SESO STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

# THE CHARACTERIZATION OF PARETO-EFFICIENT POINTS IN NONLINEAR VECTOR-MAXIMUM PROBLEMS

W. PAUWELS

werknota 7547

augustus 1975

Universitaire Faculteiten St.-Ignatius
Prinsstraat 13 - 2000 Antwerpen

D/1975/1169/15

The purpose of this paper is to study properties of Pareto-efficient points in non-linear vector-maximum problems. The importance of Pareto-efficiency for economic theory is well-known. Many economic situations do not allow an evaluation of alternative decisions by a scalar-valued criterion, but require the simultaneous application of several optimality criteria which cannot be given an *a priori* weighting. For such problems, Pareto-efficiency is a commonly used efficiency concept.

However, in addition to Pareto-efficiency, there exist other, related, efficiency concepts such as proper Pareto-efficiency, undominated points, and absolutely cooperative points. The purpose of this paper is to study the interrelationships among these various efficiency concepts. This will be done in two ways. First, conditions will be derived under which particular efficient points can be obtained as solutions of appropriately defined scalar-valued problems. Secondly, the interrelationships can also be made clear in terms of the consistency or inconsistency of certain linear systems of equalities or inequalities. At least for relatively simple economic applications, both approaches lead to easily applicable criteria for determining whether or not a particular point is, in some sense, efficient.

The paper is organized as follows. Section 1 fixes the notation and gives various definitions of efficient points. In the next three sections, three efficiency concepts are studied: proper Pareto-efficiency, undominated points, and absolutely cooperative points. Section 5 summarizes the main results. Finally, some examples and economic applications are given in section 6.

## I. EFFICIENT POINTS: DEFINITIONS AND NOTATION

Let the criterion function of the vector-maximum problem be denoted by f which is a function from $E^m$ into $E^n$, $f: E^m \to E^n$. The components of f will be written as $f^1, \dots, f^n$.
If f is differentiable in $E^m$, $\nabla f^i(x)$ will denote the gradient of $f^i$ evaluated at $x \varepsilon E^m$. Let $\partial f(x)/\partial x$ denote the n x m matrix, the rows of which are given by $\nabla f^1(x), \dots, \nabla f^n(x)$.

Let the constraints be given by a function g from $E^m$ into $E^k$, $g: E^m \to E^k$, with components $g^1, \dots, g^k$. The feasible set X is then defined as

$$X = \{x \varepsilon E^m \mid g(x) \geqq 0\}^{(1)}$$

If g is differentiable in $E^m$, $\nabla g^i(x)$ will denote the gradient of $g^i$ evaluated at x.
The k x m matrix $\partial g(x)/\partial x$ has the vectors $\nabla g^i(x)$ as its rows.

We can now define four concepts of efficiency.

<u>Definition 1.</u> A vector $f(\hat{x}) \varepsilon f(X)$ is <u>Pareto-efficient</u> (PE) iff there does not exist a vector $f(x) \varepsilon f(X)$ such that $f(x) \underline{>} f(\hat{x})$

<u>Definition 2.</u> A vector $f(\hat{x}) \varepsilon f(X)$ is <u>properly Pareto-efficient</u> (PPE) iff there exists a vector $p \varepsilon E^n$, $p > 0$, such that $pf(\hat{x}) \geqq pf(x)$ for all $x \varepsilon X$.

---

(1) We use the following ordering relations in $E^n$. A vector $y = (y_1, \dots, y_n) \varepsilon E^n$ is non-negative, denoted by $y \geqq 0$, iff $y_i \geqq 0$ for all i. A vector y is semipositive, denoted by $y \underline{>} 0$, iff $y \geqq 0$ and $y \neq 0$. A vector y is positive, denoted by $y > 0$, iff $y_i > 0$ for all i.

Definition 3. A vector $f(\hat{x}) \varepsilon f(X)$ is undominated (UD) iff there does not exist a vector $f(x) \varepsilon f(X)$ such that $f(x) > f(\hat{x})$

Definition 4. A vector $f(\hat{x}) \varepsilon f(X)$ is absolutely cooperative (AC) iff, for all $x \varepsilon X$, $f(x) \leqq f(\hat{x})$.

Definition 1 is the usual definition of Pareto-efficiency. The notion of proper Pareto-efficiency was first introduced in /4/. Definition 3 is related to the concept of dominance as used in cooperative n-person game theory. Absolutely cooperative points have been studied in /7/. It is clear that any AC point is also a noncooperative Nash equilibrium. Figures 1 and 2 illustrate the various definitions for $n = 2$. In figure 1, point A is UD, but not PE. Points B and C are PE, but not PPE. Points D, E, F are PPE. Point G in figure 2 is PE, PPE, UD and AC.

---

Figure 1 and Figure 2: see p. 26.

---

## 2. PROPERLY PARETO-EFFICIENT POINTS

First note that PPE points are defined in terms of solutions of scalar-valued objective functions, viz.

$$\max_{x \varepsilon X} pf(x) \qquad (1)$$

for $p > 0$. The following theorem states that the set of PPE points is a subset of the set of PE points.

<u>THEOREM 1. If $f(\hat{x}) \in f(X)$ is PPE, then $f(\hat{x})$ is PE.</u>

The proof is easy by contradiction. See, e.g. /1/ p. 109. As is illustrated by points B and C in figure 1, the reverse of this theorem need not be true: a point $f(\hat{x})$ can be PE without being PPE. This can occur even when f is strictly concave as is illustrated by example 1 in section 6.

Let us now look at the Kuhn-Tucker conditions of problem (1). If we let

$$A(\hat{x}) = \left\{ i \in \{1, \ldots, k\} \mid g^i(\hat{x}) = 0 \right\},$$

these conditions can be written as (see e.g. /8/p. 40).

$$\left.\begin{aligned} & p \frac{\partial f(\hat{x})}{\partial x} + \sum_{i \in A(\hat{x})} \lambda_i \nabla g^i(\hat{x}) = 0 \\ & \lambda_i \geqq 0, \; i \in A(\hat{x}) \end{aligned}\right\} \qquad (2)$$

Applying Tucker's theorem of the alternative (see, e.g., /5/ p. 29), the consistency of (2) for $p > 0$ and $\lambda_i \geqq 0$, $i \in A(\hat{x})$, is equivalent to the inconsistency of

$$\frac{\partial f(\hat{x})}{\partial x} d \geq 0$$

$$\nabla g^i(\hat{x}) d \geqq 0, \quad i \in A(\hat{x})$$

The following theorem then easily follows. Let KTCQ stand for the Kuhn-Tucker constraint qualification.

