STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

ON THE MAXIMUM RATE OF PROFITS

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A B S T R A C T

In Sraffa's treatment of single product industries, some finite value of the profit rate is considered as the highest possible from the economic standpoint. In order to examine this notion of a maximum rate of profits, we analyze the properties of various divisions of the surplus by means of some theorems and examples. The actual surplus is chosen as the numéraire and nonbasics are explicitly taken into account. This procedure prevents us from using the simplifying properties of Sraffa's standard commodity and indecomposable systems, but it is worth while because it reveals some complications that otherwise would be obscured.

The purpose of this paper is to offer a mathematical discussion of the notion of a maximum rate of profits, introduced by Sraffa in his "Production of Commodities by Means of Commodities" /13, p. 94/. We shall use the theory of nonnegative matrices; it is well-known that this is an efficient tool to verify many of Sraffa's verbal arguments: see, for example, Blakley and Gossling /1/, Burmeister /2/ and Newman /7/.

Most treatments on Sraffa's model of single product industries and circulating capital assume that there exist only basics, i.e. products which enter directly or indirectly into the production of all commodities. This assumption is most often made because a system without nonbasics has an indecomposable input matrix, which facilitates the mathematical argument. Moreover, Sraffa himself takes it for granted that nonbasics play such a passive role in the determination of the relevant magnitudes of the economic system that they may be eliminated at the outset /13, § 6, § 35/. Nevertheless, Sraffa seems to realize that such an approach could lead to some freak results when the nonbasics are reintroduced at the end of the analysis /13, Appendix B/. We shall see that more complications are possible. In order to avoid prejudicing the issue, we prefer to work in a more general framework where basics and nonbasics coexist, even though this forces us to handle a decomposable input matrix.

The plan of this paper is as follows. We put together all the definitions, assumptions and mathematical prerequisites in Section 1. We prove some theorems on the characteristics of different divisions of the surplus in Section 2 and we illustrate some peculiarities by means of numerical examples in Section 3. These theorems and examples lead us to some concluding remarks on the notion of a maximum rate of profits in Section 4.

I. PRELIMINARIES

Just like Sraffa, we consider a stationary economy which uses and produces n commodities. We choose the gross total outputs as units of measurement. One unit of good j requires a_{1j}, a_{2j}, ..., a_{nj} as inputs of the 1st, 2nd, ..., nth good and L_j units of homogeneous labour. The coefficients a_{ij} form the input matrix A; the jth column of A is denoted by A_j. The symbol e denotes a column vector of ones and I is an identity matrix of appropriate order. The net product or the surplus of the system is contained in a column vector c. The row vector L is made up of the direct labour inputs L_j; we take total labour employed as unity. Then, the following equalities hold:

(1) $c = (I - A)e$

(2) $Le = 1$

We assume that the economy has the following technological characteristics:

(A1) There are k basics and n-k nonbasics, $0 < k < n$: this means that the input matrix A is semipositive[1] and partially decomposable[2]

(A2) The surplus vector is positive: $c > 0$

(A3) Direct labour is indispensable to produce any good: $L > 0$

As already mentioned in our introduction, the most important of these assumptions from the economic standpoint is (A1). It can be shown (see Gantmacker /4, pp. 74-80/ that the partially decomposable matrix A can be reduced, by a suitable numbering of the commodities, to the normal form (3), which satisfies (4), (5) and (6)

(3)
$$\begin{bmatrix} B & N_{01} & N_{02} & \dots & N_{0h} \\ 0 & N_1 & N_{12} & \dots & N_{1h} \\ 0 & 0 & N_2 & \dots & N_{2h} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & N_h \end{bmatrix}$$

(4) The principal submatrix B is an indecomposable k x k matrix, associated with the k basic commodities

(5) In each column at least one of the submatrices N_{0j}, N_{1j}, ..., $N_{j-1,j}$ is semipositive (j = 1, 2, ..., h)

(6) The principal submatrices N_j are indecomposable (j = 1, 2, ..., h)

The normal form clearly shows the pattern of technological relations in the economy; we now adopt the following definition:

(D1) The commodities corresponding to the elements of the matrix N_j are said to constitute <u>the jth group of interconnected nonbasics.</u>

The Frobenius roots[3] of the matrices A, B and N_j are denoted by f(A), f(B) and $f(N_j)$. In our framework these eigenvalues have the following properties[4]:

(7) $f(A) = \text{Max}\{f(B), f(N_1), \ldots, f(N_h)\}$

(8) $0 < f(A) < 1$

(9) $0 < f(B) < 1$

(10) $0 \leq f(N_j) < 1$

(11) $f(N_j) = 0$ iff N_j is the zero matrix of order one.

