EXISTENCE OF POSITIVE PRICES FOR
SRAFFA'S BASICS AND NON-BASICS (1)

W. Parys

Werknota 7653

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(1) The author thanks W. Pauwels for some helpful comments and L. Berlage for his remarks on an earlier draft. It is obvious that none of them bears any responsibility for the shortcomings of this paper.
Our purpose is to derive a condition that is both necessary and sufficient for the existence of positive prices for all goods, basics as well as non-basics, for every feasible distribution of the surplus, in Sraffa's model of single-product industries \(^5\), Part \(^7\).

As was repeatedly stressed by different authors, e.g. by Burmeister \(^1\) and \(^2\), the application of theorems on non-negative matrices seems the most effective method for a formal approach to Sraffa's propositions. Many mathematical treatments on neo-Ricardian economics use such theorems, but concentrate on systems of basic goods, i.e. products which enter directly or indirectly into the production of all commodities. The existence of all-positive prices in such a system is often mentioned, the proof of which is relatively easy \(^-1\), p. 84\(^7\) as the underlying input matrix is indecomposable.

In this paper we shall consider a system where basics and non-basics coexist. Sraffa himself \(^5\), pp. 90-91\(^7\) realized that in this more general framework the prices of some commodities may become non-positive. Nevertheless he did not offer a rigorous analysis of this situation.
We consider a society which produces a set of $n$ commodities: $k$ basics and $n-k$ non-basics ($1 \leq k < n$); $x_j$ units of commodity $j$ are produced by means of $a_{ij}x_j$ units of commodity $i$ and $1_jx_j$ units of homogeneous labour. The coefficients $a_{ij}$ form the $n \times n$ input matrix $A$; $1$ denotes the $1 \times n$ vector of direct labour requirements per unit; $x$ and $y = x - Ax$ are $n \times 1$ vectors of total gross and total net product of the system.

We say that a non-negative (2) $n \times n$ matrix $A$ is decomposable iff (if and only if) there is a non-empty proper subset $J$ of \{1, 2, ..., $n$\} such that $a_{ij} = 0$, $i \notin J$, $j \in J$. Otherwise we shall call $A$ indecomposable (e.g. whenever it is $1 \times 1$). It is obvious that in our case the input matrix is decomposable \cite{1}, p.1097.

By a suitable numbering of the commodities, this decomposable input matrix can be reduced to its so-called normal form \cite{3}, p. 757: (3)

\[
\begin{bmatrix}
B & D_{01} & D_{02} & \cdots & D_{0m} & \cdots & D_{0q} \\
0 & D_1 & D_{12} & \cdots & D_{1m} & \cdots & D_{1q} \\
0 & 0 & D_2 & \cdots & D_{2m} & \cdots & D_{2q} \\
\vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 & \cdots & D_m & \cdots & D_{mq} \\
0 & 0 & 0 & \cdots & 0 & \cdots & D_q
\end{bmatrix} \tag{1/}
\]

(2) Vectors and matrices are called non-negative (denoted by $\geq 0$) iff all their elements are non-negative, semi-positive ($\geq 0$) iff all elements are non-negative and some are positive, positive ($> 0$) iff all elements are positive.

(3) I repeat that we consider an economy with basics and non-basics. In Gantmacher's notation, this implies $g = 1$ and $s - g > 0$. 
In our normal form, the principal submatrix B is an indecomposable \( k \times k \) -matrix associated with the \( k \) basic goods; the principal submatrices \( D_1, D_2, \ldots, D_m, \ldots, D_q \) are indecomposable and are associated with groups of non-basics; in each column at least one submatrix \( D_{0g}, D_{1g}, \ldots, D_{g-1,g} \) (\( g = 1, 2, \ldots, q \)) is semi-positive. The dominant root (Frobenius-Perron eigenvalue) of the matrices \( A, B, D_m \) is denoted by \( f(A), f(B), f(D_m) \). Let \( I \) be the identity matrix and \( h \) a numerical parameter. It is well known \( 4, \) pp. 95, 102, 107 \( \) that the matrix \( hI - A \) has a non-negative inverse \( (hI - A)^{-1} \) iff \( h > f(A) \), that this inverse is positive if \( A \) is indecomposable and that the system with input matrix \( A \) is workable iff \( 1 > f(A) \).