THEOREM 2. Let f and g be differentiable in $E^m$

(a) Let the KTCQ hold at $\hat{x}$. If $f(\hat{x})$ is PPE, then the system

$$\left.\begin{aligned} & p\,\frac{\partial f(\hat{x})}{\partial x} + \lambda\,\frac{\partial g(\hat{x})}{\partial x} = 0 \\ & \lambda g(\hat{x}) = 0 \\ & p > 0, \quad \lambda \geq 0 \end{aligned}\right\} \quad (3)$$

must be consistent, or, equivalently, the system

$$\left.\begin{aligned} & \frac{\partial f(\hat{x})}{\partial x}\,d \geq 0 \\ & \nabla g^i(\hat{x})\,d \geq 0, \; i \varepsilon A(\hat{x}) \end{aligned}\right\} \quad (4)$$

must be inconsistent.

(b) Let X be convex, and let f be concave in X. If, at a point $\hat{x} \varepsilon X$, (3) is consistent, or, equivalently, (4) is inconsistent, then $f(\hat{x})$ is PPE.

Proof

(a) If the KTCQ is satisfied at $\hat{x}$, the Kuhn-Tucker conditions (2) are necessary for a solution of (1). As $\lambda_i = 0$ for $i \notin A(\hat{x})$, conditions (3) are the same as (2). As we have seen, the inconsistency of (4) is equivalent to the consistency of (2).

(b) If f is concave over the convexset X, the Kuhn-Tucker conditions are sufficient for a maximum of pf(x) over X. Q.E.D.

If $\hat{x}$ is interior to X, the following corollary immediately follows.

COROLLARY: Let $\hat{x}$ belong to the interior of X, and let f be differentiable in $E^m$

(a) If $f(\hat{x})$ is PPE, then

$$\frac{\partial f(\hat{x})}{\partial x} d \geq 0 \qquad (5)$$

must be inconsistent, or, equivalently,

$$p \frac{\partial f(\hat{x})}{\partial x} = 0, \quad p > 0 \qquad (6)$$

must be consistent.

(b) If $f(\hat{x})$ is PPE, the rank of $\partial f(\hat{x})/\partial x$ must be smaller than n

(c) If $\partial f(\hat{x})/\partial x$ has a nonzero column all of whose nonzero elements have the same sign, then $f(\hat{x})$ is not PPE.

(d) Let X be convex, and let f be concave in X. If (5) is inconsistent, or, equivalently, if (6) is consistent, then $f(\hat{x})$ is PPE.

## 3. UNDOMINATED POINTS

It is clear from the definitions in section 1 that any PE point must also be UD. The reverse is not true: an UD point need not be PE, as is illustrated by point A in figure 1. The following theorem gives a sufficient condition under which UD points are also PE.

THEOREM 3. Let X be convex, and let f be strictly quasi-concave in X. If then $f(\hat{x}) \varepsilon f(X)$ is UD, $f(\hat{x})$ must also be PE.

Proof

Suppose the implication is not true. Then there exists a point $f(\hat{x}) \varepsilon f(X)$ which is UD, but not PE. If $f(\hat{x})$ is not PE, there must exist a vector $f(x) \varepsilon f(X)$ such that $f(\bar{x}) \geq f(\hat{x})$. As $\bar{x} \neq \hat{x}$ and f is strictly quasi-concave, we have, for all $\lambda$, $0<\lambda<1$,

$$f\{\lambda \bar{x} + (1-\lambda)\ \hat{x}\} > f(\hat{x})$$

As $\lambda \bar{x} + (1-\lambda)\ \hat{x} \ \varepsilon\ X$, this contradicts the assumption that $f(\hat{x})$ is UD. Q.E.D.

COROLLARY. Let X be convex, and let f be strictly quasi-concave in X. Then $f(\hat{x}) \varepsilon f(X)$ is UD iff $f(\hat{x})$ is PE.

It is clear that, in the above theorem and corollary, one can replace strict quasi-concavity by strict concavity. As will be illustrated in example 2 in section 6, theorem 3 is no longer true if f is merely concave.

Under what conditions can UD points be generated as solutions of scalar-valued problems of the type used in (1) ? The answer is given by the following two theorems:

THEOREM 4. Let $p \geq 0$. If $\hat{x}$ maximises $pf(x)$ over X, then $f(\hat{x})$ is UD.

Proof

Suppose the implication is false. Then there must exist a vector $\hat{x} \varepsilon X$ such that $pf(\hat{x}) \geqq pf(x)$, for all $x \varepsilon X$, while $f(\hat{x})$ is not UD.

If $f(\hat{x})$ is not UD, there must exist a vector $\tilde{x} \varepsilon X$ such that $f(\tilde{x}) > f(\hat{x})$. As $p \geq 0$, it follows that $pf(\tilde{x}) > pf(\hat{x})$. This contradicts the assumption that $\hat{x}$ maximises $pf(x)$ over X. Q.E.D.

THEOREM 5. Let X be convex, and let f be concave in X. If $f(\hat{x}) \varepsilon f(X)$ is UD, there must exist a vector $p \geq 0$ such that $\hat{x}$ maximises $pf(x)$ over X.

This theorem is called the generalized Gordan Theorem, and is proved, e.g., in /5/ p. 65. See also /3/ p. 217.

COROLLARY. Let X be convex, and let f be strictly concave in X. Then $f(\hat{x}) \varepsilon f(X)$ is PE iff there exists a vector $p \geq 0$ such that $\hat{x}$ maximises $pf(x)$ over X.

