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# A SURVEY OF INTERREGIONAL PROGRAMMING MODELS AND A REHABILITATION OF THE POTENTIAL MODEL

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Potential models, originally developed by STOUFFER, ZIPF and STEWART (1), have found widespread application in applied transportation studies. Though primarily conceived to explain passenger movements, these models have also proved to be useful in the field of commodity transport (2) or location problems (3).

Especially with regard to commodity transport however, the validity of potential models is increasingly being doubted (4). The model is said not to be founded on sound economic grounds and to be unrelated with general equilibrium theory. It is felt desirable to replace it by a linear or non-linear programming model that explicitly refers to spatial price equilibrium.

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(1) S. STOUFFER, "Intervening opportunities: A theory relating mobility and distance", American Sociological Review", Vol.5, nr.4, Dec. 1940, pp.845-867.
G.K. ZIPF, "The $P_1P_2/D$ Hypothesis: The case of Railway Express", Journal of Psychology, 11, 1946.
J.Q. STEWART, "Empirical Mathematical Rules Concerning the Distribution and Equilibrium of Population", Geographical Review, July 1947, pp.461-485.

(2) E.g. J.R. MEYER & M.R. STRASZHEIM, Techniques of Transport Planning, Vol.1: Pricing and Project Evaluation, Washington, 1971, pp.165-169 (Further references in this book on pp.326-328), T. PEETERS, "Application du modèle gravitationnel à la structure des échanges internationaux de biens d'équipement", Statistische Studies en Enquêtes, 1970, nr.1, pp..63-94, and last but not least H. LINNEMAN, An econometric study of international trade flows, Amsterdam, 1966.

(3) E.g. W. KAU, Theorie und Anwendung raumwirtschaftlicher Potentialmodelle, Tübingen, 1970.

(4) Typical examples are G.K. SLETMO, Demand for air cargo, Bergen 1972, p.21, J.M. HARTWICK, "The gravity hypothesis and transportation cost minimization", Regional and Urban Economics, 1972, pp.297-308 and I.G. HEGGIE, "Are Gravity and Interactance Models a Valid Technique for Planning Regional Transport Facilities?", Operational Research Quarterly, Vol.20, nr.1, pp.93-110.

It is true that the theoretical development of potential models has greatly been inspired by vague analogies to fysical gravitation laws or by a priori assumptions on the probability of interaction (1). These expositions admittedly have created the impression that a potential model is merely a mechanical device.

In this paper however we will show that there exists a clear relation between the general equilibrium analysis of commodity flows and the potential model. The procedure to derive the latter from the former does not require more restrictive assumptions than those which are implied by the usual type of econometric models. Moreover, in practical applications the assumptions of the mpotential model may prove to be more attractive than the simplifications of the interregional programming models that are now becoming more and more popular.

This paper consists of 5 paragraphs. In §1 a general model of spatial price equilibrium is presented. In §2 we show how applied programming models may be derived from this general equilibrium analysis. In §3 the same thing is done for the potential model. In §4 some further comments on potential models are made. The paper is briefly summarized in §5.

## §1. SPATIAL PRICE EQUILIBRIUM

Along the lines of traditional analysis, the general equilibrium of an economy with n commodities can be defined by a system of n simultaneous equations in n prices, requiring the equality of demand and supply. Such a system can easily be adapted to cover explicitly the case of equilibrium in a spatial economy. For this purpose, it is sufficient to distinguish between spatially different commodities: We only have to treat" as two distinct commodities the good which is available at two distinct places" (2).

(1) Cfr. G.A.P.CARROTHERS, "An historical Review of the gravity and potential concepts of human interaction", Journal of the American Institute of Planners, 1956, vol.22, pp.94-102, W. ISARD, Methods of Regional Analysis, New York-London, 1960, pp.494-504. For more recent developments, cfr.A.G.WILSON, "The use of entropy maximising methods", Journal of Transport Economics and Policy, vol.III, 1969, pp.108-112.

(2) E.MALINVAUD, Leçons de théorie économique, Paris, 1969, p.6.

The total number of goods will then equal $n=I.K$ with I standing for the number of locations (1) and K for the number of commodities that differ in an other way than by location. The general equilibrium of this spatial market is defined by $n=I.K.$ simultaneous equations in $n=I.K.$ prices, expressing that at each of the I locations demand and supply of the K commodities have to be equal.

It is convenient to write this system in the following form:

$$S_k^i(P_1^1, \ldots, P_K^I) = \sum_{j=1}^{I} x_k^{ij} - \sum_{j=1}^{I} x_k^{ji} \qquad (i=1\ldots I,\ k=1\ldots K) \quad /\overline{1}.1\overline{/}$$

with $S_k^i$ the excess supply (the excess demand if negative), of commodity k at location i;
with $P_k^i$ the price of commodity k at location i;
with $x_k^{ij}$ the transport flow of commodity k from place i to place j.

The I.K. equations /1.1/ are balance equations, expressing that each positive $S_k^i$ must be "exported" to other locations and that each negative $S_k^i$ must be "imported". The equations require that after transport there remains no excess supply nor excess demand.

As it has been written however, system /1.1/ is not complete. It does not yet take account of the behaviour of the transport industry, which does not supply transport without a certain payment. We have to add the transport supply relations:

$$x_k^{ij} = x_k^{ij}(P_1^1 \ldots P_K^I) \qquad (i,j=1\ldots I, k=1\ldots K)$$

which can be specified as follows:

$$x_k^{ij} = 0 \quad \text{if} \quad P_k^j - P_k^i < c_k^{ij}\,(P_1^1 \ldots P_K^I) \qquad /1.2/$$

$$x_k^{ij} = \infty \quad \text{if} \quad P_k^j - P_k^i > c_k^{ij}\,(P_1^1 \ldots P_K^I) \qquad /1.3/$$

$$x_k^{ij} = \text{any desired non negative quantity} \quad \text{if} \quad P_k^j - P_k^i = c_k^{ij}\,(P_1^1 \ldots P_K^I) \qquad /1.4/$$

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(1) We assume this number to be finite. This assumption is essential, both in interregional programming models and in the potential model.

In these expressions $c_k^{ij}$ stands for the transfer cost, i.e. the total of transport, insurance, package, intrest, obsolescense and other costs of transferring one unit of commodity k from i to j (1). This transfer cost of course depends on the various prices $P_1^1 \ldots P_K^I$. The conditions [1.2] express that no transport takes place when the price difference $P_k^j - P_k^i$ does not cover the cost $c_k^{ij}$. The conditions [1.3] express that transporters or arbitragists would keep sending goods without any limit as long as $P_k^j - P_k^i$ exceeds $c_k^{ij}$, and finally [1.4] states that for $P_k^j - P_k^i = c_k^{ij}$ any desired quantity $x_k^{ij}$ will be transported, as long as it is not negative.

Transport supply is thus depicted by [1.2] to [1.4]. The counterpart of these supply relations are the demand functions for transport inputs, which can be written

$$ {}_tS_k^i = {}_tS_k^i(P_1^1 \ldots P_K^I,\ x_1^{11} \ldots x_K^{II}) \qquad [1.5] $$

in which ${}_tS_k^i < 0$ is a negative component in $S_k^i$, representing the factor demand for commodity k at location i by the transport industry.
We can write $S_k^i = {}_tS_k^i + {}_oS_k^i$, with ${}_oS_k^i$ negative or positive and standing for production and consumption outside the transport industry.