One of Sraffa's achievements was the construction of an elegant numéraire (his standard composite commodity), but for our purposes nothing essential will be lost \( \) if we simply normalize prices so that the value of the actual net product equals one. This net product is distributed post factum among workers (wages) and capitalists (profits).

**Definition 1:** A \( 1 \times n \) -vector \( p \) is said to be a Sraffa price vector iff there exist non-negative scalars \( w \) (wage rate) and \( r \) (uniform rate of profits) such that the following equalities hold:

\[
pA (1 + r) + lw = p \quad /2/
\]

\[
p^y = 1 \quad /3/
\]

A price vector \( p \) will also be denoted by \( (p_B, p_{D1}, p_{D2}, \ldots, p_{Dm}, \ldots, p_{Dq}) \), the vector of direct labour inputs also by \( (l_B, l_{D1}, l_{D2}, \ldots, l_{Dm}, \ldots, l_{Dq}) \).

The notation is obvious: e.g. the \( 1 \times k \) -vector \( p_B \) contains the prices of the basics. Using this notation, the price relations \( /2/ \) can be written as: \[\text{\ldots} \]
\[ p_B B(1 + r) + l_B w = p_B \]

\[ (p_B^{D_0} + p_{D_1}^{D_1})(1 + r) + l_{D_1} w = p_{D_1} \]

more generally, for the \( m \)th group of non-basics:

\[ (p_B^{D_{0m}} + p_{D_1}^{D_{1m}} + p_{D_2}^{D_{2m}} + \ldots + p_{D_m}^{D_{mm}})(1 + r) + l_{D_m} w \]

\[ = p_{D_m} \]

We use the following weak (4) **assumptions**: labour is indispensable to produce any good \((i > 0)\), all \( n \) commodities have positive gross outputs \((x > 0)\) and there is a surplus to be distributed \((y > 0)\); the system is workable i.e. \( f(A) < 1 \) and thus a fortiori \( f(B) < 1 \) and \( f(D_i) < 1 \) all \( i \).

**Lemma**: The existence of all-positive prices implies that Sraffa's maximum rate of profits (i.e. the value of \( r \) if \( w = 0 \)) equals \( 1/f(B) - 1 \).

**Proof**: If the lemma is false, putting \( w = 0 \) in equation /4/ implies \( p_B B(1 + r) = p_B \) for a vector \( p_B > 0 \) and \( 1 + r \neq 1/f(B) \).

This means that the indecomposable matrix \( B \) has a positive eigenvector associated with an eigenvalue other than \( f(B) \). This is impossible \( \sum 3, p. 69 \).

We denote \( 1/f(B) - 1 \) by \( R \). Note that \( R \) is positive as \( 0 < f(B) < 1 \). As relations /2/ and /3/ consist of \( n + 1 \) equations with \( n + 2 \) unknowns, some combinations of \( w \) and \( r \) make the equations inconsistent.

(4) I mean: not restrictive from an economic point of view.
Therefore we introduce the following terminology:

**Definition 2**: A pair \((\mathbf{\bar{w}}, \mathbf{\bar{r}})\), \(\mathbf{\bar{w}} \geq 0\), \(0 \leq \mathbf{\bar{r}} \leq \mathbf{R}\), is said to be a **feasible distribution** iff there exist a Sraffa price vector that solves equations /2/ and /3/ for \(\mathbf{w} = \mathbf{\bar{w}}\) and \(\mathbf{r} = \mathbf{\bar{r}}\).

Note that in definitions 1 and 2, nothing is specified about the sign of the price vector.

We are able now to formulate our theorem that holds in the general neo-Ricardian model outlined above

**THEOREM**: With any feasible distribution there is associated a **positive** Sraffa price vector if and only if the normal form of the input matrix satisfies:

\[
\mathbf{f(B)} > \mathbf{f(D_i)} \quad i = 1, 2, \ldots, q \quad /7/
\]

**Proof:**

a) **necessity**

We shall prove by contradiction that without condition /7/ no positive Sraffa price vector exists for \(\mathbf{w} = 0\).

