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W. PAUWELS

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Universitaire Faculteiten St.-Ignatius
Prinsstraat 13 - 2000 Antwerpen

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W. Pauwels (°)

The purpose of this paper is to prove a new necessary and sufficient condition for the workability and profitability of Leontief systems. Let $A = (a_{ij})$, $a_{ij} \geqq 0$, be the nxn matrix of input coefficients of a Leontief system. This system is said to be workable if, for any nonnegative (1) n-vector $c \geqq 0$ of final demands (net outputs), there exists a unique nonnegative n-vector $x \geqq 0$ of gross outputs such that x solves

$$(I - A)x = c.$$

Various equivalent workability conditions have been formulated. They can be summarized as follows (see /2/, pp. 88-95):

(I) $(I-A)x > 0$ has a semipositive solution $x \geqslant 0$.

(II) For any nonnegative vector $c \geqq 0$, there exists a unique nonnegative vector $x \geqq 0$ such that $(I - A)x = c$.

(III) (Hawkins-Simon) All leading principal minors of (I-A) are positive.

(IV) All principal minors of (I - A) are positive.

(°) The author is Assistant Professor at the University Faculties St.-Ignace, University of Antwerp.

(1) We use the usual ordering relations in E^n : $x = (x_i) \varepsilon E^n$ is nonnegative, denoted by $x \geqq 0$, iff $x_i \geqq 0$, i: 1, ..., n, $x = (x_i) \varepsilon E^n$ is positive, denoted by $x > 0$, iff $x_i > 0$, i: 1, ..., n, $x = (x_i) \varepsilon E^n$ is semipositive, denoted by $x \geqslant 0$, iff $x \geqq 0$ and $x \neq 0$.

It is well-known that the above workability conditions can easily be translated in terms of dual profitability conditions. A Leontief system is said to be profitable if, for any nonnegative vector $v \geqq 0$ of profits (values added), there exists a unique nonnegative price vector $p \geqq 0$ such that p solves

$$p' (I - A) = v'.$$

In this paper we will prove a new equivalent workability condition which can be formulated as

(V) $(I - A)x \leqq 0$ has no semipositive solution $x \geqslant 0$.

The corresponding profitability condition is then

(V)' $p'(I - A) \leqq 0$ has so semipositive solution $p \geqslant 0$.

In order to prove the equivalence of (I) - (IV) with (V), we first prove the following lemma.

<u>LEMMA</u>: Let $B = (b_{ij})$ be a real kxk matrix with $b_{ij} \leqq 0$, $i \neq j$, and let Δ_i be the i-th leading principal minor of B. If then $\Delta_i > 0$, i: 1,..., k-1, there exists a real kxk lower triangular matrix T such that
(1) TB is upper triangular.
(2) The diagonal elements of TB are given by Δ_1, Δ_2/Δ_1, ..., Δ_k/Δ_{k-1}.
(3) All off-diagonal elements of TB are nonpositive.

Proof:
Results (1) and (2) are standard results of Gauss's elimination method for solving linear equations (see /1/, pp. 23-31). To make the proof selfcontained, however, we will prove all three properties.

Since $\Delta_1 = b_{11} > 0$, we can define the kxk matrix

$$T^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -b_{21}/b_{11} & 1 & 0 & \cdots & 0 \\ -b_{31}/b_{11} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{k1}/b_{11} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Calling $B = B^{(1)}$, we define a matrix $B^{(2)}$ as

$$B^{(2)} = T^{(1)}B^{(1)} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1k} \\ 0 & b_{22}^{(2)} & b_{23}^{(2)} & \cdots & b_{2k}^{(2)} \\ 0 & b_{32}^{(2)} & b_{33}^{(2)} & \cdots & b_{3k}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_{k2}^{(2)} & b_{k3}^{(2)} & \cdots & b_{kk}^{(2)} \end{bmatrix}$$

where

$$b_{ij}^{(2)} = b_{ij} - (b_{i1}/b_{11})\, b_{1j}, \quad i, j\colon 2, 3, \ldots, k. \tag{1}$$