In order to facilitate our exposition, we state here the following list of useful propositions[5] on semipositive matrices:

(P1) $\rho I - A$ has a semipositive inverse, viz. $(\rho I - A)^{-1} \geq 0$ iff $\rho > f(A)$

(P2) To the Frobenius root f(A) of the partially decomposable matrix A there belongs a positive row eigenvector iff the normal form (3) satisfies $f(A) = f(B) > f(N_j)$ for all j

(P3) If D is an indecomposable matrix and $\rho > f(D)$, then $(\rho I - D)^{-1} > 0$

(P4) If D is indecomposable and the system $y(\rho I - D) \geq 0$ has a semipositive solution, then $\rho > f(D)$

(P5) If D is indecomposable, $0 < f(D) < 1$ and $0 < \delta_1 < \delta_2 < 1/f(D)$, then $(I - \delta_1 D)^{-1} < (I - \delta_2 D)^{-1}$

(P6) To the Frobenius root f(D) of the indecomposable matrix D there belongs a positive row (column) eigenvector; the matrix D cannot have two linearly independent semipositive row (column) eigenvectors.

Considering these propositions, one could say that indecomposability entails a sharpening of the general results of semipositive matrices. In the proof of some theorems in Section 2, we shall need such sharp results. In that case, a typical trick is to consider in turn each of the indecomposable principal submatrices of the normal form (3). In some other cases, less sharp results will do; then it will be sufficient to use the following smaller partition of the input matrix:

(12) $$\begin{bmatrix} B & N_0 \\ 0 & N \end{bmatrix}$$

In this partition all nonbasics are considered as one block, i.e. in one principal submatrix N, even though N could be further decomposable. It is easy to derive the following equalities:

(13) $f(A) = \text{Max} \{ f(B), f(N) \}$

(14) $f(N) = \text{Max} \{ f(N_1), \ldots, f(N_h) \}$

It is well-known that an economy with input matrix A is able to generate a surplus iff $f(A) < 1$. In loose language one could say that the higher is the positive difference between 1 and f(A) (or the higher is the value of $1/f(A) - 1$) the more productive is the economy (or the greater is its surplus production). This leads us to the following terminology:

(D2) We say that $R(A) = 1/f(A) - 1$ is the rate of surplus production of the whole economy; $R(B) = 1/f(B) - 1$ is the own rate of surplus production of the block of basics; $R(N) = 1/f(N) - 1$ ($+\infty$ if N is a zero matrix) is the own rate of surplus production of the block of nonbasics; $R(N_j) = 1/f(N_j) - 1$ ($+\infty$ if $N_j = 0$) is the own rate of surplus production of the jth group of interconnected nonbasics.

It is easy to see that $R(A)$, $R(B)$, $R(N)$ and $R(N_j)$ are all positive, that $R(A) = \text{Min}\{ R(B), R(N)\}$ and $R(N) = \text{Min}\{ R(N_1), .., R(N_h)\}$. Later on in this paper it will become clear that some complications may be excluded in those cases where the following inequality holds:

REGULARITY CONDITION: $R(B) < R(N_j)$ for all j

Just like Sraffa, we assume that the surplus of the economy is divided at the end of the production period between wages and profits. Sraffa examines the effects of changes in the division of this given surplus on the prices of the individual commodities. One could also say that he compares stationary states with the same inputs and outputs, but with a different distribution. Sraffa first chooses the surplus as the numéraire /13, § 12/. Later on, however, he adopts the net product of a hypothetical standard system as his value unit, because the latter procedure leads to a transparent linear relation between the wage and the profit rate /13, Chapter IV/. With another value unit, this linear relation cannot be guaranteed, unless when there is equal organic composition of capital. We refer to Burmeister /2/ for a proof of this statement, and to Parys /10/ for a simple criterion for the presence of equal composition: in the economy under consideration, there is equal organic composition of capital in all industries iff all ratio's LA_j/L_j are equal[6]. This clearly is a very special case; there is thus no doubt that Sraffa's standard system is a useful auxiliary construction for some purposes. However, its surplus (Sraffa's standard net product) is

made up of basics only. We shall see that this could obscure some peculiarities of the division of the surplus of the actual system, which is made up of both basics and nonbasics. This is the reason why we adopt the actual net product as our numéraire in definition (D3) below. Moreover, it is the division of this actual surplus, and not that of a hypothetical standard system, which is ultimately the object of investigation.

