_t)x_{2t}
\]
\[
< = >
\]
\[
b \frac{z_t}{a-b w_t(1+s_t)} < x_{2t}
\]
\[
< = >
\]
\[
x_{2t} > x_{2t}
\]

(1)

We can summarize as follows:

<table>
<thead>
<tr>
<th>(x_{2t}^+)</th>
<th>(\frac{x_{2t}}{1-b \ w_t(1+s_t)})</th>
<th>(x_{2t}^D)</th>
<th>(\frac{x_{2t}}{1-b \ w_t(1+s_t)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{d y'}{d x_{2t}})</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(\frac{d \ x_{1t}}{d x_{2t}})</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

We can illustrate this graphically on figure 1.

(1) When we use the minimum condition \(x_{2t}^+ = \min\{x_{2t}^S, x_{2t}^D\}\) we can state that \(x_{2t}^+ < x_{2t}^D\) and therefore \(y_{s_t}' / x_{2t}^+\) will always be nonnegative. Working with the condition \(x_{2t}^+ = \min\{x_{2t}^S, x_{2t}^D, x_{2t}^D'\}\), \(x_{2t}\) can be greater than \(x_{2t}^D\) if \(x_{2t}^D' > x_{2t}^D\) and \(x_{2t}^D < x_{2t}^D' < x_{2t}'\).
If the number of hours of work per worker is set below $x^D_{2t}$, unemployment will increase.

Another way to get round the non-loglinearity is to use a Taylor expansion.

Let $f(\ln x^0_{2t}) = \ln (w_t(1+s_t)x^0_{2t} + s_t)$

Using a first degree Taylor expansion around $\ln x^D_{2t}$ gives

$$f(\ln x_{2t}) = \ln \left( w_t(1+s_t)x^D_{2t} + s_t \right) + \frac{w_t(1+s_t)}{w_t(1+s_t)x^D_{2t} + s_t} \left( \ln x^0_{2t} - \ln x^D_{2t} \right)$$
As $x_{2t}^D = \frac{b}{a-b} \frac{z_t}{w_t(1+s_t)}$ we can write

$$f(\ln x_{2t}^D) = \ln a - \ln \frac{b}{a-b} + \ln z_t + \frac{a-b}{a} \frac{w_t(1+s_t)}{z_t} \frac{1}{\ln x_{2t}^D}$$

$$- \frac{(a-b)}{a} \ln \frac{b}{a-b} \frac{w_t(1+s_t)}{z_t} - \frac{a-b}{a} \frac{w_t(1+s_t)}{z_t} \ln \frac{w_t(1+s_t)}{z_t}$$

The effective functions are then

$$\ln x_{1t}^D = \frac{1}{1-a} \ln A + \frac{1}{1-a} \ln a - \frac{1}{1-a} \ln \frac{b}{a-b} + \frac{\lambda}{1-a} \ln p_t - \frac{1}{1-a} \ln z_t$$

$$+ \frac{(a-b)}{a(1-a)} \ln \frac{b}{a-b} \frac{w_t(1+s_t)}{z_t} + \frac{a-b}{a(1-a)} \ln \frac{w_t(1+s_t)}{z_t}$$

$$+ \frac{b}{(1-a)a} - \ln x_{2t}^D$$

$$\ln y_t^{S'} = \frac{1}{1-a} \ln A + \frac{a}{1-a} \ln a - \frac{1}{1-a} \ln \frac{b}{a-b} + \frac{\lambda}{1-a} \ln p_t - \frac{a}{1-a} \ln z_t$$

$$+ \frac{(a-b)}{1-a} \ln \frac{b}{a-b} \frac{w_t(1+s_t)}{z_t} + \frac{a-b}{1-a} \ln \frac{w_t(1+s_t)}{z_t}$$

$$+ \frac{b}{1-a} - \ln x_{2t}^D$$

The sign of the spill-over coefficients is no longer a function of $x_{2t}^D$. We cannot directly use this spill-over coefficients. In estimation, they can always be used to determine the real sign of the spill-over as we can solve $a$ and $b$ from them.
2.1.2.4. The producer is rationed in the commodity market and on the number of workers.

The effective commodity supply is equal to /14/, the effective demand for workers coincide with /12/. The effective demand for hours is

$$\ln x_{2t}^{D'} = - \frac{1}{b} \ln A - \frac{\lambda}{b} t - \frac{a}{b} \ln x_{1t} - \frac{1}{b} \ln y_t$$  /19/

The effective demand for average hours of work per worker contains two spill-over effects

$$\frac{\partial \ln x_{2t}^{D'}}{\partial \ln x_{1t}} = - \frac{a}{b} - \frac{a}{b} < - 1$$

$$\frac{\partial \ln x_{2t}^{D'}}{\partial \ln y_t} = \frac{1}{b} \quad 1 < \frac{1}{b}$$

As $a < 1$ we have $\frac{1}{b} > \frac{a}{b}$

If both $\ln x_{1t}$ and $\ln y_t$ change with the same quantity the spill-over effect from the commodity market will dominate.

2.1.2.5. The producer is rationed on the commodity market and on the average hours of work per worker.

The effective commodity supply is equal to /16/ and the effective demand of hours of work is equal to /11/.

The effective demand for workers is

$$\ln x_{1t}^{D'} = - \frac{1}{a} \ln A - \frac{\lambda}{a} t - \frac{b}{a} \ln x_{2t} + \frac{1}{a} \ln y_t$$  /20/

In this case the effective demand for workers contains two spill-over effects

$$\frac{\partial \ln x_{1t}^{D'}}{\partial \ln x_{1t}} = 1$$

$$\frac{\partial \ln x_{1t}^{D'}}{\partial \ln x_{2t}} = - \frac{b}{a} \quad - 1 < - \frac{b}{a} < 0$$

$$\frac{\partial \ln x_{1t}^{D'}}{\partial \ln y_t} = \frac{1}{a} \quad \frac{1}{a} > 1$$
If both $\ln x_{2t}$ and $\ln y_t$ change with the same quantity the spill-over effect from the commodity market will dominate.

2.1.2.6. The producer is rationed on the number of workers and on the average hours of work per worker.

The effective demand for workers is equal to /15/ and the effective demand for hours is equal to /13/.

The effective commodity supply is

$$\ln y_{t}^{S'} = \ln A + \lambda t + a \ln x_{1t} + b \ln x_{2t} \quad /21/$$

The spill-over effects on the effective commodity supply are

$$\frac{\partial \ln y_{t}^{S'}}{\partial \ln x_{1t}} = a \quad 0 < a < 1$$

$$\frac{\partial \ln y_{t}^{S'}}{\partial \ln x_{2t}} = b \quad 0 < b < 1$$

with $a < b$.

The spill-over elasticities are simply the elasticities of output with respect to the number of workers and the average hours of work per worker.

