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DECOMPOSABILITY, SRAFFA'S PRICE SYSTEM AND THE STANDARD COMMODITY

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A B S T R A C T

We establish some mathematical theorems on the number of linearly independent semipositive resp. positive eigenvectors of a decomposable nonnegative matrix. These theorems may have various interesting economic applications; in this paper they are used to analyse a Sraffian single product system that contains both basic and nonbasic commodities. We formulate propositions about the Sraffa price system characterizing its solutions for all nonnegative values of the rate of profits. We also consider Sraffa's standard system and solve questions such as "How many different semipositive resp. positive standard commodities exist in a decomposable system" ? Moreover, our results have an economically meaningful interpretation in terms of the mutual dependence of the different blocks of industries and their internal rate of surplus production.

DECOMPOSABILITY, SRAFFA'S PRICE SYSTEM AND THE STANDARD COMMODITY.

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1. INTRODUCTION

Many authors have shown that the mathematical theory of nonnegative matrices is an effective tool for a formal treatment of Sraffa's model of single product industries.[2] Usually they assume that there exist only basics, i.e., commodities that enter directly or indirectly into the production of all commodities. Such an assumption is most often made because a system without nonbasics has an indecomposable input matrix, which facilitates the mathematical argument. Besides, Sraffa himself, probably influenced by writings of Ricardo and von Bortkiewicz, takes it for granted that nonbasics play a rather passive role in the determination of the relevant magnitudes of his economic model and he temporarily eliminates the nonbasics to simplify his discussion [1960,pp.7-8,25]. Nevertheless, Sraffa realizes that such an approach could lead to some freak results in his price system when the nonbasics are reintroduced at the end of the analysis [1960,pp.90-91]. We shall see that more complications are possible. In order to avoid prejudicing the issue, we prefer to work in a more general framework where basics and nonbasics coexist. We shall establish some new theorems on decomposable matrices and use them to analyse a decomposable model of single product industries.

[1] I wish to thank Wilfried Pauwels for his remarks on an earlier version, especially with regard to the mathematical argument in the first part of this paper. I am also indebted to Lode Berlage for his comments on previous drafts. It is obvious, however, that none of them bears any responsability for the shortcomings of the present text.

[2] See Sraffa [1960]. We refer to Pasinetti [1977] for a detailed matrix algebraic introduction to linear models of production. We also draw special attention to a recent book by Roncaglia [1978], who provides an extensive bibliography of about 500 items relating to Sraffa's work.

We formulate some results on Sraffa's price system which are more exhaustive than those of Newman |1962| and Zaghini |1967|. We pay special attention to quantity systems that produce a so called "standard composite commodity". The literature on Sraffa's standard commodity includes papers by Blakley and Gossling |1967|, Burmeister |1968|, Eatwell |1975| Miyao |1977|, Newman |1962| and many others. In most of these papers, however, few attention is paid to specific problems arising in decomposable systems, whereas our theorems enable us to solve questions such as "How many different semipositive resp. positive standard commodities exist in a system with basics and nonbasics?". Moreover, we formulate our answers in terms which have an economically meaningful interpretation.

The plan of this paper is as follows. In Section 2 we prove some new mathematical theorems that allow us to determine the number of linearly independent semipositive resp. positive eigenvectors of any decomposable nonnegative matrix. Ourresults on row eigenvectors are helpful to analyse some properties of Sraffa's price system in Section 3; our theorems on column eigenvectors are useful to formulate some propositions about Sraffa's standard system in Section 4. A few peculiarities are commented upon and illustrated by means of numerical examples in Section 5. Some concluding remarks end the paper.

2. EIGENVECTORS OF DECOMPOSABLE MATRICES

In this Section we wish to establish some new mathematical theorems on the eigenvectors of decomposable matrices. These theorems are interesting in themselves and they may also have various economic applications outside the Sraffa world. Some aspects of their proofs may help to get a better insight into the structure of Sraffa's or any other linear system. Therefore it seems worth while to include the proofs in the present Section instead of using an appendix. We start with some preliminaries concerning notation, terminology and the like in Section 2.1; we then have to follow a rather lenghty reasoning in order to determine the number of linearly independent semipositive column vectors of a decomposable nonnegative matrix in Section 2.2; fortunately the same reasoning is also useful when considering the case of positive column eigenvectors in Section 2.3 and that of row eigenvectors in Section 2.4. Impatient readers may note that the theorems are formulated at the end of the Sections 2.2, 2.3 and 2.4.

2.1. Some preliminaries

We consider a nonnegative[3] and decomposable[4] square matrix A with generic element a_{ij} (i-th row, j-th column : input of the i-th commodity to produce a unit of output of the j-th), where i and j belong to the set of industries {1,2, ..., n}. By a suitable permutation of the rows and the corresponding

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[3] Note that throughout this paper we consider only scalars, vectors and matrices that are real. We use the usual ordering relations. For example, we say that vectors and matrices are nonnegative (denoted by $\geqq 0$) iff (if and only if) all their elements are nonnegative, semipositive ($\geq$) iff all elements are nonnegative and at least one is positive, positive (> 0) iff all elements are positive.

[4] A square matrix A of order n is said to be decomposable iff the set {1,2, ..., n} can be partitioned into two subsets I and J such that $a_{ij} = 0$ if $i \in I$, $j \in J$. In all other cases, we call the square matrix A indecomposable; observe that the zero matrix of order one is called indecomposable throughout this paper.

columns (i.e., by a suitable numbering of the industries) we can put A into the normal form[5]

$$(1) \quad A = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1m} \\ 0 & A_2 & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_m \end{bmatrix};$$

where the submatrices A_j $(j=1,2,\ldots,m)$ on the diagonal are all square and indecomposable. In accordance with the normal form the set of industries is partitioned into m subsets. We use the following terminology.

DEFINITION 1. The industries associated with the submatrix A_j are said to constitute block j. We say that block k depends on another block j iff the normal form contains a chain of nonzero submatrices A_{jq}, A_{qr}, ..., A_{st}, A_{tk}, i.e. a chain from j to k. If k does not depend on j and vice versa, then k and j are called mutually independent.

Observe that block k is said to depend on another block j iff the industries of k require directly or indirectly some inputs from industries of j. This is possible only if $j<k$, i.e., only if block j is "earlier" than block k in the ordering of the normal form.

The dominant root (Frobenius eigenvalue, spectral radius) of the matrix A is denoted by s, that of the matrix A_j by s_j. It is well known that s equals the maximum of s_1, s_2, ..., s_m and that $s_j>0$ unless A_j is the zero matrix of order one (then $s_j=0$, of course). Throughout this paper we often use the following propositions[6] that hold resp. for a Decomposable (D) matrix, say A, and an Indecomposable (I) matrix, say A_j.

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[5] The notion of a normal form of a decomposable matrix has been discussed by Debreu and Herstein [1953, p.600]; Dorfman, Samuelson and Solow [1958, pp.254-260]; Gantmacher [1959, pp.74-80]; Solow [1952, pp.33-35] and many others.

