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A GENERAL FRAMEWORK FOR COMPARATIVE STATICS ANALYSIS

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In 1974 E. Silberberg /4/ proposed a general frame work for deriving qualitative comparative statics properties of equilibrium systems obtained from the maximization of an objective function. This framework was based on the properties of a "primal-dual" problem, derived from the original maximization problem. It was then claimed that all the known comparative statics theorems concerning maximization models are contained in the positive semi-definiteness over the parameter space of a (constrained) quadratic form.
This quadratic form is based on the second partial derivatives of the Lagrangean function associated with the primal-dual problem.

The purpose of this note is, first, to show that Silberberg's main results can also easily be derived without introducing the primal-dual problem, and, secondly, to sharpen and correct some of his results. In particular, we will characterize the subspace of the parameter space in which the above mentioned quadratic form is positive definite, and not merely positive semi-definite. This is not unimportant. For example, E. Silberberg's results do not allow to prove the negativity condition of consumer demand theory.
Finally, E. Silberberg formulates the Le Châtelier principle as the negative definiteness of an unconstrained quadratic form.
It will be shown that only negative semi-definiteness can hold.

In a first section we will prove some preliminary results. The next two sections contain a general discussion of qualitative comparative statics results and of the Le Châtelier principle. In the final section a simple example is analyzed.

1. Some Preliminary Results

Throughout this paper, the transpose of a matrix A will be indicated by A', and its rank by $\rho(A)$. 0 will be used to indicate the zero matrix of appropriate order. I_k indicates the kxk identity matrix.

LEMMA I. Consider a symmetric non-singular (r+n)x(r+n) matrix

$$\begin{bmatrix} 0 & B_1 \\ B_1' & A \end{bmatrix} \tag{1}$$

Let the inverse of this matrix be given by the (symmetric) (r+n)x(r+n) matrix

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{12}' & C_{22} \end{bmatrix} \tag{2}$$

If $r < n$, then

a) $\rho\ (C_{12}) = r$ (3)

b) $\rho\ (C_{22}) = n-r$ (4)

Proof

As (2) is the inverse of (1), we have

$$B_1C_{12}' = I_r \tag{5}$$

$$B_1C_{22} = 0 \tag{6}$$

$$B_1'C_{11} + AC_{12}' = 0 \tag{7}$$

$$B_1'C_{12} + AC_{22} = I_n \tag{8}$$

a) Proof of (3). From (5) we know that

$$\rho(B_1C_{12}') = r \leqq \text{Min}\{\ \rho\ (B_1),\ \rho(C_{12})\} \quad = \text{Min}\ \{r,\ \rho(C_{12})\}$$

so that $\rho(C_{12}) \geqq r$. As we must also have that $\rho(C_{12}) \leqq r$, it follows that $\rho(C_{12}) = r$.

b) Proof of (4). Let $\rho(C_{22}) = n-k$. We will show that $k=r$. As $\rho(B_1) = r$, it follows from (6) that $\rho(C_{22}) \leqq n-r$, so that

$$k \geqq r \tag{9}$$

On the other hand, if $\rho(C_{22})$ is $n-k$, there must exist a nxk matrix D, $\rho(D) = k$, such that $C_{22}D = 0$. Consider then

$$\begin{bmatrix} C_{11} & C_{12} \\ C'_{12} & C_{22} \end{bmatrix} \begin{bmatrix} 0 \\ D \end{bmatrix} = \begin{bmatrix} C_{12}D \\ 0 \end{bmatrix}$$

It follows that

$$\rho(C_{12}D) = k \leqq \text{Min } \{\rho(C_{12}), \rho(D)\} = \text{Min } \{r,k\}$$

so that

$$k \leqq r \tag{10}$$

(9) and (10) imply that $k = r$. Q.E.D.

The nullspace of C_{22}, denoted by $N(C_{22})$, is defined as

$$N(C_{22}) = \{u \in R^n / \ C_{22}\, u = 0\}$$

$N(C_{22})$ is a linear subspace of R^n with dimension r. The following corollary then follows from (6).

<u>Corollary</u> $N(C_{22})$ is spanned by the row vectors of B_1.