2.1.2.7. The producer is rationed in all the markets.

The effective functions are given by /19/, /20/ and /21/.

They contain two spill-over effects.
2.2. The consumer side of the model

We consider a representative consumer (or a body of consumers) who has for every period $t=0,\ldots,\infty$ a utility function

$$U_t = \frac{\beta_1 v_t}{\alpha_1} + \frac{\beta_2 x_{2t}}{\alpha_2} + \frac{\beta_3 M_t}{\alpha_3 P_{c,t}}$$

where $\alpha_1, \alpha_2, \alpha_3 < 1$

$\beta_1, \beta_3 > 0$

$\beta_2 < 0$

The budget restriction is given by

$$P_{c,t} + M_t = \omega_t (1-q_t)x_{2t} + N_t (1-v_t) + M_{t-1}$$

with $P_{c,t}$ the consumer price index

$M_t$ the nominal stock of money at the end of period $t$

$q_t$ the average coefficient for calculating the contribution to the social security

$N_t$ non-labour income

$v_t$ average personal income tax rate

We will assume that money has an indirect utility. The utility function for the consumer over the period $0,\ldots,\infty$ will be given by

$$U = \sum_{t=0}^{\infty} l_t U_t$$

The factor $l_t$ is a discount factor which gives less importance to future utilities.

2.2.1. The Walrasian demand and supply function

In order to determine the Walrasian commodity demand and the supply of average hours of work, we have to solve following programme
Max \( U = \sum_{t=0}^{\infty} \ell_t \left\{ \frac{\beta_1 y_t}{\alpha_1} \alpha_1 + \frac{\beta_2 x_{2t}}{\alpha_2} \alpha_2 + \frac{\beta_3 M_t/P_{c,t}}{\alpha_3} \alpha_3 \right\} \)

s.t. \( P_{c,t} v_t + M_t = w_t (1-q_t) (1-v_t) x_{2t} + (1-v_t) M_t + M_{t-1} \)

\( t=0, \ldots, \infty \)

The Lagrange function, with \( \mu_t \) as multiplier, is

\[ h = \sum_{t=0}^{\infty} \ell_t \left\{ \frac{\beta_1 y_t}{\alpha_1} \alpha_1 + \frac{\beta_2 x_{2t}}{\alpha_2} \alpha_2 + \frac{\beta_3 M_t/P_{c,t}}{\alpha_3} \alpha_3 \right\} \]

\[ + \sum_{t=0}^{\infty} \mu_t \left\{ w_t (1-q_t) (1-v_t) x_{2t} + (1-v_t) M_t + M_{t-1} - P_{c,t} v_t - M_t \right\} \]

The first order conditions are

\[ \ell_t \frac{y_t}{\alpha_1} = P_{c,t} \mu_t \quad /2.1/ \]

\[ \ell_t \frac{x_{2t}}{\alpha_2} = -w_t (1-q_t) (1-v_t) \mu_t \quad /2.2/ \]

\[ \ell_t \frac{M_t/P_{c,t}}{\alpha_3} = \mu_t \quad /2.3/ \]

\[ P_{c,t} v_t + M_t = w_t (1-q_t) (1-v_t) x_{2t} + (1-v_t) M_t + M_{t-1} \quad /2.4/ \]

Substituting 2.3 in 2.1 and 2.2 gives

\[ \begin{cases} \frac{y_t}{\alpha_1} = \frac{M_t/P_{c,t}}{\alpha_3} \\ \frac{x_{2t}}{\alpha_2} = -w_t (1-q_t) (1-v_t) \frac{M_t/P_{c,t}}{\alpha_3} - \frac{1}{P_{c,t}} \end{cases} \]

\[ \iff \]

\[ \begin{cases} y_t = \beta_1 \beta_3 \frac{M_t}{1-\alpha_1} \\ x_{2t} = -\beta_2 \beta_3 \frac{-w_t (1-q_t) (1-v_t)}{P_{c,t}} \end{cases} \]
\begin{equation}
\begin{align*}
\ln y^G_t &= \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_1} \ln \frac{M_t}{P_c,t} \\
\ln x^S_{2t} &= \ln(-\beta_2) - \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_3 - \frac{1}{1-\alpha_2} \ln \frac{w_t}{P_c,t} (1-q_t)(1-v_t) + \frac{1-\alpha_3}{1-\alpha_2} \ln \frac{M_t}{P_c,t}
\end{align*}
\end{equation}

If we assume that the money stock is known, we have found the Walrasian commodity demand and supply of average hours of work per worker.

We still have to find the supply of the number of workers or the labour force participation. Leonell C. Andersen (1978) constructed a model of labour force participation. He starts from the assumption that the decision to be in the labour market depends on an individual's reservation wage rate relative to his decision wage rate. Andersen gives following definitions

"...the reservation wage rate is an individual's marginal" "rate of substitution of goods and services for leisure when" "all available time is allocated to leisure. The decision" "wage rate is the perceived amount of goods and services" "the individual can purchase if a unit of time is shifted" "from leisure to work." (1)

When the decision wage rate is greater than the reservation wage rate, the individual will be in the labour market. Andersen states for the reservation wage rate

\[RW_{ij} = f_{ij} (N_j^*, WE_j^*, v_j^*, q_j^*, B_j, N_j, WE_j, v_j, q_j, B_j)\]

where \(RW_{ij}\) is the reservation wage rate of the \(i\)th number of the \(j\)th household

\(WE_j\) is the perceived average real wage rate of employed members of the \(j\)th household

\(B_j\) is the size of the \(j\)th household

The asterisks indicate the permanent component of the variable. The transitory component is the difference between the actual value and the permanent component.

(1) p.10
For the decision wage rate it is assumed that

$$DW_{ij} = s_{ij} \{ \frac{w^*}{\mathcal{P}_c}, v_j^*, q_j^*, (\frac{w^*}{\mathcal{P}_c})^*_j, v_j^*-v_j^*, q_j^*-q_j^* \}$$

where $DW_{ij}$ is the decision wage rate of the $i^{th}$ member in the $j^{th}$ household.

The individual's participation in the labour market ($LF_{ij}$) is then

$$LF_{ij} = 1 \iff RW_{ij} < DW_{ij}$$
$$LF_{ij} = 0 \iff RW_{ij} \geq DW_{ij}$$

The total number of members of labour force age (14 years and over) in a household participating in the labor market ($LF_j$) is the sum of the total number of individuals whose decision wage rates are not smaller than their reservation wage rates. Andersen gives following function

$$LF = h_j[N^*, \frac{w^*}{\mathcal{P}_c}, v^*, q^*, P^*, N-N^*, \frac{w^*}{\mathcal{P}_c}, v-v^*, q-q^*, B-B^*, PL]$$

He assumes that the relationship is linear in natural logarithms.