[6] These propositions can easily be compiled from Debreu and Herstein [1953] and Gantmacher [1959].

(D1) Associated with its dominant root s the matrix A has a semipositive column eigenvector.

(I1) Associated with its dominant root s_j the matrix A_j has a positive column eigenvector; A_j does not have two linearly independent semipositive column eigenvectors.

(D2) $(\beta I - A)^{-1} \geq 0$ iff $\beta > s$.

(I2) $(\alpha I - A_j)^{-1} > 0$ iff $\alpha > s_j$.

(D3) If for a column vector $x \geq 0$, $\beta x > Ax$, then we have $\beta > s$.

(I3) If for a column vector $y \geq 0$, $\alpha y \geq A_j y$, then we have $\alpha > s_j$.

Intuitively speaking, an indecomposable matrix is easier to handle and has sharper properties than a decomposable one. In order to obtain results on decomposable matrices that are as sharp as possible, one has to use such a fine partitioning as in the normal form, where only indecomposable matrices appear on the diagonal. An often succesful strategy, which was used as early as 1912 by Frobenius /1912, pp.472-474/, is to consider in turn each of the blocks of the normal form instead of the whole matrix A at once. It is thus not surprising that the properties of the m blocks, especially their mutual dependence and the value of their s_j, will play a crucial role in our paper. For our purposes, it is very important to note that <u>any block k belongs to one and only one of the following three sets:</u>

$$
\begin{aligned}
(2) \qquad & G = \{k : s_k > s_j \quad \text{for all } j < k\} \\
& H = \{k : s_k \leqq s_j \quad \text{for some } j < k \text{ and such that } k \text{ depends on } j\} \\
& L = \{k : s_k \leqq s_j \quad \text{for some } j < k \text{ but } k \text{ does not depend on any such } j\}
\end{aligned}
$$

The set G always contains block 1. The sets H and L may be empty. The number of elements (cardinality) of these sets is denoted by $|G|$, $|H|$ and $|L|$. Hence $|G| + |H| + |L| = m$.

2.2. Semipositive Column Eigenvectors

We want to determine the number of linearly independent semipositive column eigenvectors of the decomposable matrix A, i.e., the number of linearly independent solutions q of the system

$q \geq 0$;

$\lambda q = Aq$ for some scalar λ.

In accordance with the normal form (1) we partition the column vector q into m subvectors. We distinguish the subvectors by subscripts. We use the symbol e to denote a vector (of appropriate order) exclusively made up of ones. Such a summation vector is introduced here only for normalization purposes. The number of linearly independent solutions q does not change if we concentrate on the following more manageable system:

$$q = (q_1 \quad q_2 \quad \dots \quad q_m) \geq 0 \ ;$$

$$\text{if } q_k \geq 0 \text{ and } q_j = 0 \text{ for all } j > k, \text{ then } eq_k = 1;$$

(3)

$$\lambda q_j = A_j q_j + \sum_{h=j+1}^{m} A_{jh} q_h \quad (j=1,2,\dots,m-1) \ ;$$

$$\lambda q_m = A_m q_m \text{ for some scalar } \lambda.$$

LEMMA 1. Let the column vector q be a solution of the system (3) and denote its last nonzero subvector by q_k. Then $\lambda = s_k$, and q_k is positive and unique.

PROOF. If q_k is the last nonzero subvector, then $\lambda q_k = A_k q_k$. The rest follows from proposition (I1).

This simple lemma leads us to the following strategy to obtain the maximal number of linearly independent solutions of the system (3) above. We consider each block k in turn in natural order (i.e.,we first put k=1, then k=2, ..., finally k=m) and we try to associate with each block k (i.e., with the eigenvalue s_k) semipositive eigenvectors by a sort of "backward procedure" as follows:

1) we set $q_j = 0$ for all $j>k$.
2) we fix q_k (positive and unique) such that $s_k q_k = A_k q_k$ and $eq_k=1$.
3) we investigate whether successively subvectors q_{k-1}, q_{k-2}, ..., q_1 can be constructed so that a complete solution vector q is obtained.

In the following pages we shall specify in more detail how to compute these subvectors. To distinguish eigenvectors q corresponding to different eigenvalues s_k we shall use superscripts (but only when necessary, because of the heavy notation). For example, the "backward procedure" will associate with block 1 a column eigenvector $q^1=(q_1^1\ 0\ 0\ \ldots\ 0)$; with block 2 we shall try to associate a column eigenvector $q^2=(q_1^2\ q_2^2\ 0\ \ldots\ 0)$; etc. It is obvious that we use the word "backward" because the subvector q_2^2 is computed before the subvector q_1^2, but we stress again that we first put k=1, then k=2, etc. so that the complete eigenvector q^1 has already been constructed before we try to form q^2. When applying a "backward procedure" as outlined above, we shall have to distinguish three possible cases according as block k belongs to the set G, H or L. We shall see that for each block k of G and L we can construct exactly one semipositive column eigenvector that is linearly independent of eigenvectors constructed earlier.

<u>CASE 1 : k ε G</u>

After $q_j=0$ for all $j>k$, and $q_k \gg 0$, have been fixed as described above, we obtain q_{k-1} by solving

(4) $$s_k\, q_{k-1} = A_{k-1}q_{k-1} + A_{k-1,k}q_k.$$

From $s_k>s_{k-1}$ (as $k \varepsilon G$) and proposition (I2), it follows that s_kI-A_{k-1} has a positive inverse. Hence we have

(5) $$q_{k-1} = (s_kI - A_{k-1})^{-1}\, A_{k-1,k}q_k\ ,$$

which means that q_{k-1} is unique and such that (see Appendix)

$$q_{k-1} > 0 \text{ if } k \text{ depends on } k-1 ;$$
(6)
$$q_{k-1} = 0 \text{ if } k \text{ does not depend on } k-1.$$

Proceeding in this way we find successively unique subvectors q_j for $j=k-2, k-3, \ldots, 1$:

(7) $$q_j = (s_k I - A_j)^{-1} \sum_{h=j+1}^{k} A_{jh}\, q_h ,$$

where it can easily be shown by induction (see Appendix) that

$$q_j > 0 \text{ if } k \text{ depends on } j ;$$
(8)
$$q_j = 0 \text{ if } k \text{ does not depend on } j.$$

In this way we can associate with every element k of G exactly one column eigenvector

(9) $$q^k = (q_1^k \ \ldots \ q_k^k \quad 0 \ \ldots \ 0) ,$$

so that we obtain $|G|$ linearly independent semipositive eigenvectors of the matrix A. (They correspond to $|G|$ different eigenvalues.)