Taking [1.1] to [1.5] together, we obtain the complete system [1.6], defining spatial price equilibrium:

$$ {}_tS_k^i(P_1^1 \ldots P_K^I, x_1^{11} \ldots x_K^{II}) + {}_oS_k^i(P_1^1 \ldots P_K^I) = \sum_{j=1}^{I} x_k^{ij} - \sum_{j=1}^{I} x_k^{ji} \qquad (1.6.1) $$

$$ (i=1 \ldots I, k=1 \ldots K) $$

$$ P_k^j - P_k^i \leq c_k^{ij}(P_1^1 \ldots P_K^I) \qquad (i,j=1 \ldots I,\ k=1 \ldots K) \qquad (1.6.2) $$

$$ [P_k^j - P_k^i - c_k^{ij}(P_1^1 \ldots P_K^I)]\, x_k^{ij} = 0 \qquad (i,j=1 \ldots I,\ k=1 \ldots K) \qquad (1.6.3) $$

$$ x_k^{ij} \geq 0 \qquad (i,j=1 \ldots I,\ k=1 \ldots K) \qquad (1.6.4) $$

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(1) Cfr. B. OHLIN, Interregional and International Trade, Cambridge, 1933 and D. L'HUILLIER, Le coût de transport, Paris 1965, pp.49-52.

The equations (1.6.1) are the familiar balance equations /1.1/, $S_k^i$ being divided into its two components, ${}_tS_k^i$ and ${}_oS_k^i$. The remaining equations or inequalities have been added to define the flows $x_k^{ij}$ in accordance with the behavior of transport supply, as described by /1.2/ to /1.4/. The inequalities (1.6.2) can be called price limits. They express that a price difference may never exceed the transfer cost. (Otherwise we would obtain an infinite quantity $x_k^{ij}$, as stated by /1.3/). The equations (1.6.3) can be called transport prohibitions. They require $x_k^{ij}$ to be zero (in accordance with /1.2/) as soon as the strict inequality holds in the price limit (1.6.2). The transport prohibitions are inactive when the equality holds in (1.6.2). In this case they do not define $x_k^{ij}$ (which is in accordance with /1.4/).

The system /1.6/ is a fairly general description of spatial price equilibrium. It is formulated in terms of demand and supply functions and constitutes a counterpart of the spatial equilibrium analysis that LEFEBER and von BÖVENTER have developed in terms of utility and transformation functions (1).

It should be noted that the solution of /1.6/ is complicated. The source of the trouble are the specifications of transport supply in /1.2/ to /1.4/. For an adequate description of reality however, such cumbrous specifications cannot be avoided.

The system /1.6/ can in theory be solved by a blind trial and error procedure. A better alternative however is its reformulation as a programming problem (2). This can be done very easily: using (1.6.2) and (1.6.4), we can conclude that the product $(P_k^j - P_k^i - c_k^{ij}).x_k^{ij}$ is never positive. Its maximum value is zero. Zero is also the maximum value of the sum of $I^2K$ terms $Z_K^I = \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{k=1}^{K} (P_k^j - P_k^i - c_k^{ij}) x_k^{ij}$. If we maximize $Z_k^I$ (i.e. make $Z_k^I$ zero), each term in the summation will be zero and the transport prohibitions (1.6.3) will be satisfied. Thus instead of the transport prohibitions we may impose to system /1.6/ the artificial objective function Max $Z_K^I$.

(1) L. LEFEBER, Allocation in Space, Amsterdam, 1958.
E. von BÖVENTER, Theorie des räumlichen Gleichgewichts, Tübingen, 1962.

(2) This procedure is not new in economics. In general one returns to some economic maximum principle from which the equilibrium conditions are deduced (cfr. P.A. SAMUELSON, Foundations of Economic Analysis, Cambridge 1966, p. ). In the present case an explicitation of this maximum principle is not needed.

We then obtain the programming problem /1.7/.

$$\text{Max } Z_K^I = \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{k=1}^{K} [P_k^j - P_k^i - c_k^{ij}(P_1^1 \ldots P_K^I)] x_k^{ij} \qquad (1.7.1)$$

subject to

$$_tS_k^i(x_1^{11} \ldots x_K^{II}, P_1^1 \ldots P_K^I) + {_oS_k^i}(P_1^1 \ldots P_K^I) = \sum_{j=1}^{I} x_k^{ij} - \sum_{j=1}^{I} x_k^{ji} \qquad (1.7.2)$$

$$(i=1 \ldots I,\ k=1 \ldots K)$$

$$P_k^j - P_k^i \leqslant c_k^{ij}(P_1^1 \ldots P_K^I) \qquad (i,j=1 \ldots I, k=1 \ldots K) \qquad (1.7.3)$$

$$x_k^{ij} \geqslant 0 \qquad (i,j=1 \ldots I, k=1 \ldots K) \qquad (1.7.4)$$

This programming problem differs from the original system /1.6/ only by the fact that an objective function Max $Z_K^I$ has been introduced instead of the transport prohibitions (1.6.3). The maximum of $Z_K^I$ is never positive since (1.7.3) and (1.7.4) require each term in this sum to be negative or zero. If Max $Z_K^I$ is zero all terms $(P_k^j - P_k^i - c_k^{ij}) x_k^{ij}$ are zero and the original transport prohibitions are satisfied. If Max $Z_K^I < 0$, the original system /1.6/ is inconsistent.

The programming variables in /1.7/ are the prices $P_k^i$ (i=1...I,k=1...K) and the flows $x_k^{ij}$ (i,j=1...I, k=1...K). If the functions $c_k^{ij}$, $_tS_k^i$ and $_oS_k^i$ are linear in these variables, Max $Z_K^I$ is a quadratic programming problem with a bilinear objective function and linear constraints. Problems of this kind can be solved with existing computer programs.

## §2. APPLIED SPATIAL PROGRAMMING MODELS

In the preceding paragraph we have shown that spatial price equilibrium can be defined equivalently by a system of equations and inequalities /1.6/ or by a programming problem /1.7/. In this paragraph we will show that the existing applied interregional programming models are special cases of the general models /1.6/ or /1.7/. A common simplification in all these cases is the assumption that the transfer costs $c_k^{ij}$ (i,j=1...I, k=1...K), which depend upon prices in the general model, are autonomous constants. The other assumptions however are different from one model to another.

Consecutively we will treat the Samuelson-Smith-model, the Moses-model and the Koopmans-Hitchcock-model.

## 1. The Samuelson-Smith-model

The simplifications introduced by Enke, Samuelson and Smith, are the following (1):

1. Partialisation: Only one commodity k is being considered. The excess supply $S_k^i$ of this commodity at location i is only a function of $P_k^i$.
2. The transfer costs are autonomous constants.

Clearly our general programming model /1.7/ may be aplied to this simplified case if we set K equal to 1, each $c_1^{ij}$ equal to a given constant and write $S_1^i(P_1^i)$ in the left hand side of (1.7.2). If $S_1^i(P_1^i)$ (i=1...I) are linear functions, the programming problem is quadratic.

In the Samuelson-Smith approach other programming formulations are used: A typical (2) formulation would be /2.1/.

$$\text{Max } S = -\sum_{i=1}^{I} \int P_1^i(S_1^i)dS_1^i - \sum_{i=1}^{I}\sum_{j=1}^{I} x_1^{ij}c_1^{ij} \qquad (2.1.1)$$

subject to $x_1^{ij} \geqslant 0.$ (i,j=1...I) (2.1.2)

(1) The problem was formulated but not solved analytically by S. ENKE, "Equilibrium among spatially separated markets: Solution by electric analogue", Econometrica, 1951, nr.1, pp.40-47. Samuelson made the first step to a programming solution in "Spatial Price Equilibrium and Linear Programming", American Economic Review, Vol.XLII, nr3, pp.283-303. The problem was further elaborated by V.L.SMITH, "Minimization of Economic Rent in Spatial Price Equilibrium", Review of Economic Studies, Vol.XXX,nr.1,1963, pp.24-31. A clear reformulation can be found in E.SILBERBERG, The demand for waterway transportation, Ann Arbor, 1964, pp.44-47.