For the formation of perceptions of permanent components we assume the following adjustment process (1).

$$(1-\delta)N_t^* = \lambda N_{t-1} \quad \text{with } \delta = 1 - \lambda$$
$$(1-\delta)\left(\frac{w}{\mathcal{P}_c}\right)_t^* = \lambda \left(\frac{w}{\mathcal{P}_c}\right)_{t-1}$$
$$(1-\delta)v_t^* = \lambda v_{t-1}$$
$$(1-\delta)q_t^* = \lambda q_{t-1}$$
$$(1-\delta)B_t^* = \lambda B_{t-1}$$

The function $LF$ is

$$\text{(1) D is the delay operator such that } D^iX_t = X_{t-i}. $$
\[ \ln \text{LF}_t = \alpha_0 + \alpha_1 \ln N_t^* + \alpha_2 \ln \left( \frac{W_{t-1}}{P_{c,t}} \right)^* + \alpha_3 \ln \nu_{t-1}^* + \alpha_4 \ln q_{t-1}^* + \alpha_5 \ln B_t^* \\
+ \alpha_6 (\ln N_t - \ln N_t^*) + \alpha_7 (\ln \left( \frac{W_t}{P_{c,t}} \right)^* - \ln \left( \frac{W_{t-1}}{P_{c,t}} \right)^* ) + \alpha_8 (\ln \nu_t - \ln \nu_{t-1}^*) \\
+ \alpha_9 (\ln q_t - \ln q_{t-1}^*) + \alpha_{10} (\ln B_t - \ln B_t^*) + \alpha_{11} \ln \text{PL}_t \]

\[
(1-\delta)\ln \text{LF}_t = (1-\delta)\alpha_0 + \lambda_1 \ln N_{t-1} + \lambda_2 \ln \left( \frac{W_{t-1}}{P_{c,t-1}} \right)^* + \lambda_3 \ln \nu_{t-1} + \lambda_4 \ln q_{t-1} \\
+ \lambda_5 \ln B_{t-1} + \alpha_6 (\ln N_t - \ln N_{t-1}) + \alpha_7 (\ln \left( \frac{W_t}{P_{c,t}} \right)^* - \ln \left( \frac{W_{t-1}}{P_{c,t-1}} \right)^* ) \\
+ \alpha_8 (\ln \nu_t - \ln \nu_{t-1}^*) + \alpha_9 (\ln q_t - \ln q_{t-1}^*) + \alpha_{10} (\ln B_t - \ln B_{t-1}) \\
+ \alpha_{11} \ln \text{PL}_t - \delta \alpha_{11} \ln \text{PL}_{t-1} \]

\[
\ln \text{LF}_t = \lambda_0 + \lambda_1 \ln N_{t-1} + \alpha_2 \ln \left( \frac{W_{t-1}}{P_{c,t-1}} \right)^* + \alpha_3 \ln \nu_{t-1} + \alpha_4 \ln q_{t-1} + \alpha_5 \ln B_{t-1} \\
+ \alpha_6 (\ln N_t - \ln N_{t-1}) + \alpha_7 (\ln \left( \frac{W_t}{P_{c,t}} \right)^* - \ln \left( \frac{W_{t-1}}{P_{c,t-1}} \right)^* ) + \alpha_8 (\ln \nu_t - \ln \nu_{t-1}^*) \\
+ \alpha_9 (\ln q_t - \ln q_{t-1}^*) + \alpha_{10} (\ln B_t - \ln B_{t-1}) + \alpha_{11} \ln \text{PL}_t \\
- (1-\lambda)\alpha_{11} \ln \text{PL}_{t-1} + (1-\lambda)\ln \text{LF}_{t-1} \]

We shall consider this function as the labour force participation function.

\[ \ln x_{1t}^S = \alpha_0 + \alpha_1 \ln N_{t-1} + \alpha_2 \ln \left( \frac{W_{t-1}}{P_{c,t-1}} \right)^* + \alpha_3 \ln \nu_{t-1} + \alpha_4 \ln q_{t-1} + \alpha_5 \ln B_{t-1} \\
+ \alpha_6 (\ln N_t - \ln N_{t-1}) + \alpha_7 (\ln \left( \frac{W_t}{P_{c,t}} \right)^* - \ln \left( \frac{W_{t-1}}{P_{c,t-1}} \right)^* ) + \alpha_8 (\ln \nu_t - \ln \nu_{t-1}^*) \\
+ \alpha_9 (\ln q_t - \ln q_{t-1}^*) + \alpha_{10} (\ln B_t - \ln B_{t-1}) + \alpha_{11} \ln \text{PL}_t + \alpha_{12} \ln \text{PL}_{t-1} \\
+ \alpha_{13} \ln \text{LF}_{t-1} \]
2.2.2. The effective demand and supply functions

In order to have functions that are linear in natural logarithms we cannot always use a direct derivation from maximization of the utility function. We shall have to make additional assumptions.

2.2.2.1. The consumer is rationed in the commodity market

The effective commodity demand is equal to the Walrasian demand. The quantity of money is in this case the Walrasian quantity \( \frac{M_t}{P_{c,t}} \).

So we can write

\[
\ln y_{t}^{D'} = \ln y_{t}^{D} = \ln \gamma_1 + \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_1} \ln \left( \frac{M_t}{P_{c,t}} \right)^W
\]

In order to derive the effective supply of hours of work we make the following assumption. The rationing on the commodity market is reflected in the money stock. We assume

\[
\ln \left( \frac{M_t}{P_{c,t}} \right)^W = \ln \left( \frac{M_t}{P_{c,t}} \right)^W + \gamma_1 \left( \ln y_{t}^{D} - \ln \bar{y}_t \right)
\]

As the consumer can buy less commodities, he will save more so that we assume \( \gamma_1 > 0 \).

We can now replace \( \ln y_{t}^{D} \) to become:

\[
\ln \left( \frac{M_t}{P_{c,t}} \right)^W = \ln \left( \frac{M_t}{P_{c,t}} \right)^W + \gamma_1 \ln \beta_1 - \gamma_1 \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \gamma_1 \frac{1-\alpha_3}{1-\alpha_1} \ln \left( \frac{M_t}{P_{c,t}} \right)^W - \gamma_1 \ln \bar{y}_t
\]

This expression will now be replaced into the supply of average hours

\[
\ln \bar{x}_{t}^{S} = \ln \left( \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_2} \ln \left( 1-\alpha_3 \right) \ln \left( 1-\alpha_4 \right) \right) + \frac{1-\alpha_3}{1-\alpha_2} \frac{1-\alpha_4}{1-\alpha_1} \ln \left( \frac{M_t}{P_{c,t}} \right)^W
\]

\[
= \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_2} \gamma_1 \ln \beta_3 - \frac{1-\alpha_3}{1-\alpha_2} \gamma_1 \ln \bar{y}_t
\]