<u>LEMMA II</u>. Let A be a symmetric nxn matrix, let B_1 be a rxn matrix with $\rho(B_1) = r$, and let

$$\begin{bmatrix} 0 & B_1 \\ B'_1 & A \end{bmatrix}^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C'_{12} & C_{22} \end{bmatrix}$$

If then $v'Av < 0$ for all $v \in R^n$, $v \neq 0$, which satisfy $B_1 v = 0$, then $v'Av < 0$ for all $v \in R^n$, $v \neq 0$, which satisfy $B_1 v = 0$, then

$$\text{a) } \forall u \in N(C_{22}),\quad u'\, C_{22}\, u = 0 \tag{11}$$

$$\text{b) } \forall u \notin N(C_{22}),\quad u'\, C_{22}\, u < 0 \tag{12}$$

<u>Proof</u>

Note that, under the given assumption, the indicated inverse must exist. See e.g. G. Debreu /1/, or Y. Murata /2, pp. 58-59/. Property (11) is trivial. To prove (12), we have using (8)

$$C_{22}B'_1C_{12} + C_{22}AC_{22} = C_{22}$$

As $C_{22}B_1' = 0$ (from (6)), it follows that

$$C_{22}AC_{22} = C_{22}.$$

Consider then the quadratic form

$$u'C_{22}AC_{22}u = u'C_{22}u$$

or, defining $v = C_{22}u$,

$$v'Av = u'C_{22}u.$$

For all $u \in R^n$ we have from (6) that

$$B_1v = B_1C_{22}u = 0$$

More over, for all $u \notin N(C_{22})$ we have

$$v = C_{22}u \neq 0$$

Hence, for all $u \notin N(C_{22})$, we must have that

$$v'Av = u'C_{22}u < 0 \qquad \text{Q.E.D.}$$

LEMMA III. Let A be a symmetric nxn matrix
B_1 be a rxn matrix, with $\rho(B_1) = r$
B_2 be a sxn matrix, with $\rho(B_2) = s$
$\rho(B_1', B_2') = r+s < n$.

If $v'Av < 0$ for all $v \in R^n$, $v \neq 0$, which satisfy $B_1v = 0$, and if we denote

$$\begin{bmatrix} 0 & B_1 \\ B_1' & A \end{bmatrix}^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}' & C_{22} \end{bmatrix} \qquad (13)$$

$$\begin{bmatrix} 0 & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ [B_1', B_2'] & A \end{bmatrix}^{-1} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}' & D_{12} \end{bmatrix} \qquad (14)$$

then the matrix

$$\left[\begin{array}{c|c} \left[\begin{array}{c|c} C_{11} & 0 \\ \hline 0 & 0 \end{array}\right] - D_{11} & \left[\begin{array}{c} C_{12} \\ \hline 0 \end{array}\right] - D_{12} \\ \hline \left[\begin{array}{c|c} C'_{12} & 0 \end{array}\right] - D'_{12} & C_{22} - D_{22} \end{array}\right] \tag{15}$$

is negative semi-definite, and has rank s.

<u>Proof</u>

If $v'Av < 0$ for all $v \in R^n$, $v \neq 0$, which satisfy $B_1 v = 0$, then also $v'Av < 0$ for all $v \in R^n$, $v \neq 0$, which satisfy $B_1 v = 0$ and $B_2 v = 0$.

It follows that the inverse matrix (14) must exist. As

$$\begin{bmatrix} 0 & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \begin{bmatrix} B'_1, B'_2 \end{bmatrix} & A \end{bmatrix}$$

is non singular and symmetric, it follows that (15) is negative semi-definite if and only if the matrix

$$\begin{bmatrix} 0 & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \begin{bmatrix} B'_1, B'_2 \end{bmatrix} & A \end{bmatrix} \left[\begin{array}{c|c} \left[\begin{array}{c|c} C_{11} & 0 \\ \hline 0 & 0 \end{array}\right] - D_{11} & \left[\begin{array}{c} C_{12} \\ \hline 0 \end{array}\right] - D_{12} \\ \hline \left[\begin{array}{c|c} C_{12} & 0 \end{array}\right] - D_{12} & C_{22} - D_{22} \end{array}\right] \begin{bmatrix} 0 & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \begin{bmatrix} B'_1, B'_2 \end{bmatrix} & A \end{bmatrix} \tag{16}$$

is negative semi-definite. Also, matrices (15) and (16) must have the same rank.