(2) This formulation slightly differs from the existing ones. Where it uses an indefinite integral in the objective function, Samuelson and Silberberg specify the limits of integration, in order to obtain a welfare interpretation. The difference with Smith's model is not fundamental either. The equivalence of the approaches can be shown. An explicit treatment however of Smith's model would carry us toofar here.

The functions $P_1^i(S_1^i)=P_1^i(\sum_{j=1}^{I} x_1^{ij} - \sum_{j=1}^{I} x_1^{ji})$ are the inverse functions of the excess supply functions $S_1^i(P_1^i)$. Their integral is the function which after differentiation with respect to $S_1^i$ (this means to $x_1^{ij}$ or to $-x_1^{ji}$) reproduces the price $P_1^i$.

From the definition of $S_1^i \equiv \sum_{j=1}^{I} x_1^{ij} - \sum_{j=1}^{I} x_1^{ji}$ and from the Kuhn-Tucker conditions for the maximum of /2.1/.

$$\frac{\partial S}{\partial x_1^{ij}} = P_1^j - P_1^i - c_1^{ij} \leq o \qquad (i,j=1...I)$$

$$\frac{\partial S}{\partial x_1^{ij}} x_1^{ij} = (P_1^j - P_1^i - c_1^{ij}) x_1^{ij} = o \qquad (i,j=1...I)$$

$$x_1^{ij} \geq o \qquad (i,j=1...I)$$

it is clear that /2.1/ coincides with the general model /1.6/. The qualifications however are important: especially the partialisation to one single commodity (1).

## 2. The Moses-model

The Moses-model (2) is obtained by specifying the excess-supply functions in such a way that input-output concepts can be used. It makes the following assumptions:

1) The transfer costs $c_k^{ij}$ are autonomous constants;

2) the excess supply functions are of the form

$$S_k^i = X_k^i(P_1^i...P_K^i) - \sum_{h=1}^{K} a_{kh}^i X_h^i (P_1^i...P_K^i) - Y_k^i$$

(1) A multi-commodity extension has been worked out by TAKAYAMA T. and JUDGE G.G., "Spatial Equilibrium and Quadratic Programming", Journal of Farm Economics, Vol.46, February 1964, pp.67-93, Id., "Equilibrium among spatially separated markets: A Reformulation", Econometrica, Vol.32, nr.4, October 1964, pp.510-524; TAKAYAMA T. and WOODLAND A.D., "Equivalence of Price and Quantity Formulations of Spatial Equilibrium", Econometrica, Vol.38, nr.6, November 1970, pp.889-906.

(2) Leon N. MOSES, "A general equilibrium model of production, interregional trade and location of industry", Review of Economics and Statistics, 1960, November, pp.373-397.

with $X_k^i$ home production of commodity k in region i;

$a_{kh}^i$ fixed Leontief technical coefficient, representing the input of commodity k per unit of commodity h, produced in region i.

$Y_k^i$ final demand for commodity k in region i (autonomously given).

Excess supply is thus defined by substracting from the home production, the intermediate home demand and the final home demand, both being subject to the traditional specification in input-output-analysis.

3) The home production $X_k^i$ depends as follows on prices:

$$X_k^i = 0 \quad \text{for } P_k^i < w_k^i + \sum_{h=1}^{K} P_h^i \, a_{hk}^i$$

$$X_k^i = C_k^i \quad \text{for } P_k^i > w_k^i + \sum_{h=1}^{K} P_h^i \, a_{hk}^i$$

$$0 \leqslant X_k^i \leqslant C_k^i \quad \text{for } P_k^i = w_k^i + \sum_{h=1}^{K} P_h^i \, a_{hk}^i$$

with $C_k^i$ the production capacity of commodity k in region i. In the above expressions $P_k^i$ is compared with the cost of production ($w_k^i$ representing a fixed cost per unit of primary production factors, $\sum_{h=1}^{K} P_h^i a_{hk}^i$ representing the cost of intermediate production factors).

Spatial price equilibrium is then defined by the linear programming problem /2.2/:

$$\text{Max } M = -\sum_{i=1}^{I} \sum_{k=1}^{K} w_k^i X_k^i - \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{k=1}^{K} c_k^{ij} \, x_k^{ij} \qquad (2.2.1)$$

subject to

$$X_k^i - \sum_{h=1}^{K} a_{kh}^i X_h^i - Y_k^i = \sum_{j=1}^{I} x_k^{ij} - \sum_{j=1}^{I} x_k^{ji} \quad (i=1\ldots I, k=1\ldots K) \qquad (2.2.2)$$

$$X_k^i \leqslant C_k^i \quad (i=1\ldots I, k=1\ldots K) \qquad (2.2.3)$$

$$X_k^i, \; x_k^{ij} \geqslant 0 \quad (i,j=1\ldots I, k=1\ldots K) \qquad (2.2.4)$$

This model minimises the total of primary production costs and transfer costs. The programming variables are the home productions $X_k^i$ (i=1... I, k=1...K) and the flows $x_k^{ij}$ (i,j=1...I, k=1...K), whereas $c_k^{ij}$, $w_k^i$, $a_{kh}^i$, $Y_k^i$, $C_k^i$ (i,j=1...I, k,h=1...K) are given coefficients.

It is easily seen that /2.2/ coincides with our general equilibrium model /1.6/. Restrictions (2.2.2) correspond with the balance equations (1.6.1) and the remaining propositions in /2.2/ are equivalent with the remainder of /1.6/, as can be shown by means of the Kuhn-Tucker conditions for Max M: if L is the Langrange-function, if $\lambda_k^i$ is the multiplier associated with the condition on $X_k^i$ in (2.2.2) and if $\pi_k^i$ is the multiplier associated with the condition on $X_k^i$ in (2.2.3),the Kuhn-Tucker conditions are:

$$\frac{\partial L}{\partial x_k^{ij}} = -c_k^{ij}-\lambda_k^i+\lambda_k^j \leqslant o \ , \ \frac{\partial L}{\partial x_k^{ij}} x_k^{ij}=o \quad (i,j=1...I,k=1...K) \qquad (2.2.5)$$

$$\frac{\partial L}{\partial X_k^i} = -w_k^i+\lambda_k^i-\sum_{h=1}^{K} \lambda_h^i a_{hk}^i-\pi_k^i \leqslant o \ , \ \frac{\partial L}{\partial X_k^i}.X_k^i=0 (i=1...I,k=1...K) \qquad (2.2.6)$$

$$\frac{\partial L}{\partial \lambda_k^i} = \sum_{j=1}^{I} x_k^{ij}-\sum_{j=1}^{I} x_k^{ji}-X_k^i+\sum_{k=1}^{K} a_{kh}^i X_h^i+Y_k^i=o \ ..(i=1...I,h=1...K) \qquad (2.2.7)$$

$$\frac{\partial L}{\partial \pi_k^i} = C_k^i-X_k^i \geqslant o \ , \ \frac{\partial L}{\partial \pi_k^i} \pi_k^i = o \qquad (i=1...I, k=1...K) \qquad (2.2.8)$$

Interpreting $\lambda_k^i$ as the price $P_k^i$ of commodity k at place i, we find that (2.2.5) indeed represents the price limits (1.6.2) and the transport prohibitions (1.6.3). The remaining Kuhn-Tucker conditions either repeat the balance equation (2.2.7), or specify that at prices $\lambda_k^i$ home production is indeed in equilibrium: Using (2.2.8) we find that this production never exceeds capacity. The (non-negative)multiplier $\pi_k^i$ can only exceed zero when capacity is fully employed. In this case we find in (2.2.6) that $\lambda_k^i$ may exceed the production cost $w_k^i+\sum_{k=1}^{I} \lambda_h^i a_{hk}^i$, whereas with unused capacity ($X_h^i<c_h^i$), $\pi_k^i$ is zero and $\lambda_k^i$ exactly equals production cost (or is less than production cost, in which case $X_k^i$=o). All this is in perfect accordance with assumption 3 on the supply of home production.