(D3) A row vector p is said to be a vector of Sraffa prices iff there exist scalars $r \geq 0$ and w, $0 \leq w \leq 1$, such that the following equalities hold:

(15) $pA(1 + r) + Lw = p$

(16) $pc = 1$

Because of our choice of measurement units, w represents both the wage rate and the share of wages in the surplus; r is the uniform rate of profits. The following expression, derived from (1), (2), (15) and (16), clearly illustrates how the surplus is divided into two portions (profits and wages):

(17) $1 = pc = rpAe + w$

(D4) A pair (r, w), $r \geq 0$, $0 \leq w \leq 1$, is called a division of the surplus iff there exists a row vector p such that r, w and p solve (15) and (16); such a vector p is said to be a vector of Sraffa prices associated with the division (r, w).

The n + 1 price equations (15) - (16) contain n + 2 variables: in definition (D4) we have singled out combinations of r and w that are not inconsistent with each other as objects of further investigation. Note that in the above formalization nothing is specified yet about the positivity of the Sraffa prices. This topic will be treated in the next sections, where we shall consider the properties of different divisions of the surplus, in order to discuss the notion of a maximum value of r.

II. THEOREMS

In this section we analyze some properties of the various divisions of the surplus and their associated Sraffa prices. To this end, we partition the price equations (15) in the same way as the normal form (3) of the input matrix. Then we may write the equations for the block of basics, for the 1st and more generally for the jth group of interconnected nonbasics as follows[7]:

$$(18) \quad p_B B(1 + r) + L_B w = p_B$$

$$(19) \quad (p_B N_{01} + p_{N1} N_1) (1 + r) + L_{N1} w = p_{N1}$$

$$(20) \quad (p_B N_{0j} + p_{N1} N_{1j} + \ldots + p_{Nj} N_j) (1 + r) + L_{Nj} w = p_{Nj}$$

Using the definitions and assumptions of the preceding section, we shall prove the following four theorems:

THEOREM 1. Let $0 \leq r < R(A)$. Then there is one and only one value of w such that (r, w) forms a division of the surplus. This value of w is positive and the division (r, w) is associated with a unique and positive vector of Sraffa prices.

THEOREM 2. (a) Let $r = R(A)$ and assume that the regularity condition holds. Then there is one and only one value of w such that (R(A), w) forms a division of the surplus. This value of w is zero and the division (R(A), 0) is associated with a unique and positive vector of Sraffa prices.

(b) Let $r = R(A)$ and assume that the regularity condition does not hold. Let (R(A), w) be a division of the surplus, associated with a Sraffa price vector p. This vector cannot be positive.

THEOREM 3. Let $r > R(A)$. Let (r, w) be a division of the surplus, associated with a Sraffa price vector p. This vector can not be positive.

THEOREM 4. Let $(\bar{r}, \bar{w})$ and $(\bar{\bar{r}}, \bar{\bar{w}})$ be two divisions of the surplus, associated with respective Sraffa price vectors $\bar{p}$ and $\bar{\bar{p}}$, and assume that $0 \leq \bar{r} < \bar{\bar{r}} < R(A)$. Then the following inequalities hold:

(a) $\bar{p}/\bar{w} < \bar{\bar{p}}/\bar{\bar{w}}$

(b) $\bar{w} > \bar{\bar{w}}$

PROOF OF THEOREM 1.

We take an arbitrary $\bar{r}$, $0 \leq r < R(A)$ and denote $1/(1 + \bar{r})$ by $\bar{\rho}$. Hence, we have $0 < f(A) < \bar{\rho} \leq 1$ and thus a fortiori $f(B) < \bar{\rho}$ and $f(N_j) < \bar{\rho}$ for all j.

From proposition (P1), the matrix $\bar{\rho}I - A$ has a semipositive inverse. Hence, there are unique values $\bar{w}$ and $\bar{p}$ that solve the price equations (15) - (16) if $r = \bar{r}$: $\bar{p}$ equals $L\bar{w}\bar{\rho}\ (\bar{\rho}I - A)^{-1}$, and $\bar{w}$ and $\bar{p}$ are normalized such that $\bar{p}c = 1$; it is obvious that $\bar{w}$ is positive and $\bar{p}$ semipositive.

The proof is complete iff we show that $\bar{p}$ is positive. To this end, we consider each of its subvectors $\bar{p}_B$, $\bar{p}_{N1}$, ..., $\bar{p}_{Nj}$, ... in succession.