Working out (16), and making use of (13) and (14), one obtains

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_2 C_{22} B'_2 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

From lemma II and from the corollary of lemma I, we know that $B_2C_{22}B_2'$ is negative definite, and has rank s. Q.E.D.

2. Some General Comparative Statics Results

Consider the following programming problem

$$\left.\begin{array}{l}\max\limits_{x} f(x,\alpha) \\ \text{s.t. } g(x,\alpha) = 0\end{array}\right\} \quad (18)$$

where $x' = (x_1, \ldots, x_n) \varepsilon R^n$ is a vector of decision variables, $\alpha' = (\alpha_1, \ldots, \alpha_m) \varepsilon R^m$ is a vector of parameters, f is a function from $R^n x R^m$ into R, and $g' = (g^1, \ldots, g^r)$ is a function from $R^n x R^m$ into R^r. If we introduce a vector $\lambda \varepsilon R^r$ of Lagrange-multipliers, the Lagrangean function associated with (18) is given by

$$L(x,\lambda,\alpha) = f(x,\alpha) + \lambda' g(x,\alpha) \quad (19)$$

We will assume throughout that f and g are twice continuously differentiable over $R^n \times R^m$. If then there exist vectors $\hat{x}$ and $\hat{\lambda}$ such that (1)

$$L_x(\hat{x},\hat{\lambda},\alpha) = f_x(\hat{x},\alpha) + g_x'(\hat{x},\alpha)\hat{\lambda} = 0 \quad (20)$$

$$L_\lambda(\hat{x},\hat{\lambda},\alpha) = g(\hat{x},\alpha) = 0 \quad (21)$$

$$\left.\begin{array}{l}v' L_{xx}(\hat{x},\hat{\lambda},\alpha)\, v < 0 \text{ for all } v \varepsilon R^n,\ v \neq 0 \\ \quad \text{which satisfy } g_x(\hat{x},\alpha)\, v = 0\end{array}\right\} \quad (22)$$

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(1) We use the following notation. If h(x,y) is a real valued function of two vectors x and y, then $h_x(x,y)$ denotes the vector of partial derivatives of h with respect to the components of x. By $h_{xy}(x,y)$ we mean the matrix whose ij-th element is given by $\frac{\partial^2 h(x,y)}{\partial x_i \partial y_i}$. If $h(x,y) = (h^1(x,y), \ldots, h^k(x,y))$ is a vector valued function of two vectors x and y, then h_x represents a matrix, the ij-th element of which is given by $\frac{\partial h^i(x,y)}{\partial x_j}$.

then $\hat{x}$ is a strict local maximum of f over the feasible set. In addition, we will also assume that

$$\rho(g_x(\hat{x},\alpha)) = r < n \tag{23}$$

From (22) and (23) it follows that the Jacobian of the system (20)-(21) does not vanish so that these equations can be solved for $\hat{x}$ and $\hat{\lambda}$ as functions of α,

$$\hat{x} = \hat{x}(\alpha) \tag{24}$$

$$\hat{\lambda} = \hat{\lambda}(\alpha) \tag{25}$$

We are now interested in some general qualitative properties of the matrices

$$\frac{\partial\hat{x}(\alpha)}{\partial\alpha} \quad \text{and} \quad \frac{\partial\hat{\lambda}(\alpha)}{\partial\alpha}$$

Taking the total differential of (20)-(21), we obtain

$$\begin{bmatrix} 0 & g_x \\ g'_x & L_{xx} \end{bmatrix} \begin{bmatrix} d\hat{\lambda} \\ d\hat{x} \end{bmatrix} = - \begin{bmatrix} g_\alpha \\ L_{x\alpha} \end{bmatrix} d\alpha \tag{26}$$

where all partial derivatives are evaluated at $\hat{x}$ and $\hat{\lambda}$.