We may conclude that the Moses-model is another special case of the general model /1.6/. Again the qualifications are obvious. They consist mainly of simplifications of the Leontief-type.

Obviously the model can be made subject to various modifications (1). It is possible to treat the transport industry just as other industries and to replace the transfer cost $c_k^{ij}$ by a primary cost component and a cost of intermediate inputs. One can introduce dynamic input-output concepts, add capacity restrictions for the transport industry, split $w_k^i$ into capital costs and labour costs and specify restrictions on the total input of these factors, etc.

## 3. The Koopmans-Hitchcock Model

Finally a very simple case of spatial price equilibrium is represented by the Koopmans-Hitchcock transhipment problem (2). In this case one assumes not only that the transfer costs are constant but also that the excess supply quantities $S_k^i$ are autonomously given. This means that supply and demand are completely inelastic and that changes in the factor demand by transportation are neglected.

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(1) See A. GHOSH & A. CHAKRAVARTI, "The problem of location of an industrial complex" in A.P.CARTER & A.BRODY, ed., Contributions to input-output analysis, Amsterdam-London, 1970, pp.164-179, A.P.HURTER & L.N.MOSES, "Regional Investment and interregional programming", Papers of the Regional Science Association, Vol.13, 1964, pp.105-119, R. HOWES, "A test of a linear programming model of agriculture", Papers of the Regional Science Association, Vol.19, 1967, pp.123-140. The traditional "interregional input-output analysis" however is not a linear programming model but essentially an input-output model with fixed interregional technical coefficients. It is not a variant of the Moses-model which essentially allows substitution between alternative origins. See the earlier papers by L.N.MOSES, "The stability of interregional trading patterns and input-output analysis", American Economic Review, December 1955, pp.803-832 and W.ISARD, "Interregional and Regional Input-Output Analysis: A model of a Space Economy, Review of Economics and Statistics, November 1951.

(2) This "transhipment" problem is slightly different from the "transportation problem", as developed by F.L. HITCHCOCK, "The distribution of a product from several sources to numerous localities", Journal Math & Phys, nr.20, 1941, pp.224-230 and T.C.KOOPMANS, "Optimum Utilization of the Transportation System", Econometrica, Suppl., Vol.17, 1949. However it can be solved with the same fairly simple techniques (cfr. G.HADLEY, Linear Programming, Reading, Palo Alto, London 1962, pp.368-373 or A.ORDEN, "The transhipment problem", Management Science, 1956, nr.3, pp.276-285).

We obtain the simple linear programming model /2.3/

$$\text{Max } D = - \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{k=1}^{K} c_k^{ij} \; x_k^{ij} \qquad (2.3.1)$$

subject to

$$S_k^i = \sum_{j=1}^{I} x_k^{ij} - \sum_{j=1}^{I} x_k^{ji} \qquad (i=1 \ldots I, k=1 \ldots K \qquad (2.3.2)$$

$$x_k^{ij} \geqslant 0 \qquad (i,j=1 \ldots I, \; k=1 \ldots K) \qquad (2.3.3)$$

which minimizes the total of transfer costs subject to the balance equation and the non-negativity constraint (1). The programming variables are the various $x_k^{ij}$, whereas the transfer costs $c_k^{ij}$ and the excess supply quantities $S_k^i$ are given constants.

The equivalence of /2.3/ and the general equilibrium system /1.6/ is shown by means of the Kuhn-Tucker conditions in which $\lambda_k^i$ is the multiplier associated with the restriction on $S_k^i$ in (2.3.2)

$$\frac{\partial L}{\partial x_k^{ij}} = -c_k^{ij} - \lambda_k^i + \lambda_k^j \leqslant 0 \;, \; \frac{\partial L}{\partial x_k^{ij}} \cdot x_k^{ij} = 0 \; (i,j=1 \ldots I, \; k=1 \ldots K) \qquad (2.3.5)$$

$$\frac{\partial L}{\partial \lambda_k^i} = S_k^i - \sum_{j=1}^{I} x_k^{ij} + \sum_{j=1}^{I} x_k^{ji} = 0 \qquad (i=1 \ldots I, \; k=1 \ldots K) \qquad (2.3.6)$$

The conditions (2.3.6) are exactly the balance equations (1.6.1). Interpreting $\lambda_k^i$ as the price $P_k^i$ we see that (2.3.5) exactly contains the price limits (1.6.2) and the transport prohibitions (1.6.3). It is interesting to note that in this linear programming problem the multipliers $\lambda_k^i$ are only defined up to an additive constant. This of course is also a feature of the prices $P_k^i$. When demand and supply are completely inelastic, only the interregional differences between prices are defined by spatial price equilibrium. The general level of prices is irrelevant.

It is possible of course, to devise some variants of the Koopmans-Hitchcock model (2). The essential limitation however is the assumption

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(1) It can be shown that the model Max D is equivalent to K separate transhipment problems, one for each commodity.

(2) E.HEADY & A.EGBERT, "Programming Models of Interdependence among agricultural sectors and spatial allocation of crop production", Journal of Regional Science, 1962, Vol.2, nr.4, pp.1-20, M.L.GREENHOT, "Interregional Programming and the demand factor of location", Journal of Regional Science, 1967.

that the quantities $S_k^i$ are autonomously given. If these are made dependent upon prices (as in the models discussed previously) a Koopmans-Hitchcock solution is impossible. Clearly, with $S_k^i$ depending on $P_k^i (=\lambda_k^i)$ the linear programming problems /2.3/ can not be used. The restrictions (2.3.2) can not be formulated as long as the prices (= multipliers) resulting from the program are unknown: To start solving the problem we need the restrictions (2.3.2) and to formulate these restrictions we need the prices (multipliers) that are not known before the problem is solved.

The attempts by Samuelson and others (1) to apply linear programming to the general case of supply and demand quantities $S_k^i$ depending upon prices, are doomed to fail for this reason. Nevertheless they have drawn the attention to an important aspect of the Koopmans-Hitchcock model: if it were possible, by some other method, to define the equilibrium quantities $S_k^i$ in advance, then the rest of the solution (the flows $x_k^{ij}$) can be defined by a linear programming problem that simply minimises transfer costs subject to the restriction that the non-negative flows satisfy the import or èxport requirements $S_k^i$. It is indeed clear that the Kuhn-Tucker conditions (2.3.5) and (2.3.6) only require that, for the flows adding up to the given $S_k^i$, some variables $\lambda_k^i$ (=prices) must exist, satisfying the price limits and the transport prohibitions. This is nothing more and nothing less than a spatial price equilibrium.

We may conclude that the more general programming models, which we discussed previously, contain as a subproblem the minimization of transfer costs according to a Koopmans-Hitchcock model. These more general programming models perform two functions simultaneously:

1) They compute the equilibrium quantities $S_k^i$;
2) For these quantities they minimize the transfer costs, according to a Koopmans-Hitchcock transhipment problem.

We will refer to this important conclusion in the next paragraph, when discussing the aggregation properties of interregional programming problems.