CASE 2 : $k \in H$

In this case k depends on some other block j such that $s_k \leq s_j$. We denote the last (the nearest to k) of such j by g. This means that we can determine $q_j=0$ for all $j>k$, $q_k>0$, and $q_{k-1}, \ldots, q_{g+1}$ as above, until we arrive at the equation involving block g:

(10) $$s_k q_g = A_g q_g + \sum_{h=g+1}^{k} A_{gh}\, q_h .$$

The second part of the R.H.S. (right hand side) is semipositive because k depends on g. From $s_k \leq s_g$ and proposition (I3), it follows that (10) has no nonnegative solution q_g. We conclude that it is impossible to construct a semipositive column eigenvector q with last nonzero subvector q_k if $k \in H$.

CASE 3 : $k \varepsilon L$

We associate with every block $k\varepsilon L$ one semipositive column eigenvector by using the following criteria. We fix $q_j=0$ for all $j>k$ and $q_k>0$ as usual. If $j<k$, there are two possibilities: either $s_k>s_j$ or j belongs to the following set of blocks:

(11) $J_k = \{j : s_k \leqq s_j \text{ , } j < k \text{ , } k \text{ does not depend on } j\}$.

If $j\varepsilon J_k$ we put $q_j=0$ (we discuss the uniqueness of this solution immediately); if $s_k>s_j$ we can compute q_j as in (7) above. By using these criteria we associate with every $k\varepsilon L$ a complete semipositive eigenvector and this vector satisfies (8).

If $k\varepsilon L$ we sometimes could associate with the eigenvalue s_k more then one semipositive column eigenvector having the form $q=(q_1\dots q_k\ 0\ \dots 0)$ and $eq_k=1$. Indeed, if $j\varepsilon J_k$ we have put the subvector $q_j=0$, but there could be a sort of "tie", i.e., other nonnegative values of this subvector may exist. We now explain why we do not have to retain these other possible solutions. To this end, we now consider in more detail what happens when we solve the system (3) backwards in case of $k\varepsilon L$, $\lambda=s_k$, $q_k>0$, $eq_k=1$. We denote the block of J_k that is nearest to k by f, which means that all subvectors $q_{k-1}, q_{k-2},\dots,q_{f+1}$ are uniquely determined as in(7), until we arrive at the equation involving block f,

(12) $s_k\ q_f = A_f\ q_f + 0$,

where the second part of the R.H.S. is zero because k does not depend on f. (Compare (12) with (10) above). We shall show why it is sufficient to consider only $q_f=0$ and to neglect other possible values of this subvector. (Analogous arguments could then be repeated with reference to the other elements of J_k). Arriving at equation (12), we are confronted with one of the following four situations:

(i) $s_k < s_f$

In this case there is no problem of uniqueness. Indeed, we know from (I1) that $q_f=0$ is the only nonnegative solution of equation (12).

(ii) $s_k = s_f$ and $f\varepsilon G$

We note that $f<k$. Hence, we have already obtained the semipositive eigenvector q^f (whose last nonzero subvector is q_f^f, such that $s_f q_f^f = A_f q_f^f$ and $eq_f^f=1$) before we try to construct a semipositive eigenvector q^k (with last nonzero subvector q_k^k). When we construct q^k backwards and we meet equation (12), there arises a "tie", i.e., besides $q_f^k=0$ we could also consider $\bar{q}_f^k=\alpha_f q_f^f$ (where α_f denotes a positive scalar) as f-th subvector. We know that $f\varepsilon G$ implies that the matrix $s_f I-A_j$ has a positive inverse for all $j<f$. Thus, once we have decided how to fix the f-th subvector, the values of all earlier subvectors (f-1, f-2, ...,1) are uniquely determined as in formula (7). We conclude that besides the semipositive column eigenvector

(13) $$q^k = (\ldots \quad q_{f-1}^k \qquad 0 \quad q_{f+1}^k \ldots q_k^k \quad 0 \quad \ldots \quad 0),$$

we could also construct the semipositive column eigenvector

(14) $$\bar{q}^k = (\ldots \quad q_{f-1}^k + \alpha_f q_{f-1}^f \quad \alpha_f q_f^f \quad q_{f+1}^k \ldots q_k^k \quad 0 \quad \ldots \quad 0).$$

However, such an additional solution cannot be retained, because it is only a linear combination of other semipositive eigenvectors. Indeed, we have

(15) $$\bar{q}^k = q^k + \alpha_f \, q^f.$$

(iii) $s_k = s_f$ and $f\varepsilon H$

Besides the zero solution, equation (12) also has a positive solution which is unique up to a scalar multiple by proposition (I1). However, a positive value of the f-th subvector cannot be retained, because it would make it impossible to construct a semipositive eigenvector (for the same reasons as in Case 2 above).

(iv) $s_k = s_f$ and $f \in L$

In this case equation (12) again leads to a "tie". Moreover, suppose that other "ties" appear later on when we meet the equations of some other blocks of J_k, say (without loss of generality) the blocks d and b ($k>f>d>b$; $s_k=s_f=s_d=s_b$; $k\in L$, $f\in L$, $d\in L$, $b\in L$ or $b\in G$). Our criteria set set out in the first paragraph of Case 3 above, lead to the construction of an eigenvector q^k with $q^k_f=0$, $q^k_d=0$ and $q^k_b=0$, but an additional solution is the column eigenvector

(16) $\bar{q}^k = (\ldots\ \alpha_b q^b_b\ \ldots\ \alpha_d q^d_d\ \ldots\ \alpha_f q^f_f\ \ldots\ q^k_k\ 0\ \ldots\ 0)$,

where α_f, α_d and α_b denote positive scalars. Again, such a solution cannot be retained because it is only a linear combination of eigenvectors constructed earlier:

(17) $\bar{q}^k = q^k + \alpha_f q^f + \alpha_d q^d + \alpha_b q^b$

We have seen that, by using the criteria described in Case 1 and the first paragraph of Case 3, we associate with every element of G and L exactly one semipositive column eigenvector of the matrix A. Each of these $|G| + |L|$ eigenvectors has a different number of zero subvectors at its bottom following the last nonzero subvector. A linear combination of such vectors can equal zero only if all scalar coefficients of the combination are zero. Thus, these $|G| + |L|$ eigenvectors are linearly independent. We have also indicated that any other possible semipositive column eigenvector turned out to be a linear combination of some of these $|G| + |L|$ vectors. All this may be summarized as follows.

THEOREM 1. The nonnegative matrix A with normal form (1) has $|G| + |L|$ linearly independent semipositive column eigenvectors, where the sets G and L are defined in expression (2).

Note that $|G| + |L|$ equals $m-|H|$ and that an indecomposable matrix would have $m=1$, $|G| = 1$, $|H| = |L| = 0$.

2.3. Positive Column Eigenvectors

In this section we look for a similar theorem on positive column eigenvectors. From Lemma 1, it is obvious that a positive column eigenvector q, if any exists, has to correspond to the eigenvalue $\lambda=s_m$, where m denotes the last block of the normal form. This block m either belongs to G or to H or to L. When considering these three cases, we can use our arguments of Section 2.2. above.

CASE 1. $m\varepsilon G$ (and thus $s=s_m>s_j$ for all $j\neq m$). From (7) and (8) it follows that there is exactly one linearly independent positive column eigenvector q^m if m depends on all other blocks, and that otherwise no positive column eigenvector exists.