Proof

If $f(\hat{x})$ is PE, it must be UD. By theorem 5 there must exist a vector $p \geq 0$ such that $\hat{x}$ maximises $pf(x)$ over X.
Conversely, if, for $p \geq 0$, $\hat{x}$ maximises $pf(x)$ over X, then by theorem 4 and 3, $f(\hat{x})$ must be PE. Q.E.D.

The scalar-valued problem

$$\max_{x \varepsilon X} pf(x), \quad p \geq 0 \qquad (7)$$

can also be used to locate PE points without requiring that f be strictly concave.

THEOREM 6. Let $p \geq 0$. If $\hat{x}$ is the unique maximum of $pf(x)$ over X, then $f(\hat{x})$ is PE.

Proof

Suppose the implication is false. Then

$$pf(\hat{x}) > pf(x) \text{ for all } x \varepsilon X,\ x \neq \hat{x} \qquad (8)$$

While there must exist a vector $\tilde{x} \varepsilon X$ such that $f(\tilde{x}) \geq f(\hat{x})$. As $p > 0$, it follows that $pf(\tilde{x}) \geqq pf(\hat{x})$ which contradicts (8). Q.E.D.

Let us now consider the Kuhn-Tucker conditions for problem (7). These are given by

$$\left.\begin{aligned} & p \frac{\partial f(\hat{x})}{\partial x} + \sum_{i \varepsilon A(\hat{x})} \lambda_i \nabla g^i(\hat{x}) = 0 \\ & \lambda_i \geqq 0, \quad i \varepsilon A(\hat{x}) \end{aligned}\right\} \qquad (9)$$

Applying Motzkin's theorem of the alternative (see, e.g., /5/ p. 28), the consistency of (9) for $p \geq 0$ and $\lambda_i \geqq 0$, $i \varepsilon A(\hat{x})$, is equivalent to the inconsistency of

$$\frac{\partial f(\hat{x})}{\partial x} d > 0$$

$$\nabla g^i(\hat{x}) \, d \geqq 0, \quad i \varepsilon A(\hat{x})$$

The following theorem then easily follows.

THEOREM 7. Let X be convex, and let f be concave in X. Let f and g be differentiable in $E^m$.

(a) Let the KTCQ hold at $\hat{x}$. If $f(\hat{x}) \varepsilon f(X)$ is UD, then the system

$$\left.\begin{aligned} & p \frac{\partial f(\hat{x})}{\partial x} + \frac{\partial g(\hat{x})}{\partial x} = 0 \\ & \lambda g(\hat{x}) = 0 \\ & p \geq 0, \quad \lambda \geqq 0 \end{aligned}\right\} \qquad (10)$$

must be consistent, or, equivalently, the system

$$\left.\begin{array}{l}\frac{\partial f(\hat{x})}{\partial x}\, d > 0 \\ \\ \nabla g^i(\hat{x})d \geq 0,\ i\varepsilon A(\hat{x})\end{array}\right\} \qquad (11)$$

must be inconsistent.

(b) If, at a point $\hat{x}\varepsilon X$, (10) is consistent, or, equivalently, (11) is inconsistent, then $f(\hat{x})$ is UD.

Proof

(a) If $f(\hat{x})\varepsilon f(X)$ is UD, and if f is concave over the convexset X, then by theorem 5 there must exist a vector $p \geq 0$ such that $\hat{x}$ maximises $pf(x)$ over X. If then the KTCQ holds at $\hat{x}$, the Kuhn-Tucker conditions (10) are necessarily satisfied. As we have seen, the consistency of (10) is equivalent to the inconsistency of (11).

(b) If f is concave in the convexset X, the Kuhn-Tucker conditions (10) are sufficient for a maximum of $pf(x)$ over X. By theorem 4 this implies that $f(\hat{x})$ is UD. Q.E.D.

COROLLARY 1. Let $\hat{x}$ belong to the interior of X, let X be convex, and let f be differentiable and concave in X.

(a) $f(\hat{x})$ is UD iff

$$\frac{\partial f(\hat{x})}{\partial x}\, d > 0 \qquad (12)$$

is inconsistent, or, equivalently, iff

$$p\,\frac{\partial f(\hat{x})}{\partial x} = 0 \qquad p \geq 0 \qquad (13)$$

is consistent.

(b) If $f(\hat{x})$ is UD, the rank of $\partial f(\hat{x})/\partial x$ must be smaller than n.

(c) If $\partial f(\hat{x})/\partial x$ contains a row all of whose elements are zero, then $f(\hat{x})$ is UD.

(d) If $\partial f(\hat{x})/\partial x$ contains a column of nonzero elements, all of the same sign, then $f(\hat{x})$ is not UD.

The proof follows immediately from theorem 7.

COROLLARY 2. Let $\hat{x}$ belong to the interior of X, let X be convex, and let f be differentiable and strictly concave in X. Then $f(\hat{x})$ is PE iff (12) is inconsistent, or, equivalently, iff (13) is consistent.

This follows from the corollary of theorem 3, and corollary 1, (a) of theorem 7. (1)

As any PE point is also UD, the consistency of (10), or the inconsistency of (11) is also necessary for PE points, provided f is concave. In fact, these conditions are also necessary for PE points for f not concave.

THEOREM 8. Let f and g be differentiable in $E^m$, and let the KTCQ be satisfied at $\hat{x}$. If $f(\hat{x})\varepsilon f(X)$ is PE, then (10) must be consistent, or, equivalently, (11) must be inconsistent.

---

(1) In /6/ p. 4 it is contended that the inconsistency of $(\partial f(\hat{x})/\partial x)\ d \geq 0$ is necessary for an interior PE point. Our results show that this is not so. In fact, in example 2 of section 6 we will locate PE points where $(\partial f(\hat{x})/\partial x)\ d \geq 0$ is consistent.

Proof

Suppose the implication does not hold. Then $f(\hat{x}) \varepsilon f(X)$ is PE, while, by the consistency of (11), there must exist a feasible direction $d \varepsilon E^m$ such that the value of all components of f is increased. This contradicts the assumption that $f(\hat{x})$ is PE.Q.E.D.