<>
\[
\ln x_{2t}^S = \ln(-\beta_2) - \frac{(1-\alpha_3)(1-\alpha_1+\gamma_1-\alpha_3\gamma_1)}{(1-\alpha_1)(1-\alpha_2)} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_2} \gamma_1 \ln \beta_1 \\
+ \frac{(1-\alpha_3)(1-\alpha_1+\gamma_1-\alpha_3\gamma_1)}{(1-\alpha_1)(1-\alpha_2)} \ln\left(\frac{M_t}{p_{c,t}}\right)^\nu - \frac{1}{1-\alpha_2} \ln\left(\frac{w_t}{p_{c,t}}(1-q_t)(1-v_t)\right) \\
- \frac{1-\alpha_3}{1-\alpha_2} \gamma_1 \ln \bar{y}_t
\]

The spill-over elasticity is \(-\frac{1-\alpha_3}{1-\alpha_2} < 0\) (of course under the assumption that \(\gamma_1 > 0\))

If there is a stronger rationing, this means lower realised transactions, the effective supply of hours will rise. This suggests that our assumption was wrong and that \(\gamma_1 < 0\). For the effective labour force participation we will state in an ad hoc way that

\[
\ln x_{1t}^S = \ln x_{1t}^S + \gamma_2(\ln y_t^D - \ln \bar{y}_t)
\]

We do not want to make any assumption about the sign of \(\gamma_2\) although we think \(\gamma_2 < 0\).

Replacing the Walrasian quantities gives

\[
\ln x_{1t}^S = a_0 + \gamma_1 \ln \beta_1 - \gamma_1 \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + a_1 \ln M_{t-1} + a_2 \ln \left(\frac{w_{t-1}}{p_{c,t-1}}\right) + a_3 \ln v_{t-1} \\
+ a_4 \ln q_{t-1} + a_5 \ln B_{t-1} + a_6 (\ln M_{t-1} - \ln M_{t-1}) + a_7 (\ln \frac{w_t}{p_{c,t}} - \ln \frac{w_{t-1}}{p_{c,t-1}}) \\
+ a_8 (\ln v_t - \ln v_{t-1}) + a_9 (\ln q_t - \ln q_{t-1}) + a_{10} (\ln B_t - \ln B_{t-1}) + a_{11} \ln PT_t \\
+ a_{12} \ln PL_{t-1} + a_{13} \ln x_{1,t-1}^S + \frac{1-\alpha_3}{1-\alpha_1} \ln M_t - \gamma_2 \ln \bar{y}_t
\]

The spill-over elasticity is \(-\gamma_2\).
2.2.2.2. The consumer is rationed on his labour force participation

In this situation the effective labour force participation coincides with the Walrasian one.

We shall assume that the number of unemployed influences the money stock in the following way

\[
\ln \left( \frac{M_t}{P_c} \right) = \ln \left( \frac{M_t}{P_c} \right)^w + \gamma_3 (\ln x_{1t} - 1) - \gamma_3 \ln \bar{x}_{1t}
\]

We think it is reasonable to assume that an increase in unemployment leads to a decrease in the money stock and so \( \gamma_3 < 0 \).

We have for the effective functions

\[
\ln y_t = \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_2}{1-\alpha_1} \ln \left( \frac{M_t}{P_c} \right)^w + \gamma_3 \ln x_{1t} - \gamma_3 \ln \bar{x}_{1t}
\]

\[
\ln x_{2t} = \ln(\beta_2) - \frac{1-\alpha_2}{1-\alpha_3} \ln \beta_3 + \frac{1}{1-\alpha_2} \ln \frac{w_t}{(1-q_t)(1-v_t)} - \frac{1-\alpha_2}{1-\alpha_3} \ln \left( \frac{M_t}{P_c} \right)^w
\]

\[
+ \gamma_1 x_{1t} - \gamma_1 \ln \bar{x}_{1t}
\]

\[
\ln y_t^D = \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_1} \ln \left( \frac{M_t}{P_c} \right)^w + a_1 \gamma_3 (1-\alpha_1)
\]

\[
+ a_2 \gamma_3 (1-\alpha_1) \ln v_{t-1} + a_3 \gamma_3 (1-\alpha_1) \ln q_{t-1} + a_4 \gamma_3 (1-\alpha_1) \ln B_{t-1} + a_5 \gamma_3 (1-\alpha_1) \ln N_{t-1}
\]

\[
= \gamma_3 (1-\alpha_1) \ln M_t - \gamma_3 (1-\alpha_1) \ln N_t
\]

\[
\ln v_{t-1} + a_9 \gamma_3 (1-\alpha_1) \ln q_{t-1} + a_10 \gamma_3 (1-\alpha_1) \ln B_{t-1}
\]

\[
= \gamma_3 (1-\alpha_1) \ln M_t - \gamma_3 (1-\alpha_1) \ln N_t
\]
\[ \ln x_{2t}^S = \ln(-\beta_2) - \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_3 + a_3 \gamma_3 \ln(t) + \frac{1-\alpha_3}{1-\alpha_2} \ln \left( \frac{M_t}{P_{c,t}} \right) + \frac{1-\alpha_3}{1-\alpha_2} \ln \left( 1-q_t \right) \]

\[ + a_1 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln N_{t-1} + a_2 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln v_{t-1} + a_3 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln q_{t-1} + a_4 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln q_t \]

\[ + a_5 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln P_{t-1} + a_6 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \left( \ln N_t - \ln N_{t-1} \right) + a_7 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln \left( \frac{w_t}{P_{c,t}} \right) + \frac{1-\alpha_3}{1-\alpha_2} \ln \left( \frac{w_{t-1}}{P_{c,t}} \right) \]

\[ + a_8 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \left( \ln v_t - \ln v_{t-1} \right) + a_9 \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \left( \ln q_t - \ln q_{t-1} \right) + a_{10} \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln B_t \]

\[ + a_{11} \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln P_{t-1} + a_{12} \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln P_{t-1} + a_{13} \gamma_3 \frac{1-\alpha_3}{1-\alpha_2} \ln x_{1,t-1} \]

\[ - \frac{1-\alpha_3}{1-\alpha_2} \gamma_3 \ln x_{1t} \]

The spill-over elasticities are both greater than zero.

2.2.2.3. The consumer is rationed on the average hours of work.

The effective supply of hours is the Walrasian one

\[ \ln x_{2t}^S = \ln x_t^S = \ln \left( -\beta_2 \right) - \frac{1-\alpha_3}{1-\alpha_2} \ln \beta_3 - \frac{1-\alpha_3}{1-\alpha_2} \ln \left( \frac{M_t}{P_{c,t}} \right) - \frac{1-\alpha_3}{1-\alpha_2} \ln \left( 1-q_t \right) \]

To find the effective commodity demand we will make an assumption that is analogue to that in 2.2.2.1.

\[ \ln \left( \frac{M_t}{P_{c,t}} \right) = \ln \left( \frac{M_t}{P_{c,t}} \right) + \gamma_4 \left( \ln x_{2t}^S - \ln \bar{x}_{2t} \right) \]

with \( \gamma_4 < 0 \).