From (22) and (23) it follows that the matrix on the LHS of (26) is invertible. Let us write

$$\begin{bmatrix} 0 & g_x \\ g'_x & L_{xx} \end{bmatrix}^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C'_{12} & C_{22} \end{bmatrix} \tag{27}$$

we then obtain

$$\begin{bmatrix} \frac{\partial\hat{\lambda}}{\partial\alpha} \\ \frac{\partial\hat{x}}{\partial\alpha} \end{bmatrix} = - \begin{bmatrix} C_{11} & C_{12} \\ C'_{12} & C_{22} \end{bmatrix} \begin{bmatrix} g_\alpha \\ L_{x\alpha} \end{bmatrix} \tag{28}$$

Equation (28) contains all the quantitative information concerning $\frac{\partial\hat{\lambda}}{\partial\alpha}$ and $\frac{\partial\hat{x}}{\partial\alpha}$. We will now try to derive some general qualitative properties.

From (28) we obtain

$$\frac{\partial\hat{x}}{\partial\alpha} = - C_{12}' g_{\alpha} - C_{22} L_{x\alpha} \tag{29}$$

Premultiplying (29) by g_x, and making use of (27), gives us

$$g_x \frac{\partial\hat{x}}{\partial\alpha} + g_{\alpha} = 0 \tag{30}$$

which may be called the "adding up condition". Alternatively, if we premultiply (30) by $\hat{\lambda}'$, we obtain, making use of (20),

$$f_x' \frac{\partial\hat{x}}{\partial\alpha} = \hat{\lambda}' g_{\alpha} \tag{31}$$

Let $\phi(\alpha) = f(\hat{x}(\alpha),\alpha)$ (32)

denote the maximal value of f over the feasible set as a function of α. Making use of (31) we obtain

$$\phi_{\alpha}(\alpha) = \frac{\partial\hat{x}'}{\partial\alpha} f_x + f_{\alpha} = \hat{\lambda}' g_{\alpha} + f_{\alpha} \quad \text{or}$$

$$\phi_{\alpha}(\alpha) = L_{\alpha} \tag{33}$$

which, of course, is <u>Samuelson's envelope theorem.</u> See /3, p. 34/.

As (27) is a symmetric matrix, it follows from premultiplication of (28) by $\begin{bmatrix} g_{\alpha}' & L_{\alpha x} \end{bmatrix}$ that

$$\begin{bmatrix} g_{\alpha}' & L_{\alpha x} \end{bmatrix} \begin{bmatrix} \frac{\partial\hat{\lambda}}{\partial\alpha} \\ \\ \frac{\partial\hat{x}}{\partial\alpha} \end{bmatrix} \tag{34}$$

is symmetric. This may be called the "<u>symmetry condition</u>".

Define now a linear subspace V in R^m by

$$V = \{u \, \varepsilon \, R^m \mid g_\alpha \, u = 0\} \tag{35}$$

Then, for u ε V, we can consider the quadratic form

$$u' \begin{bmatrix} g_\alpha & L_{\alpha x} \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\lambda}}{\partial \alpha} \\ \frac{\partial \hat{x}}{\partial \alpha} \end{bmatrix} u = - u' \begin{bmatrix} g'_\alpha & L_{\alpha x} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C'_{12} & C_{22} \end{bmatrix} \begin{bmatrix} g_\alpha \\ L_{\alpha x} \end{bmatrix} u =$$

$$- u' L_{\alpha x} C_{22} L_{x\alpha} u \tag{36}$$

From lemma II it then follows that

$$\forall u \, \varepsilon \, V, \quad u' \begin{bmatrix} g'_\alpha & L_{\alpha x} \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\lambda}}{\partial \alpha} \\ \frac{\partial \hat{x}}{\partial \alpha} \end{bmatrix} u \geqq 0 \tag{37}$$

This is E. Silberberg's main result. It can also be expressed by the requirement that the appropriate principal minors of

$$\begin{bmatrix} 0 & g_\alpha \\ g'_\alpha & \begin{bmatrix} g'_\alpha & L_{\alpha x} \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\lambda}}{\partial \alpha} \\ \frac{\partial \hat{x}}{\partial \alpha} \end{bmatrix} \end{bmatrix}$$

are zero or are of the sign $(-1)^r$. See G. Debreu /1/, or Y. Murata /2, p. 60/.