Let \(\mathbf{D_m}\) be the principal submatrix with the largest dominant root of all \(\mathbf{D_i}\). Suppose that a positive vector

\[
\mathbf{\bar{p}} = (\mathbf{\bar{p}_B} \mathbf{\bar{p}_{D1}} \ldots \mathbf{\bar{p}_{Dm}} \ldots \mathbf{\bar{p}_{Dq}})
\]

solves equations /2/ and /3/ for \(\mathbf{w} = 0\) and that nevertheless \(\mathbf{f(D_m)} \geq \mathbf{f(B)}\).

The lemma implies that \(\mathbf{\bar{p}}\) is associated with the maximum rate of profits \(\mathbf{R} = 1/\mathbf{f(B)} - 1\). The price equation /6/ for the mth group of non-basics then becomes:

\[
\mathbf{\bar{p}_B D_{0m}} + \mathbf{\bar{p}_{D1} D_{1m}} + \ldots + \mathbf{\bar{p}_{D,m-1} D_{m-1,m}} + \mathbf{\bar{p}_{Dm} D_{m}} = \mathbf{f(B)} \mathbf{\bar{p}_{Dm}}
\]
As the sum of the first \( m \) terms forms a semi-positive vector, the system \( p_{D_m}(f(B)I - D_m) > 0 \) has a positive solution, viz. \( p_{D_m} \). As \( D_m \) is indecomposable, this implies that 
\( f(B)I - D_m \) is non-negatively invertible \( \quad \), p. 107 \( \), hence 
\( f(B) > f(D_m) \): contradiction.

b) sufficiency

As \( f(D_i) < f(B) \) for all \( i \) and as \( A \) is decomposable, it is obvious that \( f(B) = f(A) \) \( \quad \), p. 69 \( \). To the dominant root \( f(A) \) of the matrix \( A \) there belongs a positive characteristic row vector \( \quad \), p. 77 \( \). This row vector, normalized by equation \( /3/ \), provides us with a positive Sraffa price vector \( \vec{w} \) for the distribution \( w = 0, r = R \).

We claim that the distributions \( (w_1, r_1), w_1 = 0, 0 \leq r_1 < R \) and \( (w_2, r_2), w_2 > 0, r_2 = R \) are not feasible.

If \( w = w_1, r = r_1 \), equations \( /2/ \) and \( /3/ \) become:

\[
pA = \frac{1}{1 + r_1} p
\]
\[
py = 1
\]

As \( y \geq 0 \) and \( \frac{1}{1 + r_1} > \frac{1}{1 + R} = f(B) = f(A) \), this system of equations is inconsistent.

If \( w = w_2, r = r_2 \), equation \( /4/ \) becomes:

\[
p_B (f(B)I - B) = w_2 f(B) l_B
\]

With the dominant root \( f(B) \) of the indecomposable matrix \( B \) is associated a positive column eigenvector \( q_B \).

Postmultiplying both sides of the last equality by \( q_B \), we obtain:

\[
p_B (f(B)I - B) q_B = w_2 f(B) l_B q_B
\]
As the left-hand side is zero and the right-hand side is positive, \((w_2, r_2)\) is not a feasible distribution.

The proof is therefore complete if we can show that condition /7/ also ensures the existence of a positive Sraffa price vector for any given feasible distribution \((\overline{w}, \overline{r})\), \(\overline{w} > 0, 0 \leq \overline{r} < R\). We shall construct such a vector in an inductive way.

The matrix \(B\) is indecomposable and \(h = 1/(1 + \overline{r}) > 1/(1 + R) = f(B)\), thus \(hI - B\) has a positive inverse. Hence \(\overline{p}_B = hI_B \overline{w} (hI - B)^{-1}\) is a positive solution of price equation /4/.

The matrix \(D_1\) is indecomposable and \(h = 1/(1 + \overline{r}) > f(B) > f(D_1)\), thus \(hI - D_1\) has a positive inverse.

Hence \(\overline{p}_{D1} = (\overline{p}_B D_{01} + hI_{D_1} \overline{w}) (hI - D_1)^{-1}\), derived from equation /5/, provides us with positive prices for the first group of non-basics.