The matrix B is indecomposable and $f(B) < \bar{\rho}$, thus $(\bar{\rho}I - B)$ has a positive inverse by (P3). From equation (18), we then obtain[8]:

(21) $\bar{p}_B = L_B\bar{w}\bar{\rho}\ (\bar{\rho}I - B)^{-1} > 0$

The matrix N_1 is indecomposable and $f(N_1) < \bar{\rho}$, thus $(\bar{\rho}I - N_1)$ has a positive inverse, From (5) it follows that N_{01} is semipositive; we know that $\bar{p}_B > 0$ and we then obtain from equation (19) :

(22) $\bar{p}_{N1} = (\bar{p}_B N_{01} + L_{N1}\bar{w}\bar{\rho})\ (\bar{\rho}I - N_1)^{-1} > 0$

In the same way we show that $(\bar{\rho}I - N_j)^{-1} > 0$ for $j = 2, 3, ..,h$. We consider equation (20) to obtain the prices of the jth group of interconnected nonbasics:

(23) $\bar{p}_{Nj} = (\bar{p}_B N_{0j} + \bar{p}_{N1} N_{1j} + .. + \bar{p}_{N,j-1} N_{j-1,j} + L_{Nj}\bar{w}\bar{\rho})(\bar{\rho}I - N_j)^{-1}$

From (5), it follows that the positivity of $\bar{p}_B$, $\bar{p}_{N1}$, ..., $\bar{p}_{N,j-1}$ would imply the positivity of p_{Nj}. We already proved that $\bar{p}_B$ and $\bar{p}_{N1}$ are positive; hence by induction on j we know that all subvectors of $\bar{p}$ and thus $\bar{p}$ itself are positive.

Q.E.D.

PROOF OF THEOREM 2. (a)

It is given that $R(B) < R(N_j)$ for all j; this implies $f(A) = f(B) > f(N)$. From (P2) it follows that the row eigenvector of A corresponding to f(A) and normalized by (16), provides us with a positive vector of Sraffa prices associated with the division (R(A), 0).

We now prove that (R(A), 0) is the only division that has $r = R(A)$. Suppose the contrary: let (R(A), $\bar{w}$) be a division of the surplus associated with a Sraffa price vector $\bar{p}$ and suppose $\bar{w} \neq 0$. From equation (18), we derive the equality $\bar{p}_B (f(A) I - B) = L_B \bar{w} f(A)$. From (P6), we know that B has a positive column eigenvector corresponding with f(B). We denote this vector by q_B, we postmultiply both sides of the last equality by q_B to obtain a contradiction: $\bar{p}_B(f(A)I - B)q_B$ is zero (as $f(A) = f(B)$), while $L_B \bar{w}\, f(A)\, q_B$ is not.

We finally have to prove that there are no two different Sraffa price vectors associated with (R(A), 0), Suppose the contrary, let $\bar{p}$ and $\bar{\bar{p}}$ be two vectors of Sraffa prices associated with the division (R(A), 0). We partition $\bar{p}$, $\bar{\bar{p}}$ and the surplus vector c corresponding with partition (12), we use the equality f(A)= f(B) and we then obtain the following equations:

(24) $\bar{p}_B B = \bar{p}_B f(B)$

(25) $\bar{\bar{p}}_B B = \bar{\bar{p}}_B f(B)$

(26) $\bar{p}_B N_0 + \bar{p}_N N = \bar{p}_N f(B)$

(27) $\bar{\bar{p}}_B N_0 + \bar{\bar{p}}_N N = \bar{\bar{p}}_N f(B)$

(28) $\bar{p}_B c_B + \bar{p}_N c_N = 1$

(29) $\bar{\bar{p}}_B c_B + \bar{\bar{p}}_N c_N = 1$

From (P6) we know that $\bar{p}_B$ must be a scalar multiple of $\bar{\bar{p}}_B$. Suppose for a moment that $\bar{p}_B \neq \bar{\bar{p}}_B$. Then we may assume, without loss of generality, that $\bar{p}_B - \bar{\bar{p}}_B > 0$. From (26) and (27) we may derive:

(30) $(\bar{p}_N - \bar{\bar{p}}_N)\,(f(B)I - N) = (\bar{p}_B - \bar{\bar{p}}_B)\,N_0$