CASE 2. $m\varepsilon H$. In this case it is impossible to construct a positive column eigenvector.

CASE 3. $m\varepsilon L$. We suppose, without loss of generality, that exactly three other blocks, say f, d and b, have dominant roots not smaller than s_m. Then, the following three conditions are necessary for the existence of a positive column eigenvector: 1) from Section 2.2, Case 3(i), it follows that we must have $s_m=s_f=s_d=s_b$; 2) in order to avoid situations as in Section 2.2, Case 3 (iii), it is necessary that m, f, d and b are all mutually independent; 3) from (7) it follows that any other block, say j, has to be such that at least one of the blocks m, f, d or b depends on j. Actually, when these three conditions do hold, there appear three "ties" during the backward construction of the column eigenvector corresponding to s_m: from the discussion in Section 2.2, Case 3 (iv), we know that, besides the semipositive column eigenvector q^m with $q^m_f=0$, $q^m_d=0$, $q^m_b=0$, we can also construct positive column eigenvectors having the following form.

(18) $\quad q^{mi} = (\ldots\ \alpha_{bi}q^b_b\ \ldots\ \alpha_{di}q^d_d\ \ldots\ \alpha_{fi}q^f_f\ \ldots\ \alpha_{mi}q^m_m)$,

where the scalars α_{bi}, α_{di}, α_{fi}, α_{mi} are all chosen to be positive.

It is possible to construct up to four linearly independent positive eigenvectors of this type, provided that we choose the scalars in (18) such that the four vectors (α_{bi}: α_{di} α_{fi} α_{mi}), i=1,2,3,4, are linearly independent. The discussion of this section can be summarized as follows.

THEOREM 2. a) The nonnegative matrix A with normal form (1) has a positive column eigenvector if and only if there exists a nonempty set of blocks, say D, such that

1. any block j in D has $s_j=s$ and any two different blocks of D are mutually independent.
2. any block j not in D has $s_j<s$ and is such that at least one block of D depends on j.

b) When nonzero, the number of linearly independent positive column eigenvectors equals $|D|$.

c) Any positive column eigenvector corresponds to the same eigenvalue, viz. $\lambda=s$.

Actually, the first part of this theorem is similar to a theorem of Gantmacher /1959, p.77/. It might be useful to illustrate Theorem 2 by a trivial example. Consider the identity matrix of order n. Its normal form has n blocks, all of order one. These blocks all belong to D. There are n linearly independent positive column eigenvectors and they all correspond to the same eigenvalue s=1.

2.4. Row Eigenvectors

It is easy to formulate analogous results on row eigenvectors as soon as we realize that the matrix A and its transpose A^T have the same eigenvalues and that a row eigenvector of A corresponds to a column eigenvector of A^T. From the normal form (1) of A we have

$$(19) \qquad A^T = \begin{bmatrix} A_1^T & 0 & \dots & 0 \\ A_{12}^T & A_2^T & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{1m}^T & A_{2m}^T & \dots & A_m^T \end{bmatrix}$$

Reversing the order of the blocks leads to the normal form of A^T:

$$(20)\quad \begin{bmatrix} A_m^T & \dots & A_{2m}^T & A_{1m}^T \\ \dots & \dots & \dots & \dots \\ 0 & \dots & A_2^T & A_{12}^T \\ 0 & \dots & 0 & A_1^T \end{bmatrix}$$

Now theorems 1 and 2 allow us to find the number of linearly independent semipositive resp. positive column eigenvectors of A^T. For a translation in terms of theorems on row eigenvectors of A, we have to introduce the following sets:[7]

$$(21)\quad \begin{aligned} G^T &= \{k : s_k > s_j \quad \text{for all } j>k\} \\ H^T &= \{k : s_k \leqq s_j \quad \text{for some } j>k \text{ and such that } j \text{ depends on } k\} \\ L^T &= \{k : s_k \leqq s_j \quad \text{for some } j>k, \text{ but such } j \text{ do not depend on } k\} \end{aligned}$$

THEOREM 3. The nonnegative matrix A with normal form (1) has $|G^T| + |L^T|$ linearly independent semipositive row eigenvectors, where the sets G^T and L^T are defined in expression (21).

THEOREM 4. a) The nonnegative matrix A with normal form (1) has a positive row eigenvector if and only if there exists a nonempty set of blocks, say D^T, such that

1. any block j in D^T has $s_j=s$ and any two different blocks of D^T are mutually independent.
2. any block j not in D^T has $s_j<s$ and depends on at least one block of D^T.

b) When nonzero, the number of linearly independent positive row eigenvectors equals $|D^T|$.

c) Any positive row eigenvector corresponds to the same eigenvalue, viz. $\lambda=s$.

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[7]Compare (1) and (2) with (20) and (21).

3. SRAFFA'S PRICE SYSTEM

In the rest of this paper we use the preceding theorems while analysing a Sraffian single product system that contains both basics and nonbasics. The normal form (1) of the input matrix of such a Sraffian system has the following properties:

(22) the matrices A_j $(j=1,2,...,m)$ on the diagonal are all square and indecomposable;

(23) the semipositive matrix A_1 corresponds to the block of basic industries;

(24) each of the nonnegative matrices $A_2, ..., A_m$ corresponds to a block of nonbasic industries;

(25) in each column at least one of the matrices $A_{1j}, ..., A_{j-1,j}$ $(j=2,...,m)$ is semipositive (because each block of nonbasics depends on at least one other block. For example, each of them depends on block 1, i.e., the block of basics).

Just like Sraffa we assume that the net product of the economy is divided at the end of the production period between wages and profits. The actual net product is denoted by the column vector c. The row vector L is made up of the direct labour inputs per unit of the n commodities. We assume that c and L are positive vectors: this implies that the economic system is productive and that direct labour is indispensable to produce any good.[8] We choose the actual gross outputs and total labour employed as units of measurement. Let e (a vector of n ones) be the column vector of gross outputs, p a row vector of n prices, w the wage rate and r the rate of profits. Then we can write the following equalities:

[8] Weaker mathematical assumptions are possible, but they are not important from the economic point of view. For example, all the propositions of Section 3 can also be proved under the weaker assumption $L_1 \geq 0$, $L_j \geqq 0$ $(j=2,...,m)$ where L_j denotes the subvector that contains the direct labour coefficients of block j.

(26) $\bar{c} = (I - A)e$

(27) $Le = 1$

(28) $p = (1+r)pA + wL$

(29) $p\bar{c} = r\,pAe + w$

The last equation illustrates how the net product is divided into two portions. It is well known that the existence of a positive net product ($\bar{c}>0$) implies that the dominant root s is smaller than one and thus, a fortiori, $s_j<1$ for all j. In loose language one could say that the smaller is s, the more productive is the economy; one could say the same about s_j and the internal (i.e., considering only inputs from inside) surplus production within the j-th block of industries. This may explain the following terminology.