COROLLARY. Let $\hat{x}$ belong to the interior of X, and let f be differentiable in $E^m$.

(a) If $f(\hat{x})$ is PE, then (12) must be inconsistent, or, equivalently, (13) must be consistent.

(b) The results (b) and (d) of corollary 1 of theorem 7 still hold when UD is replaced by PE.

## 4. ABSOLUTELY COOPERATIVE POINTS

The following theorem offers an alternative definition of AC points.

THEOREM 9. A point $f(\hat{x}) \varepsilon f(X)$ is AC iff $\hat{x}$ maximises $pf(x)$ over X, for all $p \geq 0$.

Proof

If $f(\hat{x}) \varepsilon f(X)$ is AC, then $f(\hat{x}) \geqq f(x)$ for all $x \varepsilon X$. Take then any $p \geq 0$. It follows that $pf(\hat{x}) \geqq pf(x)$ for all $x \varepsilon X$. Conversely, let $\hat{x}$ maximize $pf(x)$ over X for all $p \geq 0$. Take then p as the i-th coordinate vector. It follows that $f^i(\hat{x}) \geqq f^i(x)$ for all $x \varepsilon X$ and all i. Q.E.D.

It also follows that any AC point must also be PPE, PE and UD.

It is clear from the definition of AC points that such points can be obtained by solving n scalar problems:

$$\max_{x \in X} f^k(x), \; k: 1, \ldots, n.$$

Application of the Kuhn-Tucker conditions and of Farkas'theorem of the alternative, then gives the following theorem:

THEOREM 10. Let f and g be differentiable in $E^m$.

(a) Let the KTCQ be satisfied in $\hat{x}$. If $f(\hat{x}) \varepsilon f(X)$ is AC, then

$$\left. \begin{aligned} &\nabla f^k(\hat{x}) + \lambda^k \frac{\partial g(\hat{x})}{\partial x} = 0, \\ &\lambda^k g(\hat{x}) = 0, \; \lambda^k \geqq 0, \; k: 1, \ldots, n. \end{aligned} \right\} \quad (14)$$

must be consistent, or, equivalently,

$$\left. \begin{aligned} &\nabla f^k(\hat{x}) \, d > 0, \quad k: 1, \ldots, n \\ &\nabla g^i(\hat{x}) \, d \geqq 0, \quad i \varepsilon A(\hat{x}) \end{aligned} \right\} \quad (15)$$

must be inconsistent.

(b) Let X be convex and f concave in X. If, for $\hat{x} \varepsilon X$, (14) is consistent, or, equivalently, (15) inconsistent, then $f(\hat{x})$ is AC.

COROLLARY. Let $\hat{x}$ belong to the interior of X, and let f be differentiable in $E^m$.

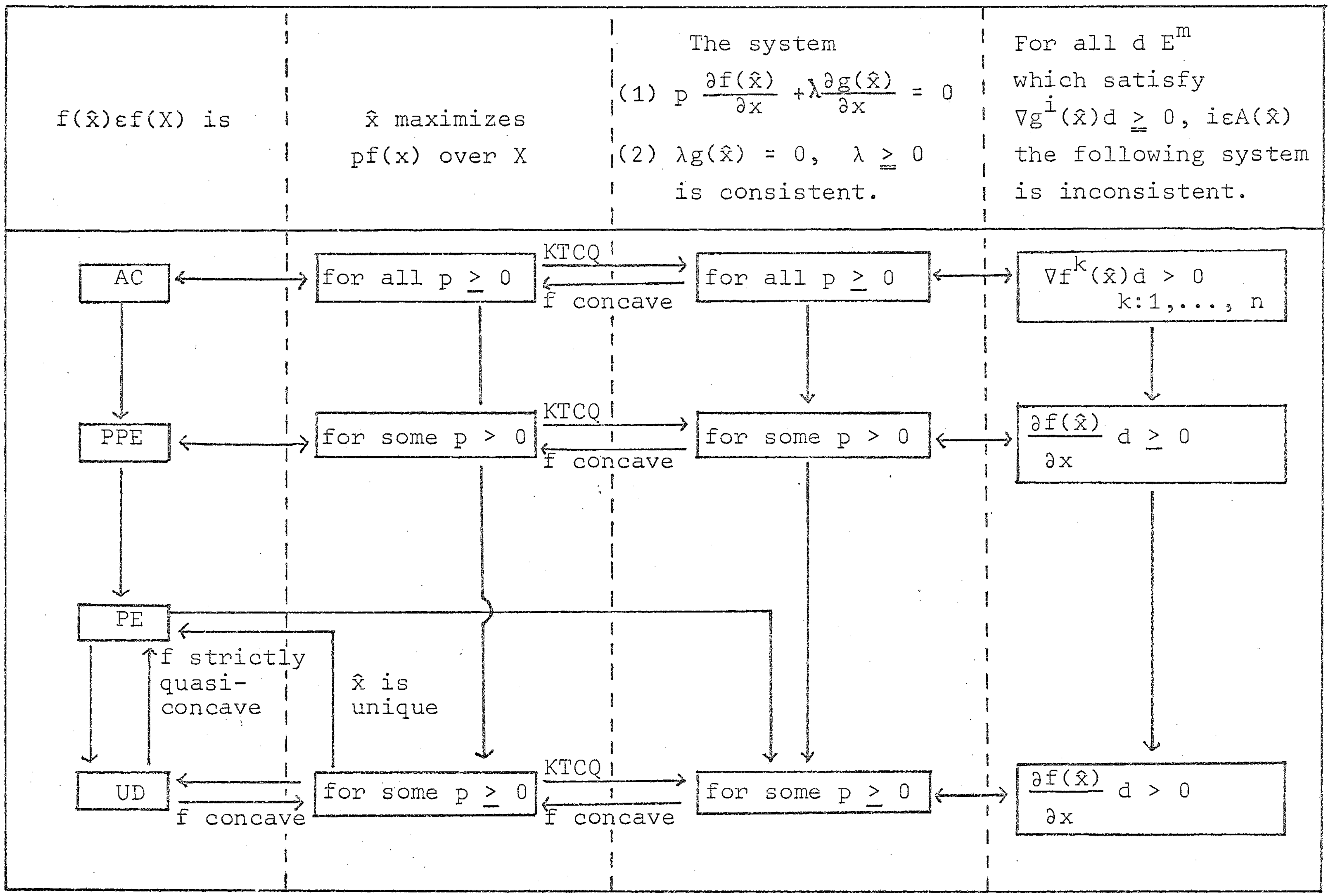
(a) If $f(\hat{x})$ is AC, then all elements of $\partial f(\hat{x})/\partial x$ must be zero.