From $f(B) > f(N)$ and (P1), it follows that the matrix $f(B)I - N$ has a semipositive inverse. Then equation (30) implies $\bar{p}_N - \bar{\bar{p}}_N \geq 0$. However, $\bar{p}_B - \bar{\bar{p}}_B > 0$, $\bar{p}_N - \bar{\bar{p}}_N \geq 0$, (28), (29) and (A2) lead to a contradiction; thus $\bar{p}_B = \bar{\bar{p}}_B$ must hold. Then equation (30) yields: $(\bar{p}_N - \bar{\bar{p}}_N)(f(B)I - N) = 0$.
As $f(B) > f(N)$, $f(B)$ is not an eigenvalue of N; hence $\bar{p}_N = \bar{\bar{p}}_N$ must hold. This shows that the two Sraffa price vectors $\bar{p}$ and $\bar{\bar{p}}$ are equal.

Q.E.D.

PROOF OF THEOREM 2. (b)

It is given that the regularity condition does not hold. This implies that $0 < f(B) \leq f(A) = f(N_m)$ for some group m of interconnected nonbasics. Suppose that a Sraffa price vector $\bar{p}$, associated with a division $(R(A), \bar{w})$, is positive.
We take into account that $1/(1 + R(A)) = f(N_m)$ and we consider the price equation for the mth group of nonbasics. Then the positive subvector $\bar{p}_{Nm}$ satisfies:

$$\bar{p}_{Nm}\,(f(N_m)\,I - N_m) = \bar{p}_B N_{0m} + \bar{p}_{N1} N_{1m} + \ldots + \bar{p}_{N,m-1}\,N_{m-1,m} + L_{Nm}\bar{w}f(N_m)$$

Hence, the system $y(\rho I - N_m) \geq 0$ has a positive solution $y = \bar{p}_{Nm}$ for $\rho = f(N_m)$ and N_m indecomposable. This is impossible by (P4). This contradiction shows that the Sraffa price vector $\bar{p}$ cannot be positive. Q.E.D.

PROOF OF THEOREM 3.

It is given that (r, w) is a division associated with a Sraffa price vector p and such that $r > R(A)$. We shall prove by contradiction that p is not positive. We denote $1/(1 + r)$ by ρ : hence $0 < \rho < 1/(1 + R(A)) = f(A)$. Suppose for a moment that p is positive. Then we obtain $p_B(\rho I - B) = L_B w\rho$, $p_B > 0$.
If $w = 0$, then $\rho = f(B)$ from (P6); if $w > 0$, then $\rho > f(B)$ from (P4). In either case $f(B) \leq \rho < f(A)$; thus $f(A) = f(N_m)$ holds for some group m of interconnected nonbasics. Just like in the proof of Theorem 2 (b), we consider the price equation of this mth group of nonbasics to obtain $p_{Nm}(\rho I - N_m) \geq 0$, $p_{Nm} > 0$, N_m indecomposable, $\rho < f(N_m)$. This contradicts (P4). Q.E.D.

PROOF OF THEOREM 4.

We first remark that $\bar{w}$, $\bar{\bar{w}}$, $\bar{p}$ and $\bar{\bar{p}}$ are all positive by Theorem 1. From equation (18), it follows that $\bar{p}_B/\bar{w} = L_B(I - (1 + \bar{r})B)^{-1}$ and $\bar{\bar{p}}_B/\bar{\bar{w}} = L_B(I - (1 + \bar{\bar{r}})B)^{-1}$. We know that $1 + \bar{r} < 1 + \bar{\bar{r}} < 1 + R(A) \leq 1/f(B)$; then we may conclude from (P5) that $\bar{p}_B/\bar{w} < \bar{\bar{p}}_B/\bar{\bar{w}}$.
The rest of the proof of Theorem 4 (a) is similar to that of Theorem 1.: we have to consider each group of interconnected nonbasics in succession. In the case of the first group, we make use of $0 < \bar{p}_B/\bar{w} < \bar{\bar{p}}_B/\bar{\bar{w}}$, of $1 + \bar{r} < 1 + \bar{\bar{r}} < 1/f(N_1)$ and (P5) if $N_1 \neq 0$, and of (5) to prove that $\bar{p}_{N1}/\bar{w} < \bar{\bar{p}}_{N1}/\bar{\bar{w}}$. We then may show, by induction on j, that $\bar{p}_{Nj}/\bar{w} < \bar{\bar{p}}_{Nj}/\bar{\bar{w}}$ holds for all j. In this way we establish the inequality $0 < \bar{p}/\bar{w} < \bar{\bar{p}}/\bar{\bar{w}}$.
The latter also implies that $\bar{r}\bar{p}Ae/\bar{w} < \bar{\bar{r}}\bar{\bar{p}}Ae/\bar{\bar{w}}$. Equation (17) can also be written as $rpAe/w + 1 = 1/w$ for $w \neq 0$. Hence, it follows that $1/\bar{w} < 1/\bar{\bar{w}}$ or $\bar{w} > \bar{\bar{w}}$.