DEFINITION 2. We say that $R_A=(1-s)/s$ is <u>the rate of surplus production of the whole economy</u> and that $R_j=(1-s_j)/s_j$ ($+\infty$ if $s_j=0$) is the <u>(internal) rate of surplus production of the j-th block of industries</u>.

Observe that R_1 is finite ($s_1\neq 0$ because A_1 is a semipositive indecomposable matrix), that all R_j are positive ($0\leq s_j<1$ for all j) and that the smallest of them equals R_A.

Using a now obvious notation we partition (28) in accordance with the normal form of A and we choose the actual net product as the numeraire.[9] We then obtain the following <u>Sraffa price system</u>:

(30.a) $p_1(I-(1+r)A_1) = wL_1;$

(30.b) $p_j(I-(1+r)A_j) = wL_j + \sum_{h=1}^{j-1} p_h A_{hj}\,(1+r) \qquad (j=2,\ldots,m)$

(30.c) $p\bar{c} = 1$

These n+1 price equations contain n+2 variables, viz. r, w and the n elements of the row vector p. The rate of profits r will be treated as the independent variable.[10] The following propositions are somewhat

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9 See Sraffa [1960, p.11]

10 See Sraffa [1960, p.33]

more exhaustive than those of Newman /1962/ and Zaghini /1967/ and they can be proven with the help of our theorems on row eigenvectors (an outline of the proofs is given in the Appendix of this paper). The propositions show what values of w and p correspond to different nonnegative values of the independent variable r in an economy with basics and non-basics.

PROPOSITION 1. To any given r in the interval $0 \leq r < R_A$ there corresponds one and only one pair (w,p) such that (r,w,p) solves the Sraffa price system; both w and p are positive.

PROPOSITION 2. a) If (R_A, w, p) solves the Sraffa price system, then $w=0$.

b) If $R_j > R_1$ for all $j>1$, there exists a unique and positive p such that $(R_A,0,p)$ solves the Sraffa price system.

c) If $R_j \leq R_1$ for some $j>1$, there exist some[11] semipositive but no positive p such that $(R_A,0,p)$ solves the Sraffa price system.

PROBOSITION 3. Let r lie in the interval $R_A < r < +\infty$ and let (r,w,p) solve the Sraffa price system. Then it is impossible that both w and p are nonnegative except in the following case: $r=R_k$ for some $k \in G^T UL^T$, $w=0$, $p \geq 0$ (but not >0).[12]

PROPOSITION 4. Let $(\bar{r},\bar{w},\bar{p})$ and $(\bar{\bar{r}},\bar{\bar{w}},\bar{\bar{p}})$ be two solutions of the Sraffa price system and assume that $0 \leq \bar{r} < \bar{\bar{r}} < R_A$. Then the following strict inequalities hold: a) $\bar{p}/\bar{w} < \bar{\bar{p}}/\bar{\bar{w}}$

b) $\bar{w} > \bar{\bar{w}}$

[11] Not necessarily unique: it is possible that $s=1/(1+R_A)$ is not a simple root and that more than one linearly independent semipositive row eigenvector p corresponds to s.

[12] $G^T UL^T$ is the union of sets defined in Section2, expression (21). Actually, this union contains all blocks k with the following property: there it no other block j that depends on k while having an equal or smaller internal rate of surplus production.

In a Sraffian system production is regarded as a circular process in which the same kind of commodities appear both among the inputs and among the outputs. Sraffa knew that in such a framework some finite value of the rate of profits is the highest possible from the economic standpoint.[13] Sraffa calls Maximum rate of profits "the rate of profits as it would be if the whole of the national income went to profits" /1960, p.17/, i.e., the value of r that corresponds to $w=0$. Now there are as many such values of r as there are (real) eigenvalues of the matrix A. From Proposition 3 we conclude that the highest value of r for which zero w and nonnegative p exist, is the maximum of R_k for all $k \in G^T U L^T$, say R_M. The point $r=R_M$, however, is "isolated" in the sense that any other value of r in a sufficiently small neighborhood cannot be associated with nonnegative w and p (unless $R_M=R_A$, of course). The same "isolation" holds for all other $R_k>R_A$. Taking all this into account, it seems reasonable to consider R_A, not R_M, as the Maximum rate of profits: smaller values of r correspond to unique and positive w and p (Proposition 1), higher values of r lead to anomalies (Proposition 3). Besides, a numerical example in Section 5 will show that the inverse relation[14] between the wage and the profit rate (Proposition 4) does not necessarily hold once the value of r exceeds R_A.

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[13] Sraffa's analysis helped Robinson to correct an error in an earlier edition of her book on capital accumulation /1969, pp.414, 426/. It is also relevant to the work of Gallaway and Shukla /1974; 1976/, who had concluded /1974, p.358/ that within two-commodity indecomposable systems "reswitching of techniques" is impossible if positive prices exist for all positive values of the profit rate. This led to replies by Sato /1976/ and Garegnani /1976/. The latter used Sraffa's results and called attention to the fact that not all positive values of the rate of profits have economic significance.

[14] Proposition 4 can be interpreted as follows: a lower rate of profits ($\bar{r}<\bar{\bar{r}}$) implies a lower value in terms of labour commanded for all commodities ($\bar{p}_i/\bar{w}<\bar{\bar{p}}_i/\bar{\bar{w}}$ for any commodity i) or a higher value of the wage in terms of any commodity ($\bar{w}/\bar{p}_i>\bar{\bar{w}}/\bar{\bar{p}}_i$ for any i); then it is also intuitively clear that a lower value of $\bar{r}$ implies a higher value of the wage when the latter is expressed in terms of a positive composite commodity like the actual net product ($\bar{w}>\bar{\bar{w}}$). Proposition 4 is often used in the literature, but do note that a proof of the strict inequalities is not trivial for an economy with basics and nonbasics, because it involves the use of the properties (22)∿(25) of the normal form of the input matrix.

It is possible to present an intuitive economic explanation of the anomalies that arise when the uniform rate of profits r exceeds R_A. We give only the following representative example. Assume that R_k is the smallest of all R_j (this implies $R_A=R_k$) and consider the following situation: $R_A=R_k<r<R_j$ and k depends on j. In such a case block j has no problems, but block k has an internal rate of surplus production R_k that is too small to generate by itself a profit rate r. Moreover, block k has to pay for the inputs it requires (directly or indirectly) from block j. Hence, block k cannot achieve such a high profit rate ... unless it "pays" negative prices for some of its means of production and possibly also a negative wage.[15]

[15] See also Sraffa's [1960, pp. 90-91] and Zaghini's [1967, pp. 259-261] explanation of negative prices in the simple case of a self-reproducing nonbasic.