We can then write

\[ \ln y_t^S = \ln \beta_1 - \frac{1-\alpha_3}{1-\alpha_1} \ln \beta_3 + \frac{1-\alpha_3}{1-\alpha_1} \ln \left( \frac{M_t}{P_{c,t}} \right) + \gamma_4 \frac{1-\alpha_3}{1-\alpha_1} \ln x_{2t}^S - \frac{1-\alpha_3}{1-\alpha_1} \ln \bar{x}_{2t} \]
\[ \ln y_t^S = \ln \beta_2 \gamma_t \left( - \ln(1 - \beta_2) - \frac{(1 - \gamma_2)}{(1 - \alpha_1)} \ln \frac{M_t}{p} \right) - \frac{(1 - \gamma_2)(1 - \gamma_3) \gamma_4}{(1 - \alpha_1)(1 - \alpha_2)} \ln \beta_3 \gamma_t \frac{1 - \alpha_3}{(1 - \alpha_1)(1 - \alpha_2)} \ln \frac{w_t}{p_{c,t}} (1 - q_t) \]

\[ (1 - \nu_t) + \frac{(1 - \alpha_3)(1 - \alpha_2 + \gamma_3 \gamma_4)}{(1 - \alpha_1)} \ln \frac{M_t}{p_{c,t}} w - \gamma_t \frac{1 - \alpha_3}{1 - \alpha_1} \ln x_{2t} \]

/29/

The sign-over elasticity is \(- \gamma_4 \frac{1 - \alpha_3}{1 - \alpha_1} > 0\).

For the effective labour force participation we will introduce, as in 2.2.2.1., an ad hoc function.

\[ \ln x_{1t}^S = \ln x_{1t}^S + \gamma_5 (\ln x_{2t}^S - \ln x_{2t}) \]

/30/

The sign of \(\gamma_5\) is not quite clear and we will not make any assumption about it.

2.2.2.4. The consumer is rationed in the commodity market and on the labour force participation

The effective commodity demand is the same as /27/ and the effective labour force participation is equal to /28/.

For the effective supply of hours of work, we make following assumption

\[ \ln \frac{M_t}{p_{c,t}} = \ln \left( \frac{M_t}{p_{c,t}} \right)^\gamma_6 (\ln \frac{y_t}{y_t} + \gamma_7 (\ln x_{1t}^S - \ln x_{1t}) \]

/29/
The effective supply of hours becomes

$$\ln x'_{2t} = \ln(-\beta_2) - \frac{1-\alpha_3}{1-\alpha_2} \ln y_t + \frac{1-\alpha_3}{1-\alpha_2} \ln \frac{w_t}{P_{c,t}} (1-q_t) (1-v_t) + \frac{1-\alpha_3}{1-\alpha_2} \ln \frac{M_t}{P_{c,t}} w + \frac{1-\alpha_3}{1-\alpha_2} \ln y^D_t + \frac{1-\alpha_3}{1-\alpha_2} \ln x^S_{1t}$$

After substituting $\ln y^D_t$ and $\ln x^S_{1t}$ this gives

$$\ln x'_{2t} = \ln(-\beta_2) + \frac{1-\alpha_3}{1-\alpha_1} \ln y_t - \frac{1-\alpha_3}{1-\alpha_2} \ln \frac{w_t}{P_{c,t}} (1-q_t) - \frac{1-\alpha_3}{1-\alpha_2} \ln \frac{M_t}{P_{c,t}} + \frac{1-\alpha_3}{1-\alpha_2} \ln \frac{N_t}{P_{c,t}}$$

We have two spill-over elasticities

$$\frac{\partial \ln x'_{2t}}{\partial \ln y_t} = -\frac{1-\alpha_3}{1-\alpha_2} \quad \text{and} \quad \frac{\partial \ln x'_{2t}}{\partial \ln y_t} = -\frac{1-\alpha_3}{1-\alpha_2}$$

2.2.2.5. The consumer is rationed in the commodity market and on the hours of work

For the effective commodity demand we have the same function as /29/.

The effective supply of hours of work is equal to /25/.

For the effective labour force participation we assume
\[
\ln x'_{1t} = \ln x^S_{1t} + \gamma_8 (\ln y^D_t - \ln \bar{y}_t) + \gamma_9 (\ln x^S_{2t} - \ln \bar{x}_{2t})
\]

Substituting \(\ln y^D_t\) and \(\ln x^S_{2t}\) gives

\[
\ln x'_{1t} = a_0 + \gamma_8 \ln \beta_1 + \gamma_9 \ln (\beta_2) - (1 - \alpha_3) \left[ \frac{\gamma_8 \gamma_9 \gamma_2 \gamma_1 - \gamma_8 \gamma_9 \gamma_2 \gamma_1 - \gamma_8 \gamma_9 \gamma_2 \gamma_1 - \gamma_8 \gamma_9 \gamma_2 \gamma_1}{(1 - \alpha_1)} \right] \ln \beta_2 - \frac{\gamma_8 \gamma_9}{1 - \alpha_2} \ln \frac{w_t}{p_{c,t}} (1 - \alpha_1)(1 - \alpha_2)
\]

\[
+ \left(1 - \alpha_3 \right) \frac{\gamma_8 \gamma_9 \gamma_2 \gamma_1 - \gamma_8 \gamma_9 \gamma_2 \gamma_1 - \gamma_8 \gamma_9 \gamma_2 \gamma_1 - \gamma_8 \gamma_9 \gamma_2 \gamma_1}{(1 - \alpha_1)(1 - \alpha_2)} \ln \left( \frac{M_t}{p_{c,t}} \right) + a_1 \ln N_{t-1} + a_2 \ln \frac{w_{t-1}}{p_{c,t}} + a_3 \ln v_{t-1}
\]

\[
+ a_4 \ln q_{t-1} + a_5 \ln \bar{v}_{t-1} + a_6 \ln (\ln N_t - \ln N_{t-1}) + a_7 \ln \frac{w_t}{p_{c,t}} - a_8 \ln \frac{w_{t-1}}{p_{c,t-1}} + a_9 \ln \frac{w_{t-1}}{p_{c,t-1}} + a_{10} \ln (\ln \bar{v}_t - \ln \bar{v}_{t-1}) + a_{11} \ln M_t + a_{12} \ln M_{t-1}
\]

\[
+ a_{13} \ln x'_{1t-1} - \gamma_8 \ln \bar{y}_t - \gamma_9 \ln \bar{x}_{2t}
\]