(b) If X is convex and f is concave in X, and if all elements of $\partial f(\hat{x})/\partial x$ are zero, then $f(\hat{x})$ is AC.

## 5. SUMMARY OF THE RESULTS

The scheme on page 15 gives a summary of the results of the previous sections. It is clear that our results do not allow a complete characterization of PE points in terms of a single set of necessary and sufficient conditions.

The properties characterizing PPE points are sufficient for PE points, while those of UD points are necessary. Only for the special case when f is strictly concave does there exist a set of conditions which are both necessary and sufficient for PE points. In case f is not strictly concave, difficulties may still arise for points which are UD but not PPE. This will be illustrated in example 2 in section 6.



f(x̂)εf(X) is
x̂ maximizes pf(x) over X
The system
(1) p ∂f(x̂)/∂x + λ∂g(x̂)/∂x = 0
(2) λg(x̂) = 0, λ ≥ 0
is consistent.
For all d E^m
which satisfy
∇g^i(x̂)d ≥ 0, iεA(x̂)
the following system
is inconsistent.
AC
PPE
PE
UD
for all p ≥ 0
for some p > 0
for some p ≥ 0
KTCQ
f concave
∇f^k(x̂)d > 0
k:1,..., n
∂f(x̂)/∂x d ≥ 0
∂f(x̂)/∂x d > 0
f strictly quasi-concave
x̂ is unique

## 6. SOME EXAMPLES AND ECONOMIC APPLICATIONS

### (a) Example 1

Let $X = E^1$, and let f be a function from $E^1$ into $E^2$, defined by

$$f^1(x) = x - x^2$$
$$f^2(x) = - x^2$$

where $x \varepsilon E^1$. Note that both functions are strictly concave, so that PE points and UD points coincide. The set $f(E^1)$ is given by the curve on figure 3.

---

Figure 3 : see p. 26.

---

We have

$$\frac{\partial f(x)}{\partial x} = \begin{pmatrix} 1 - 2x \\ - 2x \end{pmatrix}$$

It is easy to see that

$$\begin{pmatrix} 1 - 2x \\ - 2x \end{pmatrix} d \geq 0$$

is inconsistent iff $0 < x < {}^1/_2$ . Applying (a) and (d) of the corollary of theorem 2, we conclude that the set of PPE points is given by all $f(x)$, $0 < x < {}^1/_2$.
On figure 3 these points are given by the segment between A and B, A and B themselves not included.

On the other hand, it is also easy to see that

$$\begin{pmatrix} 1 - 2x \\ - 2x \end{pmatrix} d > 0,$$

is inconsistent iff $0 \leq x \leq {}^1/_2$. Applying corollary 2 of theorem 7, we conclude that the set of PE points is given by all $f(x)$, $0 \leq x \leq {}^1/_2$. On figure 3 these points are given by the segment AB, now including the points A and B. Points A and B are the only PE points which are not PPE.

(b) Example 2

Let $X = E^2$, and let $f$ be a function from $E^2$ into $E^1$ defined by

$$f^1(x_1,x_2) = -(x_2 - 1)^2$$

$$f^2(x_1,x_2) = -x_2 - (x_1 - 1)^2$$

Note that both function are concave in $E^2$, but not strictly concave. The set $f(E^2)$ is given by the shaded area in figure 4.

---

Figure 4 : see p. 26.

---

Examining the system

$$\begin{pmatrix} 0 & -2(x_2 - 1) \\ -2(x_1 - 1) & -1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We see that it is inconsistent iff $x_1 = 1$ and $x_2 < 1$. Applying (a) and (d) of the corollary of theorem 2, we conclude that the set of all PPE points is given by all $(f^1(x_1,x_2), f^2(x_1,x_2))$ for which $x_1 = 1$ and $x_2 < 1$. This set is given by the part of the boundary of $f(E^2)$ to the north-west of the point $(0,-1)$. The point $(0,-1)$ itself is not PPE.

Examining the system

$$\begin{pmatrix} 0 & -2(x_2 - 1) \\ -2(x_1 - 1) & -1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we see that it is inconsistent iff $x_2 = 1$, or $x_1 = 1$ and $x_2 < 1$.

For these values of $(x_1, x_2)$ we obtain the set of UD points. This set is given by the whole boundary of $f(E^2)$, including the straight segment below the point $(0,-1)$.

Our results of the previous sections do not allow us to determine all PE points. Points $f(x_1,x_2)$ where $x_1 = 1$ and $x_2 < 1$ are PE, but we cannot say whether points where $x_2 = 1$ are PE or not. These latter points are UD, but not PPE. Inspection of figure 4, however, reveals that only one of these points, viz., $(0,-1) = f(1,1)$, is PE.