4. SRAFFA'S STANDARD SYSTEM

After having used our theorems on row eigenvectors in the preceding section on Sraffa's price system, we now employ our mathematical results on column eigenvectors to analyse some (hypothetical) quantity systems, more precisely those systems that produce a peculiar commodity mixture of the following type:

DEFINITION 3. A column vector q is called a standard (composite) commodity if and only if it solves the system

$$q = Aq + RAq,$$

(31) $$Lq = 1$$

for some scalar R. We also say that the above system is a standard system whose standard net product is (I-A)q and whose standard ratio is R.

This definition, including the normalization Lq=1, corresponds to that of Sraffa /1960,pp.19-21/. Observe that in a standard system "the various commodities are produced in the same proportions as they enter the aggregate means of production" /1960,p.20/. Sraffa could also have used a more suggestive term like "autogood" instead of "standard commodity": in the system (31) above the gross product (=q), the aggregate input (=Aq=q/(1+R)) and the net product (=(I-A)q=RAq=Rq/(1+R)) consist of different quantities of the self-same (composite) commodity q.

Such a composite commodity has a lot of interesting properties. We mention only three of them. First, if q is a standard commodity and R its standard ratio, then postmultiplication of the price equation (28) by q yields

$$p(I-A)q = rpAq + wLq;$$

if the standard net product (I-A)q is used as the numeraire, this becomes

$$1 = r/R + w \quad ,$$

i.e., Sraffa's well-known linear relation between the wage and the profit rate in a system with a given output and technique. With another numeraire such a simple linear relation cannot be guaranteed, unless when there is equal organic composition of capital and in some other special cases.[16] Secondly, Meek /1961/, Eatwell /1975/ and many others have indicated that a standard commodity may be an interesting tool to analyse some aspects of the relation between the rate of surplus value and the rate of profit in Marxian systems. Thirdly, from a formal point of view, a solution of the standard system corresponds to a balanced growth vector, if we assume that constant returns to scale prevail and that workers receive only a fixed basket of wage goods included[17] in the input matrix A.

There is thus no doubt that a standard commodity is a useful analytical tool for many problems. However, most authors do not pay much attention to the fact that more than one semipositive standard commodity may exist. Sraffa himself sometimes gives the impression that only one value of the standard ratio R can be associated with semipositive solutions of the q-equations (31) and that any other value of R corresponds to a case where one has "to conceive of standard commodities which include negative components" /1960, p.31n/; such standard commodities are relevant only in some cases of joint production and they cannot be thought of "as having a bodily existence" /1960, p.48/. In order to solve the question of the exact number of semipositive resp. positive standard commodities, we have only to translate our mathematical results on column eigenvectors into the following propositions.[18]

[16] See, for example, Burmeister /1968/, Parys /1977/ and Miyao /1977/. The last introduces the term "general standard commodity" for any semipositive composite commodity, say x, such that the wage-profit relation is linear if the "net product" (I-A)x is used as the numeraire, Miyao clearly shows in what cases this linearity property holds. Note that his analysis is limited to indecomposable systems and that it is not always easy to interprete a "general standard commodity". It is not always an autogood, i.e., it is possible that x is not the same composite commodity as (I-A)x. The latter, not x, is used as the numeraire.

[17] Thus "entering the system on the same footing as the fuel for the engines or the feed for the cattle" (see Sraffa /1960, p.9/).

[18] Propositions 5 and 7 are translations of Theorems 1 and 2. Proposition 6 can be considered as a straightforward corollary of Proposition 5.

PROPOSITION 5. The number of different semipositive standard commodities equals the number of blocks that do not depend on another block with a smaller or equal rate of surplus production.

PROPOSITION 6. A Sraffa system with basics and nonbasics has exactly one semipositive standard commodity if and only if $R_j \geq R_1$ for all $j>1$; this standard commodity q^1 has $q_1^1>0$ and $q_j^1=0$ for all $j>1$. Otherwise, more than one semipositive standard commodity exists; any of these, say $q^k \geq 0$, contains positive quantities of all basics, i.e. $q_1^k>0$.[19]

PROPOSITION 7. a) There exists a positive standard commodity if and only if there exists a nonempty set of blocks, say D, such that

1. any block j in D has $R_j=R_A$ and any two different blocks of D are mutually independent.
2. any block j not in D has $R_j>R_A$ and is such that at least one block of D depends on j.

b) When non zero, the number of different positive standard commodities equals $|D|$.

c) Any positive standard commodity corresponds to the same standard ratio, viz. $R=R_A$.

It is possible to present an intuitive economic explanation of these propositions (and implicitly of the method for constructing column eigenvectors in Section 2) if we interpret a standard commodity as a balanced growth vector. We guess that it is sufficient to give the following heuristic considerations that explain why a semipositive balanced growth vector can be associated with every block k that does not depend on another block j with a smaller or equal rate of surplus production (Proposition 5). We assume that s_k is a simple root of A which implies that there is only one block of industries (block k) that has an internal rate of surplus production equal to R_k. We construct a balanced growth vector "associated" with block k as follows. We

[19] The subvector q_1^k is positive because any block k depends on block 1: see expressions (7), (8) and (25).

first set equal to zero the activity level q_h of any block h ($\neq$k) on which k does not depend, thus including any block h ($\neq$k) that depends on k. This implies that block k does not have to deliver anything to another block. Block k produces only for itself, so that its growth rate is equal to its internal rate of surplus production R_k. Any remaining block, say j, is such that k depends on j. If $R_k<R_j$ ($s_k>s_j$), then no problems about a uniform growth rate arise: j has a higher internal rate of surplus production than k, but j has to deliver (directly or indirectly) something of its surplus to k, so that j and k can grow at the same rate. However, if $R_k>R_j$ ($s_k<s_j$), then things become difficult: the internal rate of surplus production of j is smaller than the growth rate of k; j cannot achieve such a high growth rate ... unless it "delivers" negative quantities to k.[20]

[20] Observe that the intuitive explanation about a uniform rate of growth and negative quantities can be considered in some sense as the dual of that about a uniform rate of profits and negative prices at the end of Section 3.

5. EXAMPLES AND COMMENTS

Throughout this Section we consider an economy with the following inputs and net outputs:

$$A = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \quad , \qquad \bar{c} = \begin{bmatrix} 0.2 \\ 0.8 \\ 0.4 \end{bmatrix} \quad ;$$

here the input matrix A is already written in its normal form; all blocks consist of a single industry; commodity 1 is basic, commodities 2 and 3 are nonbasic; $s = 0.6$, $s_1=0.4$, $s_2=0.2$, $s_3=0.6$; $R_A=2/3$, $R_1=1.5$, $R_2=4$, $R_3=2/3$.