In the same way we obtain for the \(mth\) group of non-basics via relation /6/:

\[
\overline{p}_{Dm} = (\overline{p}_B D_{0m} + \overline{p}_{D1} D_{1m} + \cdots + \overline{p}_{D,m-1} D_{(m-1)m} + hI_{Dm} \overline{w})(hI - D_m)^{-1}
\]

Here \(\overline{p}_{Dm}\) is positive for the same reasons as before.

In this way a positive Sraffa price vector \((\overline{p}_B, \overline{p}_{D1}, \ldots, \overline{p}_{Dm}, \ldots, \overline{p}_{Dq})\), associated with the distribution \((\overline{w}, \overline{r})\), can be constructed.

Q.E.D.
Remark 1: The preceding reasoning reveals that condition 7/ not only guarantees the existence of a positive Sraffa price vector for any given feasible distribution (\(\bar{w}, \bar{p}\)) with \(\bar{w} > 0\) and \(0 < r < R\), but also its uniqueness. This uniqueness also holds if \(w = 0\) and \(r = R\). Assume the contrary: let there be two positive Sraffa price vectors \(\bar{p}\) and \(\bar{q}\). B is indecomposable, hence \(\bar{p}_B = \bar{q}_B\) must hold \(\Leftrightarrow 3\), p. 53.\(^7\)

Consider equation 5/:

\[
\bar{p}_B \quad D_{01} + \bar{p}_D \quad D_{01} = f(B) \bar{p}_D
\]

\[
\bar{q}_B \quad D_{01} + \bar{q}_D \quad D_{01} = f(B) \bar{q}_D
\]

This implies that \((\bar{p}_D - \bar{q}_D) (f(B)I - D_1) = 0\)

If \(\bar{p}_D \neq \bar{q}_D\), then \(f(B)\) is an eigenvalue of \(D_1\). This contradicts condition 7/, hence \(\bar{p}_D = \bar{q}_D\).

In the same way, the uniqueness of the other \(p_{D_i}\) (\(i = 2, 3, \ldots, q\)) may be proven.

Remark 2: Define the unit of labour so that total labour \(lx\) equals one. Hence, the wage rate equals total wages.

In case of zero profits, their common value equals one. In this case, the Sraffa price vector equals the vector of labour values.

The positivity of these labour values is of course necessary for the prices to be positive for any feasible distribution, but not sufficient as is sometimes erroneously suggested in the literature \(\Leftrightarrow 1\), p. 84.\(^7\). A simple example may illustrate this. Consider an economy with two basics and one non-basic; gross outputs are \(x_1 = 1, x_2 = 1, x_3 = 1\); net outputs are \(y_1 = 1/2, y_2 = 0, y_3 = 1/2\); direct labour inputs are \(l_1 = 1/4, l_2 = 1/4, l_3 = 1/2\).
The input matrix $A$ equals
\[
\begin{bmatrix}
0 & 1/2 & 0 \\
1/2 & 0 & 1/2 \\
0 & 0 & 1/2
\end{bmatrix}
\]

In its normal form $/1/$: $q = 1$, $B = \begin{bmatrix}
0 & 1/2 \\
1/2 & 0 \\
1/2
\end{bmatrix}$, $D_0 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1/2 \end{bmatrix}$.

Labour values are all positive: $1/2$, $1/2$ and $3/2$.

Nevertheless, if the whole net product goes to capitalists ($w = 0$), then we find that $1 + R = 2$ and that the price equations have no positive solution.

In the light of our theorem, this is not surprising as $f(B) = f(D_1) = 1/2$.

**Remark 3:** The proof of the sufficiency part of our theorem shows implicitly that positivity of all prices is ensured even if only one element of $l_B$ is positive and the $n - 1$ other direct labour inputs are zero. We do not insist on this stronger mathematical result, because it does not seem very important from the economic point of view.
APPENDIX: On the reduction of a decomposable input matrix to its normal form.