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(1) P.A. SAMUELSON, op.cit., and R.L. MORRILL & W.L. GARRISON, "Projections of Interregional Patterns of Trade in Wheat and Flour", Economic Geography, April 1960, pp.116-126.

## §3. THE POTENTIAL MODEL

Often the data on interregional flows are limited to a transport table of the following kind:

| | 1 | 2 | 3 | | I | Row total |
|---|---|---|---|---|---|---|
| 1 | $c^{11}$ $x^{11}$ | $c^{12}$ $x^{12}$ | $c^{13}$ $x^{13}$ | ... | $c^{1I}$ $x^{1I}$ | $R^1$ |
| 2 | $c^{21}$ $x^{21}$ | $c^{22}$ $x^{22}$ | $c^{23}$ $x^{23}$ | ... | $c^{2I}$ $x^{2I}$ | $R^2$ |
| 3 | $c^{31}$ $x^{31}$ | $c^{32}$ $x^{32}$ | $c^{33}$ $x^{33}$ | ... | $c^{3I}$ $x^{3I}$ | $R^3$ |
| .. | .. | .. | .. | ... | .. | .. |
| .. | .. | .. | .. | ... | .. | .. |
| .. | .. | .. | .. | ... | .. | .. |
| I | $c^{I1}$ $x^{I1}$ | $c^{I2}$ $x^{I2}$ | $c^{I3}$ $x^{I3}$ | ... | $c^{II}$ $x^{II}$ | $R^I$ |
| Column total | $L^1$ | $L^2$ | $L^3$ | ... | $L^I$ | X |

In this table the economic space has been split into I regions, numbered 1 to I. The cell in row i and column j contains the observed commodity flow from region i to region j and the corresponding transfer cost. The last row and column show the column- and row-totals, which in their turn sum up to the general total $X = \sum_{i=1}^{I} R^i = \sum_{j=1}^{I} L^j$.

An essential weakness of these data is their being composed of heterogeneous commodities. This remains so, even if a transport table is provided for various commodity groups, such as iron, ore, coal or wine separately. These commodity groups are still heterogeneous. Even at an extremely high cost of data collection, it is impossible to provide transport tables for each really homogeneous product separately.

This heterogenity causes the interregional programming problems to be biased, as we will show first, and may incite us to use a potential model.

## 1. The aggregation bias in interregional programming models

It is tempting to apply a Koopmans-Hitchcock model in order to explain the flows in a transport table. Ignoring the heterogenity of the commodities, this model would amount to [3.1]

$$\text{Max } D = - \sum_{i=1}^{I} \sum_{j=1}^{I} c^{ij} x^{ij} \qquad (3.1.1)$$

subject to

$$R^{i} = \sum_{j=1}^{I} x^{ij} \qquad (i=1...I) \qquad (3.1.2)$$

$$L^{j} = \sum_{i=1}^{I} x^{ij} \qquad (j=1...I) \qquad (3.1.3)$$

$$x^{ij} \geqslant 0$$

In reality however each row-total $R^i$ and column-total $L^j$ consists of various commodities. Supply and demand must be equal for each of these various commodidies and not only for their aggregate. This means that in the true problem each restriction in (3.1.2) is to be replaced by K restrictions (one per commodity) of the type $R^i_k = \sum_{j=1}^{I} x^{ij}_k$. The same thing holds for the restrictions (3.1.3).

It is obvious that the aggregated problem [3.1] is obtained from the true problem by weakening the restrictions. Instead of requiring that each homogeneous excess supply $R^i_k$ be "exported" and each homogeneous excess demand $L^j_k$ be "imported", the aggregated problem only requires this by sum over the K commodities.

This weakening of the constraints, while the objective is to minimize transfer costs, clearly tends to produce solutions with underestimated transfer costs (1).

(1) This does occur in empirical applications, as one may infer from the results in P.O.SULLIVAN, "Linear Programming as a forecasting device for interregional freight flows in Great-Brittain", Regional and Urban Economics, 1972, nr.4, pp.383-396. For the least heterogeneous commodity-groups this bias is small but it remains present.

The tendency to underestimate transfer costs is also present in more complicated interregional programming models. We have shown indeed that these models contain two sub-problems:

1. Computation of the equilibrium $S_k^i$;
2. Koopmans-Hitchcock minimization of transfer costs subject to these $S_k^i$.

The aggregation bias emerges in the second part where aggregation (weakening the constraints) leads to underestimated transfer costs and to a false preference for unexpensive routes.

It is clear that the aggregation bias in interregional programming models is often an awkward property: we obtain exaggerated estimates of the price elasticity of transport flows, exaggereted flows on low-cost relations (such as the main diagonal in the transport table ), we cannot explain cross-haulage, we compute zero-flows for relations on which in reality positive flows occur etc. These undesirable properties become stronger when the level of aggregation in the transport table increases. When only one highly aggregated table is available, containing all goods in the economy, the aggregation bias makes interregional programming an unworkable device.

## 2. The theoretical derivation of a potential model

The potential model in its simplest form is

$$x^{ij} = O^i . D^j . f(c^{ij}) . e^{\varepsilon^{ij}}$$

with $O^i$ = the origin potential of region i, expressing its importance as an origin region;

$D^j$ = the destination potential of region j, expressing its importance as a destination region;

$\varepsilon^{ij}$ = a disturbance term

f = a function that can be specified in various forms and of which the parameters have to be estimated from the transport table.

e = basis of natural logarithms.

More complicated versions, involving more than two potentials and one transfert cost as explanatory variables, will be called multiple potential models and treated at the end of this paragraph.

For the moment we will concentrate our attention on the simplest version of the potential model which is obtained by setting all potentials equal to unity:

$$x^{ij} = f(c^{ij}).e^{\varepsilon^{ij}}$$

We will show how this relates to the general spatial price equilibrium. The extension to other potentials than unity (e.g. the row and column-totals or regional productions) and to multiple potential models is straight forward. It will only be indicated briefly.

It is clear from the analysis of spatial price equilibrium that, given the excess supply functions $S_k^i(P_1^1 \ldots P_I^I)$ the introduction of a set of transfer costs $c_k^{ij}$ specifies the price limits

$$P_k^j - P_k^i \leqslant c_k^{ij} \qquad (i,j=1\ldots I,\ k=1\ldots K)$$

and the transport prohibitions

$$(P_k^j - P_k^i - c_k^{ij})\ x_k^{ij} = o$$

From these price limits and transport prohibitions the equilibrium flows $x_k^{ij}$ could in principle be calculated by means of a programming model, if it were possible to disaggregate sufficiently.

When another set of transfer costs is introduced, the price limits and the transport prohibitions are changed which results in a new equilibrium set of flows $x_k^{ij}$, such that the transport flows can be considered as functions of the set of transfer costs.

Aggregating over commodities we may write:

$$\begin{aligned} x^{11} &= f^{11}(c^{11}, c^{12}, \ldots, c^{II}) \\ x^{12} &= f^{12}(c^{11}, c^{12}, \ldots, c^{II}) \\ &\vdots \\ x^{II} &= f^{II}(c^{11}, c^{12}, \ldots, c^{II}) \end{aligned} \qquad [3.2]$$

with $x^{ij} = \sum_{k=1}^{K} x_k^{ij}$ representing the aggregated flow from region i to region j and with $c^{ij}$ as an average of the various $c_k^{ij}(k=1...K)$.

The critical reader who wants rigorous propositions may question the existence of the aggregated functions /3.2/. As a matter of fact different sets of separate transfer costs $c_k^{ij}(k=1...K)$ for the K commodities may coincide with the same average $c^{ij}$. If one value of $c^{ij}$ in /3.2/ may be composed of different commodity components $c_k^{ij}$ it may also give rise to different transport flows, such that the functions in /3.3/ are not one-valued.