(c) Example 3: Cournot-Nash Equilibria in Oligopolistic Markets

Consider the well-known Cournot-Nash equilibrium of oligopoly theory. Let there be n firms producing a homogeneous product. Let $x_i$ be the output of the i-th firm, and let $C_i(x_i)$ be its cost function. If $p = F(x_1 + \dots + x_n)$ is the market demand function, then profits of firm i are given by

$$f^i(x) = x_i F(x_1 + \dots + x_n) - C_i(x_i), \; i: 1, \dots, n.$$

A Cournot-Nash equilibrium $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ is then characterized by the property

$$\frac{\partial f^i(\hat{x})}{\partial x_i} = p - \hat{x}_i F'(\hat{x}_1 + \dots + \hat{x}_n) - C_i'(\hat{x}_i) = 0, \; i:1,\dots, n$$

letting $f = (f^1, \ldots, f^n)$, the n x n matrix $\partial f(\hat{x})/\partial x$ is given by

$$\frac{\partial f(\hat{x})}{\partial x} = \begin{pmatrix} 0 & \hat{x}_1 F' & \ldots & \hat{x}_1 F' \\ \hat{x}_2 F' & 0 & \ldots & \hat{x}_2 F' \\ \ldots & \ldots & \ldots & \ldots \\ \hat{x}_n F' & \hat{x}_n F' & \ldots & 0 \end{pmatrix}$$

Provided all $\hat{x}_i$ are positive, and $F'$ is negative, it is easy to see that the system $(\partial f(\hat{x})/\partial x)\, d > 0$ must be consistent. If f is concave in the nonnegative orthant of $E^n$, we can conclude from (a), corollary 1 of theorem 7, that $f(\hat{x})$ is not UD. Hence, it is certainly not PE.

From the definition of PPE points, it is clear that the set of PPE points can be obtained by solving the class of problems (1)

$$\max_x \sum_{i=1}^{n} \varepsilon_i \, f^i(x), \varepsilon_i > 0$$

If $\varepsilon_i = 1$ for all i, we obtain the classical collusion solution.

Similarly, the set of UD points can be obtained by solving the class of problems

$$\max_x \sum_{i=1}^{n} \varepsilon_i \, f^i(x), \; \varepsilon_i \geq 0$$

If all $f^i$ are strictly concave, we also get the set of PE points.

---

(1) We are using the symbols $(\varepsilon_1, \ldots, \varepsilon_n)$ instead of $(p_1 \ldots p_n)$ as used in (1) in order to avoid confusion with the symbol p denoting the price of the product. Of course, we can always put $\sum_{i=1}^{n} \varepsilon_i = 1$.

(d) Example 4: Externalities in production

Consider two firms in a competitive industry selling quantities $x_1$ and $x_2$ of a certain product at a given price p. Let their cost functions be given by $C_1(x_1,x_2)$ and $C_2(x_1,x_2)$. Profits are then given by

$$f^1(x_1,x_2) = px_1 - C_1(x_1,x_2)$$

$$f^2(x_1,x_2) = px_2 - C_2(x_1,x_2)$$

We will assume that $f^1$ and $f^2$ are concave.

Consider then again a Cournot-Nash equilibrium $\hat{x} = (\hat{x}_1,\hat{x}_2)$ characterized by

$$\frac{\partial f^1(\hat{x}_1,\hat{x}_2)}{\partial x_1} = p - \frac{\partial C_1(\hat{x}_1,\hat{x}_2)}{\partial x_1} = 0$$

$$\frac{\partial f^2(\hat{x}_1,\hat{x}_2)}{\partial x_2} = p - \frac{\partial C_2(\hat{x}_1,\hat{x}_2)}{\partial x_2} = 0$$

This gives

$$\frac{\partial f(\hat{x})}{\partial x} = \begin{pmatrix} 0 & -\dfrac{\partial C_1(\hat{x}_1,\hat{x}_2)}{\partial x_2} \\ -\dfrac{\partial C_2(\hat{x}_1,\hat{x}_2)}{\partial x_1} & 0 \end{pmatrix}$$

It follows from (a) of corollary 1 of theorem 7 that $f(\hat{x})$ is not UD, provided $\dfrac{\partial C_1(\hat{x}_1,\hat{x}_2)}{\partial x_2}$ and $\dfrac{\partial C_2(\hat{x}_1,\hat{x}_2)}{\partial x_1}$ are different from zero.

The sign and magnitude of these derivatives does not affect this conclusion. As $f(\hat{x})$ is not UD, it is also not PE.

It is well-known that the introduction of taxes can produce a PE point. Let tax rates be given by $t_1$ and $t_2$. Profits then become

$$\pi^1(x_1,x_2) = (p - t_1)x_1 - C_1(x_1,x_2)$$

$$\pi^2(x_1,x_2) = (p - t_2)x_2 - C_2(x_1,x_2)$$

Let $\bar{x} = (\bar{x}_1,\bar{x}_2)$ be the solution of

$$p - \frac{\partial C_1(x_1,x_2)}{\partial x_1} - \frac{\partial C_2(x_1,x_2)}{\partial x_1} = 0$$

$$p - \frac{\partial C_1(x_1,x_2)}{\partial x_2} - \frac{\partial C_2(x_1,x_2)}{\partial x_2} = 0$$

and put

$$t_1 = \frac{\partial C_2(\bar{x}_1,\bar{x}_2)}{\partial x_1} \quad \text{and} \quad t_2 = \frac{\partial C_1(\bar{x}_1,\bar{x}_2)}{\partial x_2} .$$

The Cournot-Nash equilibrium of $\pi^1$ and $\pi^2$ is then also given in the point $(\bar{x}_1,\bar{x}_2)$ where

$$p - \frac{\partial C_1(\bar{x}_1,\bar{x}_2)}{\partial x_1} = \frac{\partial f^1(\bar{x}_1,\bar{x}_2)}{\partial x_1} = t_1 = \frac{\partial C_2(\bar{x}_1,\bar{x}_2)}{\partial x_1}$$

$$p - \frac{\partial C_2(\bar{x}_1,\bar{x}_2)}{\partial x_2} = \frac{\partial f^2(\bar{x}_1,\bar{x}_2)}{\partial x_2} = t_2 = \frac{\partial C_1(\bar{x}_1,\bar{x}_2)}{\partial x_2}$$

Hence,

$$\frac{\partial f(\bar{x})}{\partial x} = \begin{pmatrix} \frac{\partial C_2(\bar{x}_1,\bar{x}_2)}{\partial x_1} & -\frac{\partial C_1(\bar{x}_1,\bar{x}_2)}{\partial x_2} \\ -\frac{\partial C_2(\bar{x}_1,\bar{x}_2)}{\partial x_1} & \frac{\partial C_1(\bar{x}_1,\bar{x}_2)}{\partial x_2} \end{pmatrix}$$

Taking $p > 0$ in (6) equal to (1,1), we can conclude from (d) of the corollary of theorem 2 that $f(\bar{x})$ must be PPE, and hence also PE.