Q.E.D.

The interpretation of the last theorem is that a higher rate of profits ($\bar{\bar{r}} > \bar{r}$) implies a higher value in terms of labour commanded for all commodities ($\bar{\bar{p}}_i/\bar{\bar{w}} > \bar{p}_i/\bar{w}$ for any commodity i)

or a lower value of the wage in terms of any commodity ($\bar{\bar{w}}/\bar{\bar{p}}_i < \bar{w}/\bar{p}_i$ for any i). Then it is also clear that a higher value of r implies a lower value of the wage expressed in terms of the surplus (which may be considered as a composite commodity): this is the content of Theorem 4 (b). We stress that in the formulation of Theorem 4 we assume $0 \leq \bar{r} < \bar{\bar{r}} < R(A)$. A numerical example in the next section will show that the above inverse relation between the wage and the profit rate does not necessarily hold once the value of r exceeds R(A).

III. EXAMPLES

We consider an economy with one basic and two nonbasics

$$A = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \qquad C = \begin{bmatrix} 0.2 \\ 0.8 \\ 0.4 \end{bmatrix}$$

The input matrix is already in its normal form; all its submatrices are 1 x 1; $f(A) = 0.6$, $f(B) = 0.4$, $f(N_1) = 0.2$, $f(N_2) = 0.6$; $R(A) = 2/3$, $R(B) = 1.5$, $R(N_1) = 4$, $R(N_2) = 2/3$.
In this numerical example the price equations are:

(31) $0.4\ p_1\ (1 + r) + L_1 w = p_1$

(32) $(0.1\ p_1 + 0.2\ p_2)(1 + r) + L_2 w = p_2$

(33) $(0.3\ p_1 + 0.6\ p_3)(1 + r) + L_3 w = p_3$

(34) $0.2\ p_1 + 0.8\ p_2 + 0.4\ p_3 = 1$

Suppose for a moment that the direct labour inputs are $L_1 = L_3 = 0.001$, $L_2 = 0.998$. If we carry out a division of the surplus with $r = 5/8 = 0.625$, then $w = 63/77 = 0.818$, $p_1 = 18/7700$, $p_2 = 9319/7700$, $p_3 = 603/7700$: these values can be computed from (31) ∿ (34) and they illustrate Theorem 1.
If the rate of profits rises beyond $R(A) = R(N_2) = 2/3$ till $r = 6/8 = 0.750$, we obtain $w = 130/157 = 0.828$, $p_1 = 26/9420$, $p_2 = 11983/9420$, $p_3 = -429/9420$.

It is interesting to consider this example in the light of Sraffa's treatment. Sraffa simplified his analysis by temporarily assuming away the nonbasics. He then considers values of the wage w varying from 1 to 0, and a rate of profits r varying from 0 to R. Sraffa's symbol R stands for our R(B), the rate of surplus production of the block of basics. Sraffa considers R = R(B) as the maximum rate of profits and then formulates a.o. propositions about the positivity of the prices and the relation between w and r. Such an approach may lead to

anomalies if the nonbasics are taken into account again at a later stage of the analysis. Sraffa himself realizes the existence of complications concerning the positivity of the prices in his Appendix B. Those who read this part of Sraffa's book, will not be surprised that our example shows a negative price $p_3 = -429/9420$ for the second nonbasic, because we consider a situation where the profit rate $r = 0.750$ exceeds the rate of surplus production of that second nonbasic ($R(N_2) = 2/3$) even if $r = 0.750$ is still smaller than Sraffa's maximum rate $R = R(B) = 1.5$. What seems remarkable, however, is that a rise of r, from 0.625 to 0.750, corresponds with a rise of w, from 0.818 to 0.828. This seems to contradict Sraffa's results on the inverse relation between w and r in § 49 of his book. There Sraffa states that a cut of the wage in terms of any arbitrarily chosen product, corresponds with a rise in the rate of profits and vice versa; this is a proposition which he considers to be valid as long as r is not higher than $R = R(B)$. Sraffa offers two proofs of this: his first proof uses a so-called reduction series of dated labour quantities, but one can show[9] that such a series does not converge for some commodities if $r > R(A)$; his second proof implicitly assumes that all prices are positive, but in our Section 2 we proved that this is impossible if $r > R(A)$. Hence, both proofs are no longer valid if $r > R(A)$ holds. This explains the outcome of our numerical example: $r = 0.750$ is only half of Sraffa's $R = 1.5$, but it exceeds $R(A) = 2/3$. It is the latter value which is relevant here: it is impossible to offer a general proof of our Theorem 4 for values of r higher than R(A).