To avoid this objection we can define the separate commodity transfer cotst $c_k^{ij}$ as well -specified functions of their common aggregated value $c^{ij}$. For instance when $c^{ij}$ is simply distance in miles we can assume the existence of K one-valued functions $f_k(c^{ij})$ defining each $c_k^{ij}$ for each $c^{ij}$. Each set of values of the explanatory variables in /3.2/ will then correspond with well-defined values of all the I.K. different transfer costs, which clearly define a spatial price equilibrium.

With this interpretation the aggregation does not prevent the existence of /3.2/. It only prevents the calculation of these functions by means of an interregional programming model which is devised for homogeneous commodities.

It is clear that $f^{11}...f^{II}$ in /3.3/ are different functions. For example it is obvious that $\frac{\partial x^{11}}{\partial c^{11}} \neq \frac{\partial x^{12}}{\partial c^{11}}$ as the own price-effect of $c^{11}$ upon its flow $x^{11}$ must be different from its cross-price-effect upon a flow $x^{12}$.

For the moment let us simplify and choose a particular form for the functions $f^{11}...f^{II}$, e.g. a linear one:

$$\begin{aligned} x^{11} &= \alpha^{11} + \beta^{11}c^{11} + \gamma^{11,12}c^{12} + \dots + \gamma^{11,II}c^{II} \\ x^{12} &= \alpha^{12} + \gamma^{12,11}c^{11} + \beta^{12}c^{12} + \dots + \gamma^{12,II}c^{II} \\ &\vdots \\ x^{II} &= \alpha^{II} + \gamma^{II,11}c^{11} + \gamma^{II,12}c^{12} + \dots + \beta^{II}c^{II} \end{aligned} \qquad /3.3/$$

This set of equations clearly refers to the general set /3.2/, implied by spatial equilibrium, with this proviso of course that a specific functional form is assumed (presently the linear one). This functional form moreover is continuous and differentiable while in reality, as shown in §1, the flows $x^{ij}$ react more or less as stepfunctions (making a step when for instance one of the commodities, aggregated in $x^{ij}$, suddenly becomes zero because of the emergence of a transport prohibition). However as the number of different commodities in $x^{ij}$ increases, the steps become less important and a continuous function becomes a closer approximation of reality. Such an approximation of discontinuous phenomena by means of a continuous function is a general practice in econometric model-building. It should not be reproached more to the potential model than to other econometric models.

In /3.3/ the parameters $\alpha^{11} \ldots \alpha^{II}$ are $I^2$ different constants, one for each flow, the parameters $\beta^{11} \ldots \beta^{II}$ represent $I^2$ different own price effects and the parameters $\gamma^{11,12} \ldots \gamma^{II,II-1}$ represent $I^2(I^2-1)$ different cross-price effects. It is clearly out of the question to estimate so many parameters from the $I^2$ observations in a transport table. However we may simplify and neglect the cross-price effects (this means leave them in a disturbance term). We then obtain:

$$\begin{aligned} x^{11} &= \alpha^{11} + \beta^{11}c^{11} + v^{11} \\ x^{12} &= \alpha^{12} + \beta^{12}c^{12} + v^{12} \\ &\vdots \\ x^{II} &= \alpha^{II} + \beta^{II}c^{II} + v^{II} \end{aligned} \qquad /3.4/$$

with $v^{ij}$ containing the neglected cross-price effects (and any other disturbance which might already be present in /3.3/ when these functions are not exact).

As far, nothing unusual has happened. No econometric analysis can afford to consider explicitely the endless number of variables that in reality influence an economic phenomenon. To obtain unbiased estimates of the dependent variable and of the parameters the disturbance term must only be uncorrelated with the explanatory variables and be zero in expected value. Moreover when its covariance-matrix is scalar the least-squares estimate of the standard errors will also be unbiased.

For the moment we will accept these assumptions on the disturbance, which are certainly not uncommon. In §4 we will investigate this matter more closely.

The set of equations /3.4/ is not yet ready for estimation from a transport table. It still contains too many parameters. We have to introduce a final simplification and substitute one single $\alpha$ for the various $\alpha^{ij}$ and one single $\beta$ for the various $\beta^{ij}$. We then finally obtain the simplest version of the potential model (linear and with all potentials equal to unity):

$$x^{ij} = \alpha + \beta\, c^{ij} + \varepsilon^{ij} \qquad (i,j=1\ldots I) \qquad /3.5/$$

The disturbance term $\varepsilon^{ij}$ now contains not only the cross price effects $v^{ij}$ of /3.4/ but also structural differences between regions.

This last simplification, producing /3.5/ with only two instead of $2I^2$ parameters, is probably the basic simplification that causes most of the criticism on potential models. One may ask whether substituting the different $\alpha^{ij}$ and $\beta^{ij}$ by one single $\alpha$ and $\beta$ is not a deliberate specification error, leading to completely false results. Much of the criticism on potential models boils down to the statement that a specification like /3.5/ is completely different from the original systems /3.4/ and /3.3/. However one can show that, apart from an "aggregation bias", the least squares estimate $\hat{\alpha}$ is in expected value equal to the mean of the various $\alpha^{ij}$, while the expected value of $\hat{\beta}$ equals the mean of the various $\beta^{ij}$. Thus, with $\beta$ in /3.5/ we do estimate the price effect $\frac{\partial x^{ij}}{\partial c^{ij}}$ for an average relation.

The proof is analogous to the analysis of aggregation properties of micro-relations, as developed by H. THEIL (1). The parameters $\alpha^{ij}$ and $\beta^{ij}$ $(i,j=1\ldots I)$ in /3.4/ are the micro-parameters, the parameters $\alpha$ and $\beta$ in /3.5/ are the macro-parameters and the relation between both can be shown. Let us represent the column vector of $I^2$ dependent flows $x^{ij}$ by the standard symbol Y and the matrix of explanatory variables by X.

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(1) H.THEIL, Linear Aggregation of Economic Relations, Amsterdam, 1954, Id., Principles of Econometrics, Amsterdam-London, 1971, pp.556-562. Also T.KLOEK, "Note on convenient matrix notations in multivariate statistical analysis", International Economic Review, Vol.2, 1961, pp.351-360.

The matrix X consists of two columns, the first one containing $I^2$ elements equal to one, the second column containing the $I^2$ transfer costs $c^{ij}$

From the transport table we obtain the least-squares estimates of $\alpha$ and $\beta$:

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (X'X)^{-1} X'Y \qquad [3.6]$$

The expected values of $\hat{\alpha}$ and $\hat{\beta}$ as functions of the micro-parameters $\alpha^{ij}$ and $\beta^{ij}$ are obtained when Y is replaced by its expected value according to [3.4]

Rewriting the cumbrous double notation ij by means of one index i (i=1...N with $N=I^2$), we have $E(Y_i)=\alpha_i+\beta_i c_i$ or in matrix notation

$$E\,[Y] = [I \,\vdots\, \hat{C}] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix}$$

The matrix $[I \,\vdots\, \hat{C}]$ is of order N x 2N and is represented in partitioned form, I standing for the NxN unit matrix and $\hat{C}$ for het NxN diagonal matrix, containing on the main diagonal the N transfer costs.