Suppose, finally, that the cost functions were given by $C_1(x_1)$ and $C_2(x_1,x_2)$. Let $\hat{x}_1$ be such that $p = \partial C_1(\hat{x}_1)/\partial x_1$. We then have

$$\frac{\partial f(\hat{x}_1,x_2)}{\partial x} = \begin{pmatrix} 0 & 0 \\ -\frac{\partial C_2(\hat{x}_1,x_2)}{\partial x_1} & p - \frac{\partial C_2(\hat{x}_1,x_2)}{\partial x_2} \end{pmatrix}$$

It follows from (c) of corollary 1 of theorem 7 that, whatever the value of $x_2$, points of the type $(\hat{x}_1,x_2)$ are always UD. Also, by (a) of the corollary of theorem 2, these points can never be PPE. It is clear from economic reasoning that only the point $(\hat{x}_1,\hat{x}_2)$ where $p = \partial C_2(\hat{x}_1,\hat{x}_2)/\partial x_2$ will be PE, at least from the firms'point of view.

(e) <u>Example 5: Decentralization in Separable Programming</u>

Let de vector x be written as $x = (x^1, x^2, \ldots, x^n)$ where $x^i \varepsilon E^{m_i}$, $\sum_{i=1}^{n} m_i = m$. Suppose the function f(x) can be written as

$$f(x) = \begin{pmatrix} f^1(x^1) \\ \vdots \\ f^n(x^n) \end{pmatrix}$$

We have then

$$\frac{\partial f(x)}{\partial x} = \begin{pmatrix} \nabla f^1(x^1) & 0 & \cdots & 0 \\ 0 & \nabla f^2(x) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \nabla f^n(x^n) \end{pmatrix}$$

Let the constraint function g(x) take the form

$$g(x) = \sum_{i=1}^{n} g_i(x^i) = \sum_{i=1}^{n} \begin{pmatrix} g_i^1\ (x^i) \\ \vdots \\ g_i^k\ (x^i) \end{pmatrix}$$

so that

$$\frac{\partial g(x)}{\partial x} = \left( \frac{\partial g_1(x^1)}{\partial x^1}, \ldots, \frac{\partial g_n(x^n)}{\partial x^n} \right)$$

We can then prove the following decentralisation theorem.

<u>THEOREM 11</u>. <u>Let the functions $f, g_1, \ldots, g_n$ be concave and differentiable in their domains of definition.</u>

(a) <u>Let $f(\hat{x}) \varepsilon f(X)$ be a PE point, and let the KTCQ be satisfied at $\hat{x}$. Then there exists a vector $p \geq 0$ and a $\lambda \varepsilon E^k$, $\lambda \geq 0$ such that</u>

(1) $\hat{x}^k$ maximizes $p_k f^k(x^k) + \lambda g_k\ (x^k)$

(2) $\lambda g(\hat{x}) = 0$

(b) Let $p > 0$, and let $\lambda \varepsilon E^k$, $\lambda \geqq 0$. If

(1) $\hat{x}^k$ maximizes $p_k f^k(x^k) + \lambda g_k(x^k)$

(2) $g(\hat{x}) \geqq 0$, $\lambda g(\hat{x}) = 0$

then $f(\hat{x})$ is PE.

Proof

(a) By theorem 8, if $f(\hat{x})$ is PE, then the system

$$p \frac{\partial f(\hat{x})}{\partial x} + \lambda \frac{\partial g(\hat{x})}{\partial x} = 0 \qquad (16)$$

$$\lambda g(\hat{x}) = 0$$

$$p \geq 0, \quad \lambda \geqq 0$$

must be consistent.

By the special structure of f and g, condition (16) can be written as

$$p_k \nabla f^k(\hat{x}^k) + \lambda \frac{\partial g_k(\hat{x}^k)}{\partial x^k} = 0 , \quad k:1, \ldots, n \qquad (17)$$

As all $f^k$ and $g_k$ are concave, the function

$$p_k f^k(x^k) + \lambda g_k(x^k) \qquad (18)$$

must also be concave for $p_k \geqq 0$ and $\lambda \geq 0$. It follows that (17) is necessary and sufficient for a maximum of (18).

(b) If $\hat{x}^k$ maximizes (18), then (17) and (16) must also hold. In addition, $g(\hat{x}) \geqq 0$ and $\lambda g(\hat{x}) = 0$. We can then conclude from (b) of theorem 2 that $f(\hat{x})$ must be PPE, and hence also PE. Q.E.D.

The economic interpretation of this theorem is obvious. It states the relationship between PE points and decentralized decisionmaking in a market economy. See, e.g., /2/ pp. 85-93, for a possible interpretation in a linear model. Theorem 11 generalizes some of Gale's results to nonlinear functions.