In such a numerical case the Sraffa price system (30) and the standard system (31) become:

(30')
$$\begin{aligned} p_1 &= 0.4\ p_1(1+r) + wL_1, \\ p_2 &= (0.1\ p_1 + 0.2\ p_2)(1+r) + wL_2, \\ p_3 &= (0.3\ p_1 + 0.6\ p_3)(1+r) + wL_3, \\ 0.2\ p_1 &+ 0.8\ p_2 + 0.4\ p_3 = 1. \end{aligned}$$

(31')
$$\begin{aligned} q_1 &= (0.4\ q_1 + 0.1\ q_2 + 0.3\ q_3)(1+R), \\ q_2 &= 0.2\ q_2\ (1+R), \\ q_3 &= 0.6\ q_3\ (1+R), \\ L_1q_1 &+ L_2q_2 + L_3q_3 = 1 \end{aligned}$$

The column vectors $x^1=(1\ 0\ 0)$, $x^2=(-1/2\ 1\ 0)$ and $x^3=(3/2\ 0\ 1)$ are three linearly independent eigenvectors of the matrix A. Hence, the system (31') leads to three standard commodities, resp. proportional to x^1, x^2 and x^3. The proportionality factor depends on the value of the direct labour inputs L_1, L_2 and L_3. For example, if $L_1=1/80$, $L_2=78/80$ and $L_3=1/80$, then we find that the following scalars and column vectors represent a standard ratio, a standard commodity and standard net product:

(32.1) $R_1 = 1.5$; $q^1 = (80 \quad 0 \quad 0)$; $(I-A)q^1 = (48 \quad 0 \quad 0)$.
(32.2) $R_2 = 4$; $q^2 = (-80/155 \quad 160/155 \quad 0)$; $(I-A)q^2 = (-64/155 \quad 128/155 \quad 0)$.
(32.3) $R_3 = 2/3$; $q^3 = (48 \quad 0 \quad 32)$; $(I-A)q^3 = (96/5 \quad 0 \quad 64/5)$.

The existence of two semipositive q is not surprising in the light of Theorem 1 and Proposition 5: observe that in our example we have $G=\{1,3\}$, $H=\{2\}$ and $L=\emptyset$. No positive q exists, which illustrates Proposition 7.

We now consider the price system (30') and we suppose for a moment that $L_1=0.001$, $L_2=0.998$ and $L_3=0.001$. If $r=0.625$, then $w=63/77=0.818$, $p_1=18/7700$, $p_2=9319/7700$ and $p_3=603/7700$: these all-positive values illustrate Proposition 1 ($r=0.625<R_A=2/3$). If $r=0.750$, then $w=130/157=0.828$, $p_1=26/9420$, $p_2=11983/9420$ and $p_3=-429/9420$: the latter value is negative, which is not surprising from Proposition 3 ($r=0.750>R_A=2/3$). It is interesting to look at the above example in the light of Sraffa's treatment. Sraffa first simplified his analysis by temporarily assuming "that only basic industries come under consideration" /1960,p.25/. He then considers the rate of surplus production of the block of basics ($R_1=1.5$ in our example) as the maximum rate of profits. Such an approach may lead to complications when the nonbasics are taken into account again at a later stage of the analysis. Sraffa himself mentions that negative prices are possible for some $r<R_1$ /1960,pp.90-91/ On the other hand, he claims that a fall of the wage in terms of any arbitrarily chosen product corresponds to a rise of r and vice versa, and he gives the impression /1960, chapter VI/ that this holds for all $r<R_1$. He offers two proofs of this. The first uses a so-called reduction series of dated labour quantities, but one can show that such a series does not converge if $r > R_A$; the second implicitly assumes that all prices are positive, but we know that this is impossible if $r>R_A$. Hence, both methods of proof are no longer valid if $r>R_A$ holds. This explains the outcome of our numerical example where a rise of r from 0.625 to 0.750 corresponds to a rise (not a fall!) of w from 0.818 to 0.828. The profit rate $r=0.750$ is only half of Sraffa's maximum rate $R_1=1.5$, but it exceeds $R_A=2/3$. It is the latter value that is relevant here. It is impossible to offer a general proof of the inverse relation between r and w (Proposition 4) for values of r higher than R_A.

Other values of the direct labour inputs may lead to other peculiarities. Note that we still assume $r=0.750$. If $L_1=L_2=L_3=1/3$, then $w=-13/84$; if $L_1=2/45$, $L_2=40/45$, $L_3=3/45$, then $w=\infty$; if $L_1=1/80$, $L_2=78/80$, $L_3=1/80$, then $w=52/49$, $p_1=52/1176$, $p_2=1886/1176$, $p_3=-858/1176$. Such anomalies concerning the fraction w of the actual net product going to wages were not mentioned in Sraffa's book. A possible explanation of this can be given by means of the last numerical example. We see that the wage ($w=52/49$) is equivalent to 24 units of commodity 1 ($p_1=52/1176$). Sraffa would consider 48 units of commodity 1 as the standard net product (see expression 32.1 above). Suppose we follow Sraffa's practice of choosing this standard net product (which contains no nonbasics) as the numeraire. Then we find that the share of wages equals 1/2 of the standard net product. At first sight, this seems to be a perfectly regular case. The choice of the net product of a hypothetical standard economy as the numeraire has obscured the anomaly that the share of wages equals 52/49 of the net product of the actual economy (which contains both basics and nonbasics). But it is the division of the actual net product that is ultimately the relevant object of investigation. Sraffa himself clearly says that "the standard system is a purely auxiliary construction" /1960, p.31/.

We are not going to speculate whether Sraffa in his 1960 book was aware of all the analytical consequences when he temporarily eliminated the nonbasics from his discussion, chose R_1 instead of R_A as the maximum rate of profits and considered a standard commodity containing only basics as the numeraire. Actually, in his 1962 correspondence with Newman, (published in full by Bharadwaj /1970, pp.424-428/) Sraffa seems to suggest that, unlike in our fictive numerical example, in a real world system the internal input coefficients of a block of nonbasics usually are rather small. In such a case it would not be implausible that the eigenvalue s_1 exceeds all other s_i and thus $R_j>R_1$ would hold for all $j>1$. Then R_1 and R_A would be equal, no r in the interval $0\leq r\leq R_1$ would lead to anomalies in Sraffa's price system and only one semipositive standard commodity would exist, with positive elements corresponding to basic commodities.

6. SOME CONCLUDING REMARKS

It should be noted that in our treatment of basics and nonbasics we considered an unchanging technology. If real (not hypothetical) changes in distribution or production are analysed, another technological matrix may arise even such that commodities that formerly were basics now "appear in this system merely as nonbasic products" (see Sraffa [1960, p.85n7]. The study of such changes is beyond the scope of our study. It is obvious that the scope of this paper is also limited in the sense that we do not want to provide any definitive argument concerning the relevance of Sraffa's work as a critique of neoclassical economics or as a rehabilitation of classical economic theory.

The purpose of this paper was to establish some mathematical theorems on decomposable matrices and to use these theorems for the solution of some analytical problems in Sraffa's system of single product industries. This led to some interesting findings, especially with regard to the determination of the number of semipositive resp. positive standard commodities in a decomposable system. Moreover, our analytical results had an interesting economic interpretation in terms of the mutual dependence of the different blocks of industries and their internal rate of surplus production. Many of our theorems may also be useful outside the Sraffa world, for example to study price movements and balanced growth vectors in input-output systems, to analyse the connection between different blocks of countries in linear models of international trade, etc. We guess that the existence of such a formal similarity between different theoretical systems is well known.