Substituting in [3.6], we obtain

$$E \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (X'X)^{-1}X'E[Y] = (X'X)^{-1}X'[I \,\vdots\, \hat{C}] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix} \qquad [3.7]$$

Because

$$[I \,\vdots\, \hat{C}] = X\,M + ([I \,\vdots\, C] - XM)$$

holds vor each M of order 2x2N, /3.7/ can be rewritten as

$$E\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = M\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\\ \beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{bmatrix} + (X'X)^{-1}\left(\left[I\,\middle|\,C\right] - XM\right)\begin{bmatrix}\alpha_1\\ 2\\ \vdots\\ \alpha_N\\ \beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{bmatrix}$$

This gives for M equal to the 2x2N matrix

$$\begin{bmatrix}\frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\end{bmatrix}$$

transforming the micro-parameters $\alpha_i$ and $\beta_i$ into their means $\bar{\alpha}$ and $\bar{\beta}$:

$$E\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\bar{\alpha}\\ \bar{\beta}\end{bmatrix} + (X'X)^{-1}X'\begin{bmatrix}\alpha_1 - \bar{\alpha} + (\beta_1-\bar{\beta})\ c_1\\ \beta_2 - \bar{\alpha} + (\beta_2-\bar{\beta})\ c_2\\ \vdots\\ \alpha_N - \bar{\alpha} + (\beta_N-\bar{\beta})\ c_N\end{bmatrix} \qquad /3.8/$$

This shows that the expected value of $\hat{\alpha}$ and $\hat{\beta}$ indeed equals the mean of the corresponding micro-parameters plus a correction term, which is completely analogous to the "aggregation bias", described by THEIL and inherent to any regression estimate of micro-relations from macro data or to any cross-section analysis estimating one common parameter from data on different economic units, possessing all an own micro-parameter.

The correction term satisfies the well-known properties of "aggregation bias" (1):

1. It depends simultaneously on corresponding and non-corresponding micro-parameters.
2. It is zero when all micro-parameters are equal ($\alpha_i=\bar{\alpha}, \beta_i=\bar{\beta}, i=1...N$)

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(1) Cfr. H. THEIL, op.cit., pp.560-562.

3. It is a linear operation upon variables, measured as deviations from their means, and therefore zero when the coefficients of this linear operation are not correlated with the variables.

A numerical calculation of the correction term is impossible when no information on the difference between micro-parameters exists. However from this point of view the potential model is not worse off than the numerous econometric models that also proceed to cross-section analysis or to aggregation, without having this information. Not less than other econometric models the potential model relies on the hope that the zero correlation of property 3 is satisfied, such that the correction term (which simply represents a regression of micro-differences upon transfer costs and a constant) vanishes asymptotically.

In practice the aggregation problem in the simplification producing [3.5] from [3.4] means, that the standard errors of $\hat{\alpha}$ and $\hat{\beta}$ must be interpreted liberally. Not only is the variance of the micro-parameters ($\alpha^{ij}$ and $\beta^{ij}$) greater than the variance of their mean. Also, when the number of cells in the transport table is not sufficient to rely on asymptotic properties, an additional uncertainty emerges in the estimation of this mean which may contain a positive or a negative bias, depending upon the unknown micro-parameters.

The proof which we have given for a simple linear functional form and for one explanatory variable plus a constant, can be easily extended. Other functional forms can be introduced. No derivation will be invalidated if, from [3.3] on, we replace x and c by their logarithms. This will produce the model:

$\log x^{ij} = \alpha + \beta \log c^{ij} + \varepsilon^{ij}$ instead of [3.5].

Analogously we can obtain semi-logarithmic specifications. The most popular specification of simple potential models in which the explanatory variable $x^{ij}$ is divided by the product of corresponding potentials $O^i.D^j$, can be obtained when in [3.3] such division of $x^{ij}$ is performed. This modification is legitimate. The theory of spatial price equilibrium, stating that the various flows $x^{ij}$ depend on the $I^2$ transfer costs, also warrants that the ratio's $x^{ij}/O^i.D^j$ will be defined by these variables.

Further there is no problem when the number of explanatory variables is extended, like in the case of multiple potential models. The content and the order of the various matrices in /3.8/ will be changed but we obtain the fundamental result:

$$E \begin{bmatrix} \hat{\alpha} \\ \beta \\ \hat{\gamma} \\ \vdots \end{bmatrix} = \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \\ \vdots \end{bmatrix} + (X'X)^{-1}X' \begin{bmatrix} \alpha_1 - \bar{\alpha} + (\beta_1 - \bar{\beta})c_1 + (\gamma_1 - \bar{\gamma})d_1 + \dots \\ \alpha_2 - \bar{\alpha} + (\beta_2 - \bar{\beta})c_2 + (\gamma_2 - \bar{\gamma})d_2 + \dots \\ \vdots \\ \alpha_N - \bar{\alpha} + (\beta_N - \bar{\beta})C_N + (\gamma_N - \bar{\gamma})d_N + \dots \end{bmatrix}$$

where $\gamma \dots$ are the additional parameters for the additional explanatory variables d ... and where X has been extended with these additional variables.

This exposition demonstrates that various specifications of the potential model are possible, always referring to micro-parameters in a spatial equilibrium system. The choice between alternative specifications will be inspired by plain logic, by the residual variance criterion, by the objectives of the model, etc. It should be noted that common-sense arguments, leading to the division of the dependent variable by potentials, can be translated in terms of the exposition above. One can argue indeed that the micro-parameters in the explanation of the ratio's $x^{ij}/O^i.D^j$ show less mutual difference than the micro-parameters explaining $x^{ij}$ directly. The effect of changing a transfer cost must indeed be more or less proportional to the importance of the connected regions. This smaller difference between micro-parameters of course affects the correction term in a favourable way.

## §4. FURTHER COMMENTS ON THE POTENTIAL MODELS

We have shown that aggregation of commodities affects the validity of the interregional programming models, which assume homogenity. At a high level of aggregation these models become completely inadequate.

On the other hand potential models do not require homogenity of the commodity flows. The continuous functional forms in these models even become better approximations of reality when the level of aggregation is increased. It is not true that these models are unrelated with general equilibrium theory. Their deviation from the general model Max $Z_K^I$ does not require uncommon assumptions and is not more heroic than the assumption of homogenity in interregional programming models.

We have applied a potential model on Belgian interregional transport data for october 1968. These data were provided by the B.E.P. (Bureau voor Economische Programmatie) and are an aggregate of all commodities transported by rail, road or inland waterways between ten major regions in the country. Applying an interregional programming model at this level of aggregation is out of the question. However we calibrated the following potential model:

$$x^{ij} = \rho^i . \lambda^j \ \alpha (c^{ij})^{\beta} . e^{\varepsilon \delta ij} . e^{u^{ij}} \qquad /4.1/$$

with $\rho^i, \lambda^j$: (i,j=1...10) potentials of the ten regions

$\alpha$: constant

$\beta$: price elasticity of transport demand

$\delta_{ij}$: Kronecker's delta ($\delta_{ij}$ = 1 if i=j
= 0 if $i \neq j$)

$c^{ij}$ : transfer cost between region i and region j (computed as 110 + 1,87 times the geographical distance between the regional centres concerned. This function was obtained by a regression of transport tariffs on associated distances. Data for this regression were quotations by transporters and official tariffs)

$e^{\varepsilon}$ : correction term, distinguishing transport within a region from interregional transport

$u^{ij}$ : disturbance term

$x^{ij}$ : observed flows (B.E.P.-data)

e : basis of natural logarithms.