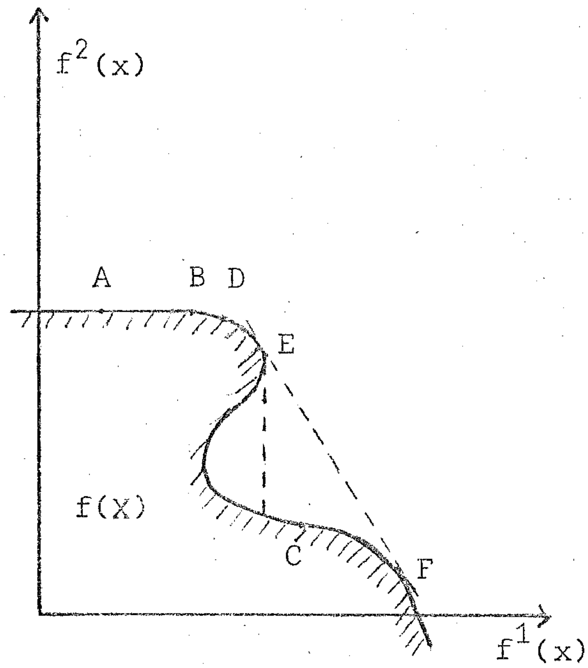



Figure 1

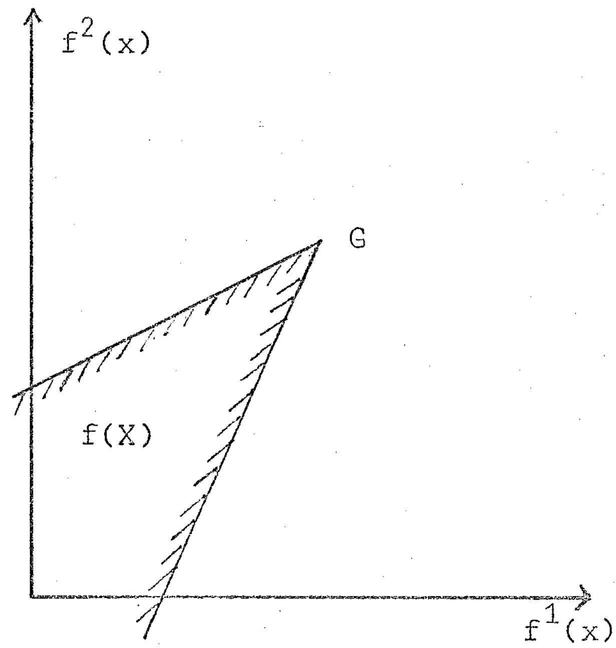



Figure 2

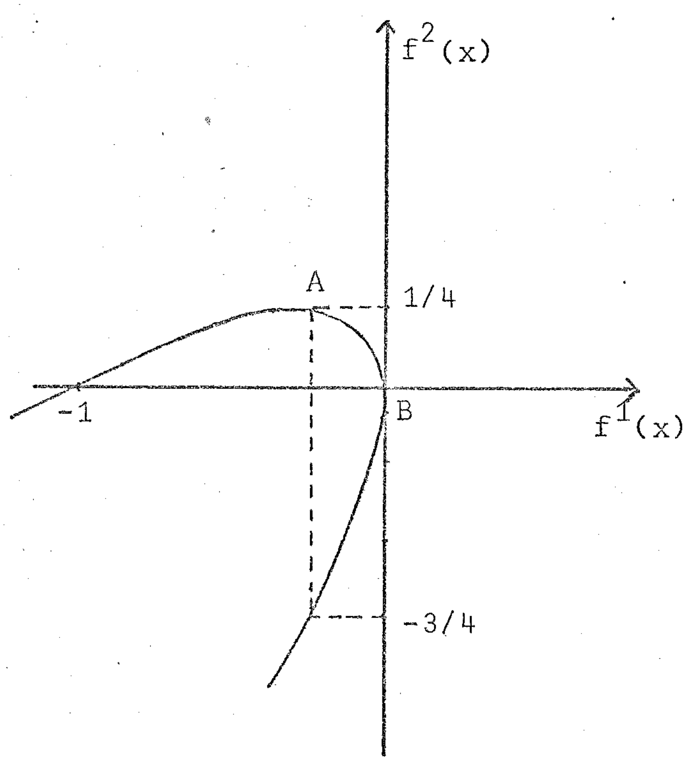



Figure 3

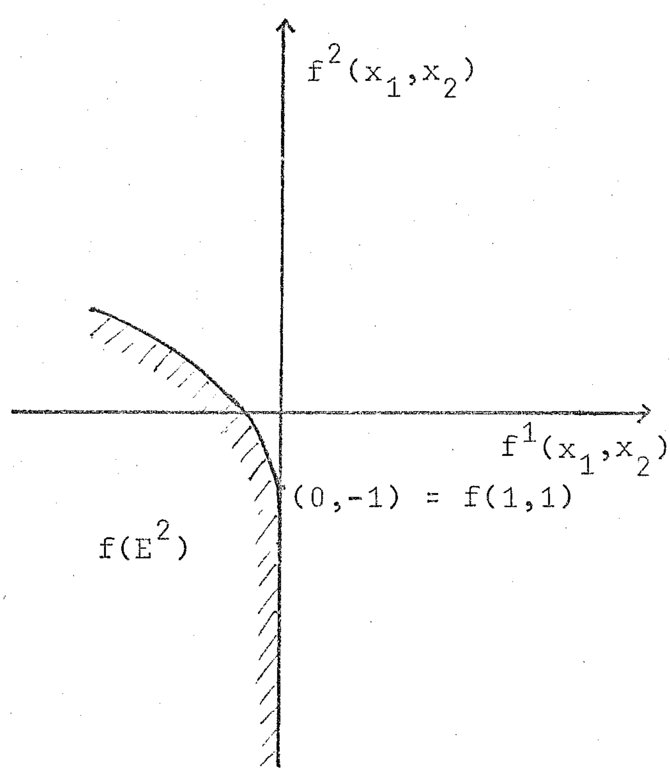



Figure 4

## REFERENCES

/1/ DA CUNHA, N.O., and E. POLAK, Constrained Minimization Under Vector-Valued Criteria in Finite Dimensional Spaces, Journal of Mathematical Analysis and Applications, Vol. 19, no. 1, July 1967, pp. 103-124.

/2/ GALE, D., The Theory of Linear Economic Models, Mc.Graw-Hill Book Company, New York, 1960.

/3/ KARLIN, S., Mathematical Methods and Theory in Games, Programming and Economics, Vol. I, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959.

/4/ KUHN, H.W., and A.W. TUCKER, Non-linear Programming, in Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Prohability, University of California Press, Berkeley, 1955, pp. 481-492.

/5/ MANGASARIAN, O.L., Non-linear Programming, Mc Graw-Hill Book Company, New York, 1969.

/6/ REKASIUS, Z.V., and W.E. SCHMITENDORF, On the non-inferiority of Nash Equilibrium Solutions, mimeographed.

/7/ ZADEH, L.A., Optimality and Non-Scalar-Valued Performance Criteria, I.E.E.E. Transactions on Automatic Control, January 1963, pp. 59-60.

/8/ ZANGWILL, W.I., Non-linear Programming. A Unified Approach, Prentice-Hall, Inc., Englewood Cliffs, 1969.