Since, by assumption, $b_{ij} \leqq 0$, $i \neq j$, and $b_{11} > 0$, it follows from (1) that

$$b_{ij}^{(2)} \leqq 0, \quad i, j\colon 2, 3, \ldots, k;\ i \neq j. \tag{2}$$

Using the usual notation for leading principal submatrices, we also have that

$$B_2^{(2)} = (T^{(1)} B^{(1)})_2 = T_2^{(1)} B_2$$

since $T^{(1)}$ is lower triangular. Taking determinants we obtain

$$\det(B_2^{(2)}) = b_{11} b_{22}^{(2)} = \det(T_2^{(1)}) \det(B_2) = \Delta_2$$

so that

$$b_{22}^{(2)} = (\Delta_2/\Delta_1) > 0 \tag{3}$$

We can then define a kxk matrix $T^{(2)}$ as

$$T^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -b_{32}^{(2)}/b_{22}^{(2)} & 1 & 0 & \cdots & 0 \\ 0 & -b_{42}^{(2)}/b_{22}^{(2)} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -b_{k2}^{(2)}/b_{22}^{(2)} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

We then obtain

$$B^{(3)} = T^{(2)} B^{(2)} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1k} \\ 0 & b_{22}^{(2)} & b_{23}^{(2)} & \cdots & b_{2k}^{(2)} \\ 0 & 0 & b_{33}^{(3)} & \cdots & b_{3k}^{(3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & b_{k3}^{(3)} & \cdots & b_{kk}^{(3)} \end{bmatrix}$$

where

$$b_{ij}^{(3)} = b_{ij}^{(2)} - (b_{i2}^{(2)}/b_{22}^{(2)})b_{2j}^{(2)}, \quad i, j : 3, 4, \ldots, k. \qquad (4)$$

From (2), (3) and (4) it follows that

$$b_{ij}^{(3)} \leqq 0, \quad i, j : 3, 4, \ldots, k, \ i \neq j.$$

Also,

$$\det (B_3^{(3)}) = \Delta_1(\Delta_2/\Delta_1)\ b_{33}^{(3)} = \det (T_3^{(2)}) \det (T_3^{(1)}) \det (B_3) = \Delta_3$$

so that

$$b_{33}^{(3)} = (\Delta_3/\Delta_2) > 0.$$

We can then continue in the same way until finally we obtain

$$B^{(k)} = T^{(k-1)}\ T^{(k-2)} \ldots T^{(1)}\ B = T\ B =$$

$$\begin{bmatrix} 1 & b_{12} & b_{13} & \ldots & b_{1k} \\ 0 & \Delta_2/\Delta_1 & b_{23}^{(2)} & \ldots & b_{2k}^{(2)} \\ 0 & 0 & \Delta_3/\Delta_2 & \ldots & b_{3k}^{(3)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \Delta_k/\Delta_{k-1} \end{bmatrix}$$

where $b_{ij}^{(i)} \leqq 0$, i: 1, 2, ..., $k-1$, $j > i$. Q.E.D.

We can now prove the main theorem of this paper. We will denote the matrix $(I - A)$ by $B = (b_{ij})$. Note that, because $a_{ij} \geqq 0$, we must have that $b_{ij} \leqq 0$, $i \neq j$.

<u>THEOREM</u>: The Leontief system $Bx = c$ is workable iff $Bx \leqq 0$ has no semipositive solution $x \geqslant 0$.

Proof:

(a) <u>Necessity</u>: We will prove that condition (II) implies the inconsistency of $Bx \leqq 0$, $x \geqslant 0$. Suppose this were not true. Then there would exist a vector $\bar{x} \geqslant 0$ such that $B\bar{x} \leqq 0$.
Consider then the system

$$Bx = B(-\bar{x}) \geqq 0.$$

Its unique solution is given by $x = -\bar{x} \leqslant 0$ which contradicts condition (II).

(b) <u>Sufficiency</u>: We will prove that the inconsistency of $Bx \leqq 0$, $x \geqslant 0$ implies condition (III). Suppose this were not true. Then at least one of the leading principal minors of B would be non-positive. Let Δ_k, $1 \leqq k \leqq n$, be the leading principal minor of B of smallest order which is nonpositive.
Hence,

$$\Delta_i > 0, \ i < k, \text{ and}$$
$$\Delta_k \leqq 0.$$

We can then partition $Bx \leqq 0$ such that

$$\begin{bmatrix} B_k & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \leqq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $z \in E^k$, $w \in E^{n-k}$.