If one follows Sraffa's approach and considers R(B) as the maximum rate of profits, one can also encounter the complication that it will be impossible to carry out a division of the surplus for some positive profit rate smaller than what one has called the maximum rate. To illustrate this, we assume that $r = 0.750 < R(B) = 1.5$ holds. From (31) ∿ (34) it follows that:

(35) $w = 13/(16\ L_2 - 164\ L_1 - 104\ L_3)$

If the direct labour inputs are $L_1 = L_3 = 1/80$, $L_2 = 78/80$, then the only formal solution for w is 52/49; if we put $L_1 = 2/45$, $L_2 = 40/45$, $L_3 = 3/45$, then w becomes infinite; if $L_1 = L_2 = L_3 = 1/3$, then $w = -13/84$. These formal solutions are hard to conceive of; it is more reasonable to follow our Definition (D4) and to say that no division of the surplus with $r = 0.750$ ($r > R(A) = 2/3$) can be carried out.

Such anomalies concerning the fraction of the surplus going to wages, were not mentioned in Sraffa's book. A possible explanation may be offered by means of our numerical example. Let $L_1 = L_3 = 1/80$, $L_2 = 78/80$ and $r = 0.750$. Then we obtained, from (35), the formal solution $w = 52/49$; we can also compute, from (31) ∿ (34), that $p_1 = 52/1176$, $p_2 = 1886/1176$, $p_3 = -858/1176$. We see that the wage is equivalent to 24 units of commodity 1. In this example, Sraffa would consider 48 units of commodity 1 as the standard net product[10]. Suppose for a moment that we follow Sraffa's practice of choosing this standard net product as the numéraire. Then we would obtain the result that the share of wages equals 1/2 of the standard net product. At first sight, this seems to be a perfectly regular case: the choice of the net product of a hypothetical standard economy as the numéraire has obscured the anomaly that the share of wages equals 52/49 of the net product of the actual economy.

IV. CONCLUDING REMARKS

Considering our mathematical results, one is inclined to single out R(A), the rate of surplus production of the whole economy, as the maximum rate of profits. We are not going to speculate whether Sraffa was aware of all the mathematical consequences when he chose R(B) instead of R(A) as the maximum rate. It is well-known that Sraffa's treatment was influenced by his neo-Ricardian vision of the economic process: in Sraffa's interpretation[11] of Ricardo's early theory of profit /12, pp. xxx-xxxiii and 13, p. 93/, corn is the sole basic and the rate of profits obtained by the grower of corn regulates that of all other trades. If one wants to stick to such a vision without losing analytical rigour, it is best to introduce explicitly something as our regularity condition[12] ($R(B) < R(N_j)$ for all j, or $R(B) < R(N)$) at the outset of the analysis. This condition implies that R(A) and Sraffa's maximum rate of profits R = R(B) are equal and it guarantees that none of the complications mentioned in the preceding sections can arise.

We finally want to make the obvious remark that the mathematics of this paper do not provide the definite argument in a discussion of the degree of economic relevance of Sraffa's approach: we have only shown how important it is to make some assumptions explicit, when one wants to avoid logical complications.

NOTES

1 We use the usual ordering relations for vectors and matrices: for example, we say that vectors and matrices are nonnegative (denoted by ≥ 0) iff (if and only if) all their elements are nonnegative, semipositive (> 0) iff all elements are nonnegative and at least one is positive, positive (> 0) iff all elements are positive.

2 A square matrix A of order n is said to be decomposable iff the set $\{1, 2, \ldots, n\}$ can be partitioned into two subsets G and J such that $a_{gj} = 0$ if $g \in G$, $j \in J$; A is said to be completely decomposable iff one can find a partition such that both $a_{gj} = 0$ and $a_{jg} = 0$ if $g \in G$, $j \in J$; decomposable but not completely decomposable matrices are called partially decomposable. In all other cases, we call the square matrix A indecomposable; thus the zero matrix of order 1 is said to be indecomposable throughout our paper.

3 See e.g. Nikaido /9, p. 118/.

4 For a proof of (7) we refer to Gantmacher /4, p. 69/. The inequality $f(A) < 1$ can be derived with the help of assumption (A2); then the other relations are obvious.