We have again two spill-over elasticities

\[
\frac{\partial \ln x'_{1t}}{\partial \ln \bar{y}_t} = -\gamma_8 \quad \text{and} \quad \frac{\partial \ln x'_{1t}}{\partial \ln \bar{x}_{2t}} = -\gamma_9
\]

2.2.2.6. The consumer is rationed on both the labour force participation and the average hours of work.

The effective labour force participation is the same as /30/ and the effective supply of hours is equal to /28/. For the effective commodity demand we make the following assumption

\[
\ln \frac{M_t}{p_{c,t}} = \ln \left( \frac{M_t}{p_{c,t}} \right) + \gamma_{10} (\ln x^S_{1t} - \ln \bar{x}_{1t}) + \gamma_{11} (\ln x^S_{2t} - \ln \bar{x}_{2t})
\]

The effective commodity demand can be written as

\[
\ln y^D_t = \ln \beta_1 - \frac{1^3}{1 - \alpha_1} \ln \beta_3 + \frac{1^3}{1 - \alpha_1} \gamma_{10} \ln x^S_{1t} + \frac{1^3}{1 - \alpha_1} \gamma_{11} \ln x^S_{2t} - \gamma_{10} \frac{1^3}{1 - \alpha_1} \ln \bar{x}_{1t}
\]

\[
- \gamma_{11} \frac{1^3}{1 - \alpha_1} \ln \bar{x}_{2t}
\]

\[
<=>
\]
\[
\ln y_t^{D'} = \ln \beta_1 + \gamma_{10}^{1-\alpha_3} a_0 + \gamma_{11}^{1-\alpha_3} \ln(-\beta_2) - \frac{(1-\alpha_3)(1-\alpha_2)x_1^{11}x_2^{13}}{(1-\alpha_1)(1-\alpha_2)} \ln \beta_3
\]
\[
- \frac{\gamma_{10}^{1-\alpha_3}}{(1-\alpha_1)(1-\alpha_2)} \ln \frac{\theta_t}{p_{c,t}} - \frac{(1-\alpha_3)(1-\alpha_2)x_1^{11}x_2^{13}}{(1-\alpha_1)(1-\alpha_2)} \ln \left(\frac{-\theta_t}{p_{c,t}}\right)^{w}
\]
\[
+ \gamma_{10}^{1-\alpha_3} \ln N_{t-1} + \gamma_{10}^{1-\alpha_3} \ln \frac{w_{t-1}}{p_{c,t-1}} + \gamma_{10}^{1-\alpha_3} \ln \frac{v_{t-1}}{p_{c,t-1}} + \gamma_{10}^{1-\alpha_3} \ln \frac{q_{t-1}}{p_{c,t-1}}
\]
\[
+ \gamma_{10}^{1-\alpha_3} \ln B_{t-1} + \gamma_{10}^{1-\alpha_3} \ln \left(\frac{N_t}{N_{t-1}}\right) + \gamma_{10}^{1-\alpha_3} \ln \frac{w_t}{w_{t-1}}
\]
\[
+ \gamma_{10}^{1-\alpha_3} \ln \frac{v_t}{v_{t-1}} + \gamma_{10}^{1-\alpha_3} \ln \frac{q_t}{q_{t-1}} + \gamma_{10}^{1-\alpha_3} \ln \frac{B_t}{B_{t-1}} + \gamma_{10}^{1-\alpha_3} \ln \frac{x_{1,t-1}}{x_{2,t}}
\]
\[
+ \gamma_{10}^{1-\alpha_3} \ln \frac{x_{1,t}}{x_{2,t}}
\]

The spill-over elasticities

\[
\frac{\partial y_t^{D'}}{\partial x_{1t}} = -\gamma_{10}^{1-\alpha_2} \quad \text{and} \quad \frac{\partial y_t^{D'}}{\partial x_{2t}} = -\gamma_{11}^{1-\alpha_3}
\]

2.2.2.7. The consumer is rationed in all the markets

We have an effective commodity demand which is the same as /33/. The effective labour force participation coincides with /32/ and the effective supply of hours is equal to /31/>. 
Conclusion

To study the commodity market, employment and the hours of work we used a disequilibrium analysis. In a first part we have defined the different disequilibrium regimes. Some of these regimes occur frequently, others are rather rare. The emphasis is placed on the influences from a disequilibrium in one market on the other markets. Those spill-over effects are first analysed in a rather general way. In a second part this is done for specific functional forms. The sign of the spill-over coefficients cannot always be determined and depends upon the chosen specifications. It appears that the spill-over coefficient can have a changing sign. This is the case when the producer is rationed on the average hours of work. The effective demand for workers may rise or fall with a rise in the rationing on the hours. Therefore a reduction in the number of hours of work will not necessarily lead to an increasing employment. This example illustrates the importance of a disequilibrium analysis.

We also derived the effective demand and supply functions starting from a Cobb-Douglas production function and a specific form of the Johanssen utility function. These functions will be used for estimation.

We have eight regimes and rather complicated functions. We suppose that the greatest problem in estimation will be the identification of the model. It will be necessary to make some simplifying assumptions concerning the number of regimes and the functional form of effective demand and supply.

This paper serves as a starting point and a guide to the construction of a disequilibrium model suitable for estimation.
Bibliography


Appendix 1

Starting with the profit function as defined in 2.1.

\[ \pi_o = \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right] \]

and using the Cobb-Douglas production function

\[ y_t = A e^{\lambda t} x_{1t}^a x_{2t}^b \quad \forall \ t = 0, \ldots, \infty \]

we will prove that, under perfect competition, an interior solution of the profit maximization will imply

\[ 0 < a < 1, 0 < b < 1, a + b < 1 \text{ and } a > b. \]

In a market with perfect competition the producer has to solve the following programme

\[
\begin{align*}
\text{Max} & \quad \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right] \\
\text{s.t.} & \quad y_t = A e^{\lambda t} x_{1t}^a x_{2t}^b \quad \forall \ t = 0, \ldots, \infty \\
& \quad y_t, x_{1t}, x_{2t} > 0 \quad \forall \ t = 0, \ldots, \infty
\end{align*}
\]

This is equivalent to

\[
\begin{align*}
\text{Max} & \quad \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t A e^{\lambda t} x_{1t} x_{2t} - w_t (1+s_t) x_{1t} x_{2t} \\
& \quad - z_t x_{1t} - c_t \right] \\
\text{s.t.} & \quad x_{1t}, x_{2t} > 0 \quad \forall \ t = 0, \ldots, \infty
\end{align*}
\]
From the Kuhn-Tucker conditions we have for every $t = o, \ldots, \infty$

\[
\begin{align*}
 p_t A e^{\lambda t} a x_{1t}^{a-1} x_{2t}^b - w_t (1 + s_t) x_{2t} - z_t &< 0 \\
 x_{1t} [p_t A e^{\lambda t} a x_{1t}^{a-1} x_{2t}^b - w_t (1 + s_t) x_{2t} - z_t] &> 0 \\
 p_t A e^{\lambda t} b x_{1t}^{a} x_{2t}^{b-1} - w_t (1 + s_t) x_{1t} &< 0 \\
 x_{2t} [p_t A e^{\lambda t} b x_{1t}^{a} x_{2t}^{b-1} - w_t (1 + s_t) x_{1t}] &> 0
\end{align*}
\]