Property (37), however, can be strengthened. Let W be a linear subspace of V defined as

$$W = \{ u \, \varepsilon \, V \mid L_{x\alpha} \, u \, \varepsilon \, N(C_{22}) \} \tag{38}$$

From lemma II it then follows that

$$\left.\begin{array}{l} \forall\ u \in W,\quad u' \begin{bmatrix} g'_\alpha & L_{\alpha x} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \hat{\lambda}}{\partial \alpha} \\ \dfrac{\partial \hat{x}}{\partial \alpha} \end{bmatrix} u = 0 \\ \forall\ u \notin W,\quad u' \begin{bmatrix} g'_\alpha & L_{\alpha x} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \hat{\lambda}}{\partial \alpha} \\ \dfrac{\partial \hat{x}}{\partial \alpha} \end{bmatrix} u > 0 \end{array}\right\} \quad (39)$$

This property may be called the "positivity condition". Note that one does not have to know C_{22} in order to know $N(C_{22})$. The corollary of lemma I tells us that $N(C_{22})$ is spanned by the row vectors of g_x.

As a special application, consider the case where there are no constraints g.

Then, instead of (33), (34) and (39), we obtain, respectively,

$$\phi_\alpha(\alpha) = f_\alpha \quad (40)$$

$$f_{\alpha x} \frac{\partial \hat{x}}{\partial \alpha} \quad \text{is symmetric} \quad (41)$$

$$\left.\begin{array}{l} \forall\ u \in N(f_{x\alpha}),\quad u' f_{\alpha x} \dfrac{\partial \hat{x}}{\partial \alpha} u = 0 \\ \forall\ u \notin N(f_{x\alpha}),\quad u' f_{\alpha x} \dfrac{\partial \hat{x}}{\partial \alpha} u > 0 \end{array}\right\} \quad (42)$$

3. The Le Châtelier Principle

Consider again problem (18), and its associated solution functions (24) and (25). Suppose now that we add to (18) s new constraints denoted by

$$g^N(x,\alpha) = 0 \quad (43)$$

where $g^N : R^n \times R^m \to R^s$. Assume in addition that

$$g^N(\hat{x},\alpha) = 0 \quad (44)$$

Denote by $\mu \in R^s$ the vector of Lagrange multipliers associated with (44), and by L^{+s} the Lagrangian function

$$L^{+s}(x,\lambda,\mu,\alpha) = f(x,\alpha) + \lambda' g(x,\alpha) + \mu' g^N(x,\alpha)$$

Then from (20) - (22) and (44) it follows that

$$L_x^{+s}(\hat{x},\hat{\lambda},0,\alpha) = f_x(\hat{x},\alpha) + g_x'(\hat{x},\alpha)\hat{\lambda} + g_x^{N'}(\hat{x},\alpha)\, 0 = 0 \qquad (45)$$

$$L_\lambda^{+s}(\hat{x},\hat{\lambda},0,\alpha) = g(\hat{x},\alpha) = 0 \qquad (46)$$

$$L_\mu^{+s}(\hat{x},\hat{\lambda},0,\alpha) = g^N(\hat{x},\alpha) = 0 \qquad (47)$$

$$\left.\begin{array}{l} v' L_{xx}^{+s}(\hat{x},\hat{\lambda},0,\alpha)\, v = v' L_{xx}(\hat{x},\hat{\lambda},\alpha)\, v < 0 \\ \text{for all } v \in R^n,\ v \neq 0,\ g_x(\hat{x},\alpha)\, v = 0 \text{ and} \\ g_x^N(\hat{x},\alpha)\, v = 0 \end{array}\right] \qquad (48)$$

It follows that $\hat{x}$ must also be a strict local maximum of f subject to both the old and the new constraints. Assume that

$$\rho \begin{bmatrix} g_x(\hat{x},\alpha) \\ g_x^N(\hat{x},\alpha) \end{bmatrix} = r + s < n \qquad (49)$$