The notion of the normal form of a decomposable matrix turned out to be indispensable to establish our conditions for all-positive prices in a neo-Ricardian economy with basics and non-basics. Our only purpose in this appendix is to sketch a method for the reduction of a matrix to its normal form. Readers who are not interested in this technical problem, may skip this appendix. As this paper deals primarily with economics, we concentrate on the input matrix of the neo-Ricardian model with basics and non-basics. This does not diminish the generality of our results, but specialists in numerical analysis or graph theory may find our terminology and some other aspects of our exposition rather peculiar.

We consider an input matrix $A$ of the $n$th order. We always assume the existence of $k$ basics and $n - k$ non-basics ($1 \leq k < n$). We represent the structural relations of the input matrix $A$ in a directed graph: every sector is associated with a vertex; there is an arrow (directed edge) from vertex $i$ to vertex $j$ iff commodity $i$ is directly required for the production of commodity $j$ (i.e. iff $a_{ij} > 0$).

There is a path from $i$ to $j$ iff commodity $i$ is directly or indirectly required for the production of commodity $j$; we also say that sector $i$ is connected with sector $j$ or that $j$ is reachable from $i$.

With the graph of the input matrix $A$ is associated an adjacency matrix $M(A)$. Its entries are denoted $m_{ij}(A)$ and defined as follows: $m_{ij}(A) = 1$ iff there is an arrow from $i$ to $j$ $= 0$ otherwise
We also define the **reachability matrix** \( R(A) \) of the graph. Its entries are denoted \( r_{ij}(A) \) and defined as follows:

\[
  r_{ij}(A) = \begin{cases} 
    1 & \text{iff there is a path from } i \text{ to } j \\
    0 & \text{otherwise}
  \end{cases}
\]

Let \( M^g(A) \) denote the \( g \)th power of the matrix \( M(A) \) and let its entries be written \( m_{ij}^{(g)}(A) \). It is easy to show that

\[
m_{ij}^{(g)}(A) > 0 \quad \text{iff there is a path of length } g \quad \text{(i.e. consisting of } g \text{ arrows)} \quad \text{from } i \text{ to } j.
\]

It is obvious that \( r_{ij}(A) = 1 \) iff \( m_{ij}^{(g)}(A) > 0 \) for at least one \( g \in \{1, 2, \ldots, n\} \).

Commodity \( i \) is basic iff \( i \) is connected with all \( j \in \{1, 2, \ldots, n\} \). Hence the rows of \( R(A) \) whose entries are all 1's, indicate the basic sectors. The indices of the \( k \) basic sectors form a set, denoted by \( K \).

We then permute (5) and partition \( A \) into the following form:

\[
\begin{bmatrix}
  B & C \\
  A_0 & D \\
\end{bmatrix}
\]

/8/

Here \( B \) is a \( k \times k \)-matrix associated with the \( k \) basics.

**Lemma 1:** The form /8/ has the following properties:

a) \( B \) is indecomposable

b) the matrix \( A_0 \) is zero

c) the matrix \( C \) is semi-positive

(5) By a permutation of a square matrix, we always mean a permutation of the rows combined with the same permutation of the columns.
Proof:

a) If $B$ is decomposable, there exist two sectors $g$ and $h$ in $K$ (not necessarily distinct), such that there is no path within $K$ from $g$ to $h$ \[ \text{Note 1, p. 109}\]. Nevertheless $g$ is a basic sector. Hence, there is a path from $g$ to $h$ which runs via a non-basic sector $m \notin K$. Via basic sector $h$, this sector $m$ is connected with all sectors. Hence $m$ is basic: contradiction.

b) Suppose $a_{mj} > 0$ for $m \notin K$, $j \in K$. Then $m$ is connected, via basic sector $j$, with all sectors. Hence $m$ is basic: contradiction.

c) If $C$ is zero, there are no basics: contradiction.

If the matrix $D$ is indecomposable, \[ \text{Note 2} \] is the normal form of $A$. If not, further manipulations of $D$ (and simultaneously of $C$, of course) are necessary.

We propose the following procedure for the reduction of the matrix $D$. Assume that $D$ is a $p \times p$ -matrix and that the indices of its sectors form the set $P$.