5 Proposition (P2) was proved by Gantmacher /4, pp. 77-78/. A proof of (P5) can easily be constructed by means of Lemma 3.6. of Schwartz /11, pp. 32-33/. The other propositions may be established with the help of various references /4, 8, 9, 14/.

6 Recently I have realized that this criterion has an obvious interpretation in terms of Sraffa's reduction to dated quantities of labour: there is equal organic composition iff the dated quantities of labour have the following regular pattern:

L_1, $L_1/(1 + R)$, $L_1/(1 + R)^2$, ... for commodity 1;
L_2, $L_2/(1 + R)$, $L_2/(1 + R)^2$, ... for commodity 2, etc...
Sraffa's standard commodity taken as an aggregate always has such a perfectly regular series /13, p. 37/ even if equal value composition does not prevail. This is yet another situation where the introduction of Sraffa's standard commodity produces some characteristics which are typical of economies with equal value composition of capital.

7 The notation is obvious: for example, p_B contains the prices of the basics, L_{N1} contains the direct labour inputs of the 1th group of interconnected nonbasics, etc...

8 Remember that all subvectors L_B, L_{N1}, ..., L_{Nj}, ... are positive because of our assumption (A3). All our proofs are also valid under the mathematically weaker assumption $L_B \geq 0$, $L_{Nj} \geqq 0$ for all j. However, the latter assumption seems less plausible from an economic point of view.

9 E.g. by means of Nikaido's results on C. Neumann series /9, pp. 127-131/.

10 Commodity 1 is the only basic; to produce a surplus of 0.6 units of it (1 - 0.4), L_1 = 1/80 direct labour units are required. Hence the standard net product equals 0.6 x 80 = 48 units of commodity 1.

11 I wrote "Sraffa's interpretation", because it is a controversial question whether Ricardo himself ever had such a structure in mind: we refer to the recent discussion between Hollander /5, 6/ and Eatwell /3/.

[12] This regularity condition provides some kind of generalization of Sraffa's Appendix B. Sraffa considers a complication due to the existence of a single nonbasic commodity which enters to a large extent into its own production. Similarly, in our more general framework, a uniform rate of profits $r<R(B)$ leads to complications when there exists a group m of interconnected nonbasics with such a low own rate of surplus production that $R(A) = R(N_m) < r < R(B)$ holds.

REFERENCES

/1/ BLAKLEY, G.R. and GOSSLING, W.F., "The Existence, Uniqueness and Stability of the Standard System", Review of Economic Studies, vol. 34, October 1967, pp. 427-431.

/2/ BURMEISTER, E., "On a Theorem of Sraffa", Economica, vol. 35, February 1968, pp. 83-87.

/3/ EATWELL, J., "The Interpretation of Ricardo's Essay on Profits", Economica, vol. 42, May 1975, pp. 182-187.

/4/ GANTMACHER, F.R., The Theory of Matrices, Vol.Two, Chelsea Publishing Company, New York, 1960.

/5/ HOLLANDER, S., "Ricardo's Analysis of the Profit Rate, 1813-'15", Economica, vol. 40, August 1973, pp. 260-282.

/6/ HOLLANDER, S., "Ricardo and the Corn Profit Model: Reply to Eatwell", Economica, vol. 42, May 1975, pp. 188-202.

/7/ NEWMAN, P., "Production of Commodities by Means of Commodities", Schweizerische Zeitschrift für Volkswirtschaft und Statistik, vol. 98, March 1962, pp. 58-75.

/8/ NIKAIDO, H., Convex Structures and Economic Theory, Academic Press, New York, 1968.

/9/ NIKAIDO, H., Introduction to Sets and Mappings in Modern Economics, North-Holland Publishing Company, Amsterdam, 1970.

/10/ PARYS, W., "A simple criterion for equal value composition of capital", Quarterly Journal of Economics, forthcoming.

/11/ SCHWARTZ, J.T., Lectures on the Mathematical Method in Analytical Economics, Gordon and Breach, New York, 1961.

/12/ SRAFFA, P. (editor), Works and Correspondence of David Ricardo, Volume One, Cambridge University Press, Cambridge, 1951.

/13/ SRAFFA, P., Production of Commodities by Means of Commodities, Cambridge University Press, Cambridge, 1960.

/14/ VARGA, R.S., Matrix Iterative Analysis, Prentice Hall, Englewood Cliffs, New Jersey, 1962.