As we assume an interior solution, we must have for every $t = o, \ldots, \infty$

\[
\begin{align*}
 p_t A e^{\lambda t} a x_{1t}^{a-1} x_{2t}^b - w_t (1 + s_t) x_{2t} - z_t &= 0 \\
 p_t A e^{\lambda t} b x_{1t}^{a} x_{2t}^{b-1} - w_t (1 + s_t) x_{1t} &= 0
\end{align*}
\]

\[\Leftrightarrow\]

\[
\begin{align*}
 p_t a \frac{y_t}{x_{1t}} &= w_t (1 + s_t) x_{2t} + z_t \\
 p_t b \frac{y_t}{x_{2t}} &= w_t (1 + s_t) x_{1t}
\end{align*}
\]

\[\Leftrightarrow\]

\[
\begin{align*}
 a &= \frac{w_t (1 + s_t) x_{1t} x_{2t} + z_t x_{1t}}{p_t y_t} \\
 b &= \frac{w_t (1 + s_t) x_{1t} x_{2t}}{p_t y_t}
\end{align*}
\]

$a > 0, b > 0, a > b$

The second-order conditions for a maximum give additional constraints

Considering the Hessian matrix for period $t$
$$H_t = \begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} a(a-1) \\ 0 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1 x_2 \end{bmatrix}$$

$$H_t$$ is negative definite

1) $$0 < a < 1$$ and $$0 < b < 1$$

2) $$ab(a-1) (b-1) - a^2 b^2 > 0$$

$$\iff$$

$$a + b < 1$$

Feldstein, however, stated in his article "Specification of the Labour Input in the Aggregate Production Function" (1967) that the elasticity of output with respect to the number of men must be smaller than the elasticity of output with respect to the average hours.

He explains this by two reasons

We quote (p.377)

"First, an increase in hours increases the flow of "
"capital services as well. Because depreciation plus "
"interest rises less than in proportion to the number "
"of hours that the capital stock is used, longer hours "
"lower the cost per unit of capital services and may "
"cause net output to rise more than in proportion "
"to hours. Second, a number of hours during each "
"working week may be regarded as a fixed "
"labour cost with no corresponding output ... "

"These paid but non-productive hours do not "
"rise proportionately with the number of hours "
"officially worked. An increase in the length "
"of the official working week therefore entails "
"a more than proportionate increase in the "

number of hours actually worked; this raises $$\eta_H$$ "

"relative to $$\eta_N$$"(1).

(1) $$\eta_H = b$$  $$\eta_N = a$$
The first reason has clearly nothing to do with the production function which gives a technological relation. The second reason must be outweighed with the fact that increasing average hours worked decrease the output per man hour, caused by fatigue.
Appendix 2: Derivation of the effective demand and supply functions

1. The producer side of the model

1.1. The producer is rationed in the commodity market

\[
\begin{align*}
\text{Max} & \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right] \\
\text{s.t.} & \quad y_t = \Lambda e^{\lambda t} x_{1t}^a x_{2t}^b \\ 
& \quad t = 0, \ldots, \infty
\end{align*}
\]

The Lagrange function is

\[
L = \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right] \\
+ \sum_{t=0}^{\infty} u_t (\Lambda e^{\lambda t} x_{1t}^a x_{2t}^b - y_t) + \sum_{t=0}^{\infty} \eta_t (y_t - y_t)
\]

where \(u_t\) and \(\eta_t\) are the Lagrange Multipliers for \(t = 0, \ldots, \infty\).

The first-order conditions are given by

\[
\begin{align*}
k_t (1-u_t) y_t &= u_t + \eta_t \quad t = 0, \ldots, \infty \quad (1.1.1) \\
k_t (1-u_t) [w_t (1+s_t) x_{2t} + z_t] &= \eta_t a \frac{y_t}{x_{1t}} \quad (1.1.2) \\
k_t (1-u_t) [w_t (1+s_t) x_{1t}] &= u_t b \frac{y_t}{x_{2t}} \quad (1.1.3) \\
y_t &= \Lambda e^{\lambda t} x_{1t}^a x_{2t}^b \quad (1.1.4) \\
y_t &= y_t \quad (1.1.5)
\end{align*}
\]

We do not need the first equation to solve the system for \(x_{1t}\) and \(x_{2t}\).
Solving (1.1.3) for $\mu_t$ gives

$$\mu_t = k_t (1-u_t) \frac{w_t (1+s_t) x_{1t} x_{2t}}{b y_t}$$

and substituting in (1.1.2) gives

$$w_t (1+s_t) x_{2t} + z_t = \frac{a}{b} w_t (1+s_t) x_{2t}$$

$$\Rightarrow$$

$$x_{2t}^D = z_t w_t^{-1} (1+s_t)^{-1} \frac{b}{a-b} \quad (1.1.6)$$

$$\Rightarrow$$

$$\ln x_{2t}^D = \ln b - \ln (a-b) + \ln z_t - \ln w_t - \ln (1+s_t)$$

Substituting (1.1.5) and (1.1.6) in (1.1.4) and solving for $x_{1t}$ gives the effective demand for men

$$x_{1t}^D = A \frac{1}{a} \frac{b}{a-b} a e^{-\lambda t} \frac{b}{a} \frac{b}{a} (1+s_t)^{b-1}$$

$$\Rightarrow$$

$$\ln x_{1t}^D = \frac{1}{a} \ln A - \frac{b}{a} \ln b + \frac{b}{a} \ln (a-b) - \frac{\lambda}{a} t + \frac{b}{a} \ln w_t$$

$$- \frac{b}{a} \ln z_t + \frac{b}{a} \ln (1+s_t) + \frac{1}{a} \ln y_t$$

1.2. The producer is rationed on the number of workers

We can immediately write the Lagrange function with multipliers $\mu_t$ and $\eta_t$

$$L = \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right]$$

$$+ \sum_{t=0}^{\infty} \mu_t \left[ A e^{-\lambda t} a x_{1t} x_{2t} - y_t \right] + \sum_{t=0}^{\infty} \eta_t (x_{1t} - x_{1t})$$

The first-order conditions for a maximum are
\[ k_t (1-u_t) p_t = \nu_t \]  
(1.2.1)

\[ k_t (1-u_t) [w_t (1+s_t) x_{2t}^3 + z_t ] = \mu_t e^{\lambda t} a x_{1t}^{a-1} x_{2t}^b - \eta_t \]  
(1.2.2)

\[ k_t (1-u_t) w_t (1+s_t) x_{1t}^2 = \mu_t A e^{\lambda t} a x_{1t}^a x_{2t}^{b-1} \]  
(1.2.3)

\[ y_t = A e^{\lambda t} x_{1t}^a x_{2t}^b \]  
(1.2.4)

\[ x_{1t} = x_{1t} \]  
(1.2.5)