Equations (45) - (47) can then be solved for $\hat{x}$, $\hat{\lambda}$ and $\hat{\mu}$ as functions of α. Denote these functions by

$$\hat{x} = \hat{x}^{+s}(\alpha) \qquad (50)$$

$$\hat{\lambda} = \hat{\lambda}^{+s}(\alpha) \qquad (51)$$

$$\hat{\mu} = \hat{\mu}^{+s}(\alpha) \qquad (52)$$

If we let

$$\begin{bmatrix} 0 & \begin{bmatrix} g_x \\ g_x^N \end{bmatrix} \\ \begin{bmatrix} g_x', g_x^{N'} \end{bmatrix} & L_{xx} \end{bmatrix}^{-1} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}' & D_{22} \end{bmatrix} \qquad (53)$$

we obtain, instead of (28)

$$\begin{bmatrix} \dfrac{\partial \hat{\lambda}^{+s}}{\partial \alpha} \\ \dfrac{\partial \hat{\mu}^{+s}}{\partial \alpha} \\ \dfrac{\partial \hat{x}^{+s}}{\partial \alpha} \end{bmatrix} = - \begin{bmatrix} D_{11} & D_{12} \\ D_{12}' & D_{22} \end{bmatrix} \begin{bmatrix} g_\alpha \\ g_\alpha^N \\ L_{x\alpha} \end{bmatrix} \tag{54}$$

Substracting (28) from (54) (after appropriately augmenting the component matrices of (28)) we obtain

$$\begin{bmatrix} \dfrac{\partial \hat{\lambda}^{+s}}{\partial \alpha} \\ \dfrac{\partial \hat{\mu}^{+s}}{\partial \alpha} \\ \dfrac{\partial \hat{x}^{+s}}{\partial \alpha} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial \hat{\lambda}}{\partial \alpha} \\ 0 \\ \dfrac{\partial \hat{x}}{\partial \alpha} \end{bmatrix} = \left[\begin{array}{c|c} \left[\begin{array}{c|c} C_{11} & 0 \\ \hline 0 & 0 \end{array}\right] - D_{11} & \left[\begin{array}{c} C_{12} \\ \hline 0 \end{array}\right] - D_{12} \\ \hline \left[\begin{array}{c|c} C_{12}' & 0 \end{array}\right] - D_{12} & C_{22} - D_{22} \end{array}\right] \begin{bmatrix} g_\alpha \\ g_\alpha^N \\ L_{x\alpha} \end{bmatrix} \tag{55}$$

Applying then lemma III to the RHS of (55), we conclude that the m x m matrix

$$\begin{bmatrix} g_\alpha' & g_\alpha^{N'} & L_{\alpha x} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \hat{\lambda}^{+s}}{\partial \alpha} - \dfrac{\partial \hat{\lambda}}{\partial \alpha} \\ \dfrac{\partial \hat{\mu}^{+s}}{\partial \alpha} \\ \dfrac{\partial \hat{x}^{+s}}{\partial \alpha} - \dfrac{\partial \hat{x}}{\partial \alpha} \end{bmatrix} \tag{56}$$

is symmetric and negative semi-definite. This property may be called the Le Châtelier principle (2).

4. A Simple Application

Consider a firm which uses n inputs, the quantities of which are given by the vector $x' = (x_1, \ldots, x_n)$. Let factor prices be given by the vector $w' = (w_1, \ldots, w_n)$. Consider then, for a given output y, the following problem

$$\left.\begin{aligned} &\underset{x}{\text{Min}}\ w'x \\ &\text{s.t. } y - g(x) = 0 \end{aligned}\right\} \quad (57)$$

where $g: R^n \to R$ is the production function. The parameters in this problem are w and y.

The adding up condition (30) states that

$$g_x \frac{\partial \hat{x}}{\partial w} = 0 \quad (58)$$

$$g_x \frac{\partial \hat{x}}{\partial y} = 1 \quad (59)$$

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(2) E. Silberberg's result states that the matrix (56) is negative definite /4, p.165/. This result, however, is too strong. Indeed, his strict inequality (<) in his equation (16) has to be replaced by a weak inequality ($\leq$). This can be illustrated by the following example. Consider the objective function $f(x_1,x_2,\alpha) = -(x_1 - \alpha)^2 - x_2^2$. We can first maximize this function without constraints, and then add the constraint $x_1 + x_2 - \alpha = 0$. It is easily seen that the maximum value of f is the same in both cases, for all values of α. In addition, his equation (17) does not follow from his equation (16).