1. if $p = 1$, $D$ is indecomposable - STOP
2. if $p > 1$, we construct the reachability matrix of $D$, denoted $R(D)$. Its elements are $r_{ij}(D)$:

\[
r_{ij}(D) = 1 \quad \text{iff there is a path from } i \text{ to } j \text{ within } P
\]
(We shall also say that $i$ is connected with $j$ within $P$)

\[
r_{ij}(D) = 0 \quad \text{otherwise}
\]

2.a) If all elements of $R(D)$ differ from zero, $D$ is indecomposable - STOP
If 2.a) is not relevant, we go to 2.b).
2. b) If \( R(D) \), and thus a fortiori \( D \) itself, contains a row with all 0's, we permute this row to the last row of \( D \). (6) We then obtain the form:

\[
\begin{bmatrix}
F & E \\
0 & D_1
\end{bmatrix}
\]

\( D_1 \) is a 1 \times 1 -matrix (with element 0) and thus indecomposable by our definition.

We recommence the whole procedure for the reduction of the matrix \( F \). If 2.b) is not relevant, we go to 2.c).

2. c) If \( R(D) \) contains \( s \) rows with all 1's, where \( 0 < s < p \), \( D \) is permuted to the form:

\[
\begin{bmatrix}
D_1 & E \\
D_0 & F
\end{bmatrix} = \begin{bmatrix}
D_1 & E \\
0 & F
\end{bmatrix}
\]

\( D_1 \) is the \( s \times s \) -matrix associated with the \( s \) rows with all 1's. That \( D_1 \) is indecomposable and that the matrix \( D_0 \) is zero, can be proved in precisely the same way as in the preceding lemma.

We recommence the whole procedure for the reduction of the matrix \( F \).
If 2.c) is not relevant, we go to 2.d).

2. d) Every row of \( R(D) \) contains at least one 0 and one 1. /9/
The only two possibilities that remain are 2.d)1. and 2.d)2.

2.d)1. There exists a sector \( j_0 \) in \( P \) such that \( r_{ij_0}(D) = 0 \) for all \( i \) in \( P \), \( i \neq j_0 \). /10/

(6) This row corresponds with a non-basic good that is never used as a means of production.
We then permute D so that the $j_0$th row becomes the first and we obtain the form:

$$
\begin{bmatrix}
D_1 & E \\
0 & F
\end{bmatrix}
$$

$D_1$ is a $1 \times 1$ matrix and thus indecomposable.

We recommence the whole procedure for the reduction of the matrix F. If 2.d)1. is not relevant, 2.d)2. must hold.

2.d)2. Every sector $j_0$ in P is such that $r_{ij_0}(D) = 1$ for at least one $i$ in $P$, $i \neq j_0$  

We consider the $z$ rows ($1 \leq z \leq p$) of $R(D)$ with the maximal (7) number of 1's. This maximal number is denoted by $d$ $(0 < d < p)$. The indices of these $z$ rows form the set $Z$. We choose an arbitrary element of $Z$:

the sector $h_0$.

We form the following subsets of $P$.

$$
V_0 = \{ h \in P \ | \ r_{hh_0}(D) = 0 \} \\
V_1 = \{ h \in P \ | \ r_{hh_0}(D) = 1 \}
$$

In lemma 2 at the end of this appendix, it is proved that both sets are non-empty. Hence they form a partition of $P$. This partition is the base for the permutation of the matrix $D$ to the form:

$$
\begin{bmatrix}
D_{V_1} & E \\
Q & D_{V_0}
\end{bmatrix} = \begin{bmatrix}
D_1 & E \\
0 & F
\end{bmatrix}
$$

(7) I decided to pick out these rows after a suggestion of C. Van Nuffelen. Of course, he is not responsible for eventual misuse of his suggestion in this context.
The square matrix $D_{V_1}$ is associated with the elements of $V_1$; $D_{V_1}$ is also denoted by $D_1$, $D_{V_0}$ by $F$.

In lemma 3 we shall prove that the matrix $D_{V_1} = D_1$ is indecomposable and that the matrix $Q$ really equals zero.

We recommence the whole procedure for the reduction of the matrix $F$.