Substituting (1.2.1) and (1.2.5) in (1.2.3) and solving for \( x_{2t} \)
gives

\[ x_{2t}' = \frac{A}{1-b} \frac{1}{b} \frac{1}{1-b} \frac{\lambda t}{1-b} p_t \frac{1}{1-b} w_t \frac{1}{1-b} (1+s_t) \frac{1}{1-b} x_{1t}^{1-a} \]  
(1.2.6)

\[ \ln x_{2t}' = \frac{1}{1-b} \ln A + \frac{1}{1-b} \ln b + \frac{\lambda t}{1-b} + \frac{1}{1-b} \ln p_t - \frac{1}{1-b} \ln w_t \]

Substituting (1.2.6) together with (1.2.5) in (1.2.4) gives the effective commodity supply

\[ y_{t}' = \frac{A}{1-b} b \frac{b}{1-b} p_t \frac{b}{1-b} w_t \frac{b}{1-b} (1+s_t) \frac{b}{1-b} x_{1t}^{b-a} \]  
\[ \ln y_{t}' = \frac{1}{1-b} \ln A + \frac{b}{1-b} \ln b + \frac{\lambda t}{1-b} + \frac{b}{1-b} \ln p_t - \frac{b}{1-b} \ln w_t \]

1.3. The producer is rationed on the average hours of work per worker

The Lagrange function for this problem is given by
\[ L = \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t} - c_t \right] \\
+ \sum_{t=0}^{\infty} \mu_t A e^{t a x_{1t} x_{2t} - y_t} - \sum_{t=0}^{\infty} \eta_t (x_{2t} - y_t) \]

The first-order conditions for a maximum are

\[ k_t (1-u_t) p_t = \mu_t \]  
(1.3.1)

\[ k_t (1-u_t) \left[ w_t (1+s_t) x_{2t} + z_t \right] = \mu_t A e^{t a x_{1t} x_{2t} - y_t} \]  
(1.3.2)

\[ k_t (1-u_t) \left[ w_t (1+s_t) x_{1t} \right] = \mu_t A e^{t b x_{1t} x_{2t} - y_t} \]  
(1.3.3)

\[ y_t = A e^{t a x_{1t} x_{2t} - y_t} \]
(1.3.4)

\[ x_{2t} = x_{2t} \]  
(1.3.5)

Substituting (1.3.1) and (1.3.5) in (1.3.2) and solving for \( x_{1t} \) gives the effective demand for workers

\[ x_{1t} = \frac{1}{1-a} A e^{t a x_{1t} x_{2t} - y_t} - \frac{1}{1-a} p_t \left[ w_t (1+s_t) x_{2t} + z_t \right] \]  
(1.3.6)

\[ \ln x_{1t} = \frac{1}{1-a} \ln A + \frac{1}{1-a} \ln a + \frac{\lambda}{1-a} t + \frac{1}{1-a} \ln p_t + \frac{b}{1-a} \ln x_{2t} \]

Substituting (1.3.5) and (1.3.6) in (1.3.4) gives

\[ y_t = \frac{1}{1-a} A e^{t a x_{1t} x_{2t} - y_t} - \frac{1-a}{1-a} p_t \left[ w_t (1+s_t) x_{2t} + z_t \right] \]  
(1.3.8)

\[ \ln y_t = \frac{1}{1-a} \ln A + \frac{a}{1-a} \ln a + \frac{\lambda}{1-a} t + \frac{a}{1-a} \ln p_t + \frac{b}{1-a} \ln x_{2t} \]

\[ - \frac{a}{1-a} \ln \left[ w_t (1+s_t) x_{2t} + z_t \right] \]
1.4. The producer is rationed in the commodity market and on the number of workers

For the effective commodity supply we have to consider only the rationing on the number of workers. Therefore we can refer to 1.2.

For the effective demand for average hours of work per worker is found by solving the following programme

\[
\text{Max } \sum_{t=0}^{\infty} k_t (1-u_t) \left[ p_t y_t - w_t (1+s_t) x_{1t} x_{2t} - z_t x_{1t}^2 - c_t \right]
\]

s.t.
\[ y_t = A e^{\lambda t} x_{1t}^a x_{2t}^b \]
\[ y_t = y_t \]
\[ x_{1t} = x_{1t} \]

This gives
\[
y_t = A e^{\lambda t} x_{1t}^a x_{2t}^b
\]
\[
x_{2t} = y_t A^{-1} e^{-\lambda t} x_{1t}
\]
\[
x_{2t}^b = y_t A^{-1} e^{-\lambda t} x_{1t}
\]
\[
\ln x_{2t}^b = -\frac{1}{b} \ln A - \frac{\lambda t}{b} x_{1t} - \frac{a}{b} \ln y_t + \frac{1}{b} \ln y_t
\]

1.5. The producer is rationed in the commodity market and on the average hours of work per worker

The effective commodity is given in 1.3., the effective demand for average hours of work is found as in 1.1.

For the effective demand for workers we have to solve the following programme
\[ \max \sum_{t=0}^{\infty} k_t (1 - u_t) \left[ p_t y_t - w_t (1 + s_t) x_1 t x_2 t - z_t x_1 t - c_t \right] \]

s.t. \[ y_t = A e^{\lambda t} x_1 t x_2 t \]
\[ y_t = y_t \]
\[ x_2 t = x_2 t \]

We have
\[ y_t = A e^{\lambda t} x_1 t x_2 t \]
\[ \Leftrightarrow \quad \frac{1}{a} \lambda t - \frac{b}{a} \frac{1}{y_t} \]
\[ x_1 t = \frac{A}{a} e^{\lambda t} \frac{1}{y_t} x_2 t \]
\[ \Leftrightarrow \quad \ln x_1 t - \frac{1}{a} \ln A - \frac{\lambda t}{a} - \frac{b}{a} \ln y_t + \frac{1}{a} \ln y_t \]

1.6. The producer is rationed on the average hours of work per worker and on the number of workers

The effective demand for workers is given in 1.3., the effective demand for average hours of work is given in 1.2.

The effective commodity supply is nothing else than the production function with given inputs \( x_1 t \) and \( x_2 t \)
\[ y_s t = A e^{\lambda t} x_1 t x_2 t \]
\[ \Leftrightarrow \quad \ln y_s t = \ln a + \lambda t + a \ln x_1 t + b \ln x_2 t \]

1.7. The producer is rationed in all the markets

The effective commodity supply is found as in 1.6., the effective demand for workers as in 1.5. and the effective demand for hours as in 1.4.