Alternatively, making use of (31), we obtain

$$\frac{\partial \hat{x}'}{\partial w} w = 0 \tag{60}$$

$$\hat{\lambda} = \frac{\partial \hat{x}'}{\partial y} w \tag{61}$$

Property (60) states that the demand functions $\hat{x}(w,y)$ are homogeneous of degree zero in w. (61) allows us to interpret $\hat{\lambda}$ as the marginal cost of y.

If we denote by $\phi(w,y)$ the minimal cost as a function of w and y, then the envelope theorem allows us to write

$$\frac{\partial \phi(w,y)}{\partial w} = \hat{x}$$

$$\frac{\partial \phi(w,y)}{\partial y} = \hat{\lambda}$$

From the symmetry condition (34) we know that

$$\begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\lambda}}{\partial w} & \frac{\partial \hat{\lambda}}{\partial y} \\ \frac{\partial \hat{x}}{\partial w} & \frac{\partial \hat{x}}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{x}}{\partial w} & \frac{\partial \hat{x}}{\partial y} \\ \frac{\partial \hat{\lambda}}{\partial w} & \frac{\partial \hat{\lambda}}{\partial y} \end{bmatrix}$$

is symmetric. In particular, this means that

$$\frac{\partial \hat{x}}{\partial w} \text{ is symmetric}$$

$$\frac{\partial \hat{\lambda}'}{\partial w} = \frac{\partial \hat{x}}{\partial y}$$

As $g_\alpha = [0, 1] \in R^{n+1}$, the subspace V defined by (35) is given by

$$V = \{ u = \begin{bmatrix} \bar{u} \\ u_{n+1} \end{bmatrix} \in R^{n+1} \mid u_{n+1} = 0 \}$$

$N(C_{22})$ is spanned by g_x. By the first order conditions, we must have that

$$w = \lambda g_x'$$

It follows that $N(C_{22})$ is also spanned by w. The subspace W, defined by (38), is therefore given by

$$W = \{ \alpha \begin{bmatrix} w \\ 0 \end{bmatrix} \varepsilon R^{n+1} \mid \alpha \varepsilon R \}$$

Condition (39), which now becomes a negativity condition because of the minimizationformat.. of (57), can now be stated as

$$\bar{u}' \frac{\partial \hat{x}}{\partial w} \bar{u} = 0 \text{ iff } \bar{u} = \alpha w \text{ for some } \alpha \varepsilon R$$

$$\bar{u}' \frac{\partial \hat{x}}{\partial w} \bar{u} < 0 \text{ iff } \bar{u} \neq \alpha w \text{ for no } \alpha \varepsilon R$$

This last inequality implies that

$$\frac{\partial \hat{x}_i}{\partial w_i} < 0, \; i = 1, \ldots, n.$$

Assume now that the firm is confronted with a new constraint, not involving w and y. Then the Le Châtelier principle (56) tells us that the (n+1) x (n+1) matrix

$$\begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \left[\frac{\partial \hat{\lambda}^{+1}}{\partial w} - \frac{\partial \hat{\lambda}}{\partial w}\right] & \left[\frac{\partial \hat{\lambda}^{+1}}{\partial y} - \frac{\partial \hat{\lambda}}{\partial y}\right] \\ \left[\frac{\partial \hat{x}^{+1}}{\partial w} \quad \frac{\partial \hat{x}}{\partial w}\right] & \left[\frac{\partial \hat{x}^{+1}}{\partial y} - \frac{\partial \hat{x}}{\partial y}\right] \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial \hat{x}^{+1}}{\partial w} - \frac{\partial \hat{x}}{\partial w} & \frac{\partial \hat{x}^{+1}}{\partial y} - \frac{\partial \hat{x}}{\partial y} \\ \frac{\partial \hat{\lambda}^{+1}}{\partial w} - \frac{\partial \hat{\lambda}}{\partial w} & \frac{\partial \hat{\lambda}^{+1}}{\partial y} - \frac{\partial \hat{\lambda}}{\partial y} \end{bmatrix}$$

is symmetric and positive semi-definite. This implies, among other things, that

$$\frac{\partial \hat{x}_i^{+1}}{\partial w_i} \geqq \frac{\partial \hat{x}_i}{\partial w_i}, \quad i = 1, \ldots, n$$

$$\frac{\partial \hat{\lambda}^{+1}}{\partial y} \geqq \frac{\partial \hat{\lambda}}{\partial y}$$

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