As 1, 2.a), 2.b), 2.c), 2.d)1. and 2.d)2. together exhaust all possible cases and as in every case an effective partition was performed, our procedure must lead up to the normal form in a finite number of steps. To complete our story, we still have to prove two lemma's.

Lemma 2: The sets $V_0$ and $V_1$, as defined for case 2.d)2. in relation /12/, are non-empty. (8)

Proof: That $V_1$ is non-empty, is trivial from /11/ and /12/.

Suppose that $V_0$ is empty and thus $V_1 = P$. Hence, every sector in $P$ is connected with $h_0$, so that $h_0$ is connected with itself. As $h_0$ belongs to $Z$, it is connected with $d$ sectors ($0 < d < p$) and thus not connected with at least one sector $h_1 \in P$. From /11/, $h_1$ is reachable from at least one element $h_2 \in P$. As $V_1 = P$, $h_2$ is connected with $h_0$ and thus also with the $d$ sectors reachable from $h_0$. Furthermore $h_2$ is connected with $h_1$ and $h_0$ is not. Hence $h_2$ is connected with at least $d + 1$ sectors: contradiction.

(8) In the proof of lemma's 2 and 3, "connected" and "reachable" always stand for "connected within $P$" and "reachable within $P$", unless the contrary is stated explicitly.
Lemma 3: The form /13/ has the following properties:
   a) $D_{V_1}$ is indecomposable
   b) the matrix $Q$ is zero

Proof: We first prove the following proposition:
   Every sector $t \in Z$ is connected with itself /14/
   From /11/, there is a sector $g \neq t$ in $P$ which is connected
   with $t$ and via $t$ with at least as many sectors as $t$.
   If $t$ is not connected with itself, $g$ is connected with
   more sectors than $t$: contradiction.

   a) Take an arbitrary sector $h_1$ in $V_1$. This sector is con-
   nected with $h_0$ and via $h_0$ with at least as many sectors
   as $h_0$. Hence $h_1$ must also belong to $Z$. Thus, we may
   conclude that $V_1 \subset Z$ and also, from /14/, that $h_1$ is
   connected with itself.

   We claim that this sector $h_1$ is connected with $h_0$ within
   $V_1$ /15/
   Suppose /15/ is false. Then the path from $h_1$ to $h_0$ runs
   via a sector $h_2 \notin V_1$. Hence $h_2$ is connected with $h_0$,
   thus $h_2 \notin V_1$: contradiction.

   We claim that the sector $h_1$ is connected with itself
   within $V_1$ /16/
   If /16/ is false, the path from $h_1$ to $h_1$ runs via a sector
   $h_3 \notin V_1$. Hence $h_3$ is connected with $h_1$ and via $h_1$ with
   $h_0$: contradiction.

   /15/, /16/ and a lemma of Nikaido /4/, p. 109 imply
   that the indecomposability of $D_{V_1}$ is established as
   soon as we can prove the following proposition:
The sector $h_0$ is connected within $V_1$ with all sectors of $V_1$. We prove by contradiction. If $h_0$ is not connected within $V_1$ with a sector $h_4 \in V_1$, then there are two possibilities:

1. A path from $h_0$ to $h_4$ runs via a sector $h_5 \notin V_1$. This implies that $h_5$ is connected with $h_4 \in V_1$ and via $h_4$ with $h_0$. Hence $h_5 \in V_1$: contradiction.

2. The sector $h_0$ is not connected with $h_4$. $V_1 \cap Z$ and imply that $h_4$ is connected with itself and thus $h_4 \notin h_0$. As $h_4 \in V_1$, it is connected with $h_0$ and thus also with the d sectors reachable from $h_0$. Hence $h_4$ is connected with at least $d + 1$ sectors: contradiction.

b) The last part of lemma 3 is proved as follows: take an arbitrary element $a_{mj}$ of the matrix $Q (m \in V_0, j \in V_1)$; $a_{mj} > 0$ implies that $m$ is connected with $j$, and via $j$ with $h_0$. Hence $m \in V_1$: contradiction.
REFERENCES


2 BURMEISTER E., "A Comment on 'This Age of Leontief ... and Who ?' "*, Journal of Economic Literature*, June 1975, pp. 454 - 457.