We will now show that there exists a semipositive vector $z \geqslant 0$ such that

$$B_k z \leqq 0.$$

Since all elements of B_{21} are nonpositive, it also follows that

$$B_{21} z \leqq 0$$

for $z \geqslant 0$. We then have

$$Bx = \begin{bmatrix} B_k & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} B_k z \\ B_{21} z \end{bmatrix} \leqq 0$$

where $x = (z, 0) \geqslant 0$, which contradicts our assumption that $Bx \leqq 0$, $x \geqslant 0$, is inconsistent.

We distinguish two cases:

(1) $\Delta_k = 0$. Consider then the system

$$B_k z = 0.$$

Applying the transformation matrix T of the lemma, we obtain

$$TB_k z = T0, \text{ or}$$

$$B_k^{(k)} z = 0. \tag{5}$$

We can arbitrarly fix $z_k = 1$. Using (5) we can then uniquely and succesively solve for z_{k-1}, z_{k-2}, ..., z_1. Since, by the preceding lemma, all offdiagonal elements of $B_k^{(k)}$ in the upper triangle are nonpositive, the resulting values of z_{k-1}, z_{k-2}, ..., z_1 are all nonnegative. Hence, $z \geqslant 0$.

(2) $\Delta_k < 0$. Consider then the system

$$B_k z = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ -1 \end{bmatrix} \leqq 0$$

Applying again the transformation matrix of the lemma, we obtain

$$TB_k z = T \begin{bmatrix} 0 \\ \cdots \\ 0 \\ -1 \end{bmatrix}, \text{ or}$$

$$B_k^{(k)} z = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ -1 \end{bmatrix} \leqq 0. \qquad (6)$$

It is now again clear that (6) has a unique semipositive solution $z \geqslant 0$. Q.E.D.

The economic interpretation of workability condition (V) is that there does not exist a semipositive vector $x \geqslant 0$ of gross outputs such that no industry has a positive net output. Or equivalently, the application of any semipositive vector $x \geqslant 0$ of gross outputs results in a vector of net outputs which has at least one positive component. The truth of this result is graphically obvious in the two-dimensional case. See figure 1.

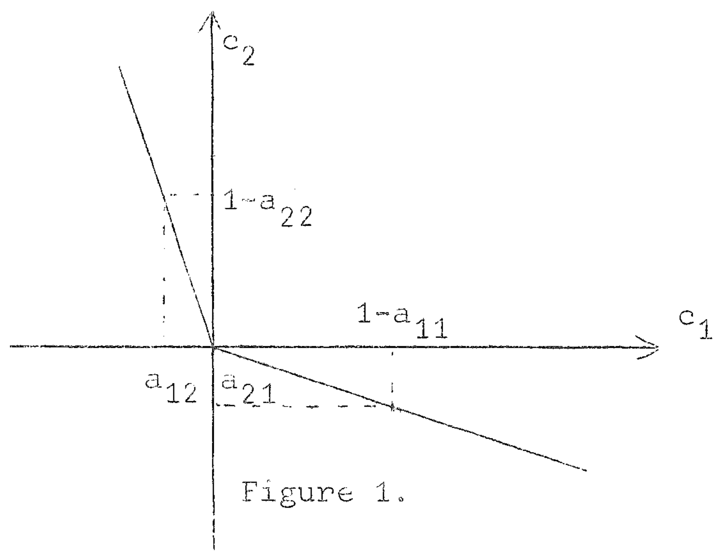

Figure 1.

The profitability condition (V)' can be interpreted similarly: there does not exist a semipositive price vector $p \geqslant 0$ such that no industry has a positive profit. Or equivalently, for any semipositive price vector $p \geqslant 0$, there is at least one industry which makes a positive profit [(1)].

(1) From a purely mathematical point of view, the theorem can be stated as follows. Let $B = (b_{ij})$ be a real nxn matrix with elements $b_{ij} \leqq 0$, $i \neq j$. Then either $Bx > 0$ has a solution $x > 0$, or $Bx \leqq 0$ has a solution $x \geqslant 0$.

REFERENCES

[1] GANTMACHER, F.R., *The Theory of Matrices*, Vol.I, Chelsea Publishing Company, New York, 1960.

[2] NIKAIDO, H., *Convex Structures and Economic Theory*, Academic Press, New York, 1968.