APPENDIX

PROOF OF RELATION (6)

Note that k-1 and k are two successive blocks. Hence, if k depends on k-1, this must be by a chain of one semipositive matrix, viz. $A_{k-1,k} \geq 0$. Then $q_k > 0$ implies $A_{k-1,k} q_k \geq 0$; premultiplication by the positive matrix $(s_k I - A)^{-1}$ in (5) then leads to $q_{k-1} > 0$. If k does not depend on k-1, then $A_{k-1,k} = 0$ and thus $q_{k-1} = 0$.

PROOF OF RELATION (8)

Actually, we just proved that (8) holds if $j=k-1$. We shall show that (8) holds in general for $j=k-i$ $(i=1,2,\ldots,k-1)$ by induction on i. To this end, we assume from now on that (8) holds for $j=k-1, k-2, \ldots, k-i+1$ and we try to show that (8) then also holds for $j=k-i$.

If k depends on j, there exists a chain of semipositive matrices connecting j to k. If there is a chain of one semipositive matrix, i.e. $A_{jk} \geq 0$, then we can prove that $q_j > 0$ by the same argument as in the proof of (6) above. If the chain from j to k consists of the semipositive matrices A_{jh}, A_{hu}, ..., A_{vw}, A_{wk} $(j<h<u<\ldots<v<w<k$, of course), then k depends on h. Hence, $q_h > 0$ because (8) is assumed to hold for $k-1, k-2, \ldots, j+1$). Then $A_{jh} q_h \geq 0$, so that (7) leads to $q_j > 0$.

If k does not depend on j, then we can prove that $q_j = 0$ by contradiction. Suppose for a moment that $q_j \geq 0$. From (7) this can hold only if $A_{jh} q_h \geq 0$ for some $h, j<h<k$. ($h=k$ is impossible, because that would imply $A_{jh} = A_{jk} \geq 0$ and thus k dependent on j). We thus have $A_{jh} \geq 0$, $q_h \geq 0$. Now $q_h \neq 0$ implies that k depends on h(because (8) is assumed to hold for $k-1, k-2, \ldots, j+1$). Hence, there exists a chain of semipositive matrices from h to k. But $A_{jh} \geq 0$ means that we can also construct a chain from j (via h) to k. This means that k depends on j, which is a contradiction. Hence, $q_j \geq 0$ is false and thus $q_j = 0$ must hold.

PROOF OF PROPOSITION 1

If $0 \leq r < R_A$, then $I-(1+r)A$ has a semipositive inverse by (D2). Hence, there are unique values w and p that solve the Sraffa price system (30): p equals $wL(I-(1+r)A)^{-1}$, and w and p are normalized such that $pc=1$. It is obvious that w is positive and p semipositive.

In order to prove that p is positive, we have to consider each of its subvectors p_j in succession as in (30). Note that the matrices $(I-(1+r)A_j)^{-1}$ are all positive by (I2). Hence, from (30.a), it follows that $p_1>0$; from (30.b) and the properties of the normal form (especially (25) is important), we can derive that $p_j>0$ for $j=k$ if it holds for all $j<k$. Hence, by induction $p_j>0$ holds for all j and thus $p>0$.

PROOF OF PROPOSITION 2

a) Suppose for a moment that (R_A, w, p) solves the Sraffa price system and that $w \neq 0$. Then we have:

$$p(I-(1+R_A)A) = wL \neq 0.$$

We know, from (D1), that A has a semipositive column eigenvector, say q, corresponding to the dominant root $s=1/(1+R_A)$. Postmultiplication of both sides of the above equation by q leads to a contradiction: $p(I-(1+R_A)A)q$ is zero, while wLq is not. Hence, we must have $w=0$.

Note that $w=0$ and $r=R_A$ imply that p is a row eigenvector of A corresponding to the dominant root s.

b) If $R_j > R_1$ $(s_j < s_1)$ for all $j>1$, then $s=s_1$ is a simple root of A. Hence, p is unique; p is also positive by Theorem 4 (the set $D^T=\{1\}$).

c) If $R_j \leq R_1$ $(s_j \geq s_1)$ for some $j>1$, then no positive row eigenvector exists by Theorem 4. Of course, at least one semipositive p exists by (D1).

PROOF OF PROPOSITION 3

Let $r>R_A$ and let (r,w,p) solve the Sraffa price system. Suppose for a moment that $w>0$ and $p\geqq 0$. We denote $1/(1+r)$ by β and we then obtain:

$$p(\beta I-A) = \beta wL > 0 \ , \ p \geqq 0,$$

or

$$\beta p > pA \ , \ p \geqq 0.$$

This implies $\beta>s$ by (D3), but that contradicts $r>R_A$ $(\beta<s)$. Hence, it is impossible that both w and p are nonnegative if $w\neq 0$. If $w=0$, then p must be a row eigenvector of A corresponding to an eigenvalue, say $\lambda=1/(1+r)$, smaller than s (as $r>R_A$). Hence, p cannot be positive by Theorem 4.c. From Theorem 3, it follows that p is semipositive only if $\lambda=s_k$ $(r=R_k)$ for some $k\varepsilon G^T UL^T$.

PROOF OF PROPOSITION 4

It should be noted that $\bar{w},\bar{\bar{w}},\bar{p}$ and $\bar{\bar{p}}$ are all positive by Proposition 1. Because $\bar{r}<\bar{\bar{r}}<R_A\leqq R_1$ holds, the semipositive indecomposable matrix A_1 satisfies the following strict matrix inequality:[21]

$$(I-(1+\bar{r})A_1)^{-1} < (I-(1+\bar{\bar{r}})A_1)^{-1}.$$

From (30.a) it then easily follows that $\bar{p}_1/\bar{w}<\bar{\bar{p}}_1/\bar{\bar{w}}$. Using (30.b) and the properties of the normal form (especially (25) is important), we can show that $\bar{p}_j/\bar{w}<\bar{\bar{p}}_j/\bar{\bar{w}}$ holds for $j=k$ if it holds for all $j<k$.[22] Hence, by induction $\bar{p}_j/\bar{w}<\bar{\bar{p}}_j/\bar{\bar{w}}$ holds for all j and thus $\bar{p}/\bar{w}<\bar{\bar{p}}/\bar{\bar{w}}$.

This inequality also implies $\bar{r}\bar{p}Ae/\bar{w}<\bar{\bar{r}}\bar{\bar{p}}Ae/\bar{\bar{w}}$. From (29) and (30.c) we know that $rpAe/w+1=1/w$. Hence, $1/\bar{w}<1/\bar{\bar{w}}$ or $\bar{w}>\bar{\bar{w}}$

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[21] See Newman /1962, p.75/.

[22] This can easily be shown in the case that A_j is the zero matrix of order one. In other cases, we use the fact that the semipositive indecomposable matrices A_j satisfy the same strict matrix inequality as the matrix A_1 above.

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