Taking natural logarithms /4.1/ can be read as:

$$\log x^{ij} = \lg\alpha + \lg\rho^{i} + \lg\lambda^{j} + \beta \lg c^{ij} + \varepsilon\delta_{ij} + u^{ij} \quad (i,j=1...10) \qquad /4.1/$$

with 21 parameters to be estimated ($\lg\alpha, \beta, \varepsilon$, $\lg\rho^{i}$ (i=2...10), $\lg\lambda^{j}$(j= 2...10 . The values of $\lg\rho^{1}$ and $\lg\lambda^{1}$ are zero by definition). The estimation of the potentials of regions 2...10 proceeds by treating them as coefficients, associated with dummy variables. The dummy of lg $^{i}$ is one for all commodity flows $x^{ij}$ leaving region i and is zero for the other observations $x^{uj}$ ($u \neq i$). The dummy of $\lg\lambda^{j}$ is one for all commodity flows $x^{ij}$ arriving in region j and zero for the other observations $x^{iv}$ ($v \neq j$).

The main least-squares-estimates are

$\hat{\beta}$ = -2,4534 with standard error 0,2455

$\hat{\varepsilon}$ = 1,3408 with standard error 0,2928

$R^2$ = 0,8939

$F(20,79)$ = 33,29,

which are highly significant results.

We pretend that $\beta$= - 2,4534 can be considered as an estimate of the price elasticity of demand for commodity transport. Moreover this estimate is a long-run estimate, as the transfer costs are closely associated with geographical distances, which have been constant since prehistoric times.

The interpretation of $\beta$ as a price elasticity necessitates some further comments. With regard to this aim of the model some objections have been made in the literature. We want to show that they cannot seriously invalidate the interpretation of $\beta$ as the price elasticity of transport demand. Three points have to be considered:

1) the selection of potentials;
2) the problem of simultaneous equations;
3) the nature of the disturbance term.

## 1. The selection of potentials

The interpretation of a simple potential model

$$x^{ij} = O^i . D^j \; f(c^{ij}) . e^{u^{ij}}$$

depends for a great deal on the selection of the potentials $O^i$ and $D^j$. Quite often these potentials are set equal to endogenous variables, such as the side-totals of the transport table, regional production or income, which of course depend themselves upon the explanatory variable $c^{ij}$. This has some awkward consequences. The parameters of $f(c^{ij})$ no longer indicate the price elasticity of $x^{ij}$ (the effect of $c^{ij}$ upon $x^{ij}$) but they give merely the price elasticity of the ratio $x^{ij}/O^i.D^j$. Instead of representing the effect $\frac{\partial x^{ij}}{\partial c^{ij}}$, they stand for a mixed effect upon both the flow $x^{ij}$ and the potentials $O^i.D^j$(1). Nearly all applications of potential models are based upon endogeneous potentials. None of these applications indeed allow an estimation of the price elasticity of transport demand.

It is clear however that our specification /4.1/ does not use endogenous potentials but estimates the potentials as parameters, which do not change when the transfer costs alter. The objection on endogenous potentials, which is valid for most applications of the potential model, does not hold for our specification /4.1/.

An additional advantage of least-squares-estimation of the potentials is of course a better explanation of the observed flows $x^{ij}$. The potentials $\rho^i$ and $\lambda^j$, calibrated by means of least-squares by definition produce a higher $R^2$ than any set of (endogenous or exogenous) potentials, selected a priori. In the Belgian case we compared the $R^2$=0,8939 in model /4.1/ with the $R^2$ obtained for potentials equal to

- side totals of the transport table;
- regional production;
- surface of the regions.

The difference in $R^2$ is always significant at 1% level.

(1) This point has been stressed by I.S.JONES, "Gravity Models and Generated Traffic", Journal of Transport Economics and Policy, Vol.4, N°2, 1970, pp.208-211 and I.G.HEGGIE, op.cit., pp.101-103.

## 2. Simultaneous equations

A further question is whether the least-squares-estimate $\hat{\beta}$ in the demand function [4.1] is not biased by the presence of simultaneous equations. One could argue indeed, that the observations on $x^{ij}$ and $c^{ij}$ in the transport table do not trace out an identified demand function but merely represent a mixture of both the demand and the supply function, from which no estimate of the price elasticity $\beta$ is possible.

The key to the identification problem is the fact that from one cell in the transport table to another we observe different geographic distances. This however is an exogenous variable appearing in the supply function but not appearing in the demand function. (The transport customer is only intrested in distance to the extent of its presence in the transfer cost $c^{ij}$. In his demand function distance does not appear beside the transfer cost. The supplier however essentially compares the price $c^{ij}$ with the cost of transporting over a certain distance). To make the picture clear, we can associate with the demand function [4.1] a supply function [4.2]

$$lgx^{ij} = lg\alpha' + \beta' lgc^{ij} + \pi lgd^{ij} + \varepsilon'\delta_{ij} + u'^{ij} \qquad [4.2]$$

with $d^{ij}$ geographic distance.

Identification of the demand function [4.1] will then result from the fact that only one endogeneous variable is present in its specification ($c^{ij}$) while an exogenous variable ($d^{ij}$) does not occur.

In order to obtain an unbiased estimate of $\beta$ in [4.1] we could apply two-stage-least-squares. In practice however this if often unnecessary. The first stage, regressing the endogenous variable $c^{ij}$ upon the predetermined variables (such as $d^{ij}$), is made superfluous by the practice of using distance $d^{ij}$ itself as a proxy for $c^{ij}$. This practice rules out estimation of the supply function [4.2] (containing two identical variables now) but provides a solution of the simultaneous-equation-problem in the demand function [4.1]. The explanatory variable $lgc^{ij}$ is made exogenous and the supply equation is no longer able to cause a correlation between the disturbance term and the explanatory variable in the demand function.

Preferably, as in our application of /4.1/, one does not use distance itself as a proxy for $c^{ij}$ but a function of distance, calibrated from a set of observations on cost and distance. The result is again the elimination of simultaneous-equation bias and this time without introducing another bias, due to the systematic difference between distance and transfer costs, which in reality are not proportional to distance but increase less.

## 3. The nature of the disturbance term

When the transfer costs and the specification of the model are treated adequately, all objections so far can be aboided. However, there remains the problem of the disturbance term $u^{ij}$. If this disturbance is uncorrelated with the explanatory variable $c^{ij}$, a specification such as /4.1/ by definition produces an unbiased least squares estimate $\hat{\beta}$ of the price elasticity of transport demand .

The problem is, that zero-correlation between $c^{ij}$ and $u^{ij}$ (which was already assumed in §1) is not warranted as such. The main objection relates to the fact that $u^{ij}$ (as explained in §3) incorporates the "other" transfer costs in the table. Now, when observing a "change" of $c^{ij}$ in the transport table, we do not remain in one cell of the table but we move to another cell. This means that a higher value of the explanatory variable $c^{ij}$ is always associated with a lower value of the "other" transfer costs, hidden in $u^{ij}$. A "change" of the explanatory variable $c^{ij}$ is not a pure change in $c^{ij}$ alone but is associated with a compensating change in "other" transfer costs. The least-squares estimate of $\beta$ represents not only the own price effect of $c^{ij}$ upon $x^{ij}$ but also a cross price effect of transfer costs $c^{uv}$ upon $x^{ij}$ ($uv \neq ij$).

The point is whether this objection, even if it is theoretically correct, is important in practice. One may assume that cross-price effects $\partial x^{ij}/\partial c^{uv}$ ($uv \neq ij$) are small, compared to $\partial x^{ij}/\partial c^{ij}$. In this case the difference between $\hat{\beta}$ and the true price elasticity is negligible.

A way to remove cross price effects from the disturbance is a specification of the type /4.2/.

$$x^{ij} = O^{i}.D^{j} f(c^{ij}) / \sum_{v=1}^{I} D^{v} f(c^{iv}).e^{u^{ij}} \qquad /4.2/$$

This is a multiple potential model (1) which removes some cross-price-effects from the disturbance. In the denominator $\sum_{v=1}^{I} D^v f(c^{iv})$ we find indeed the effects exerted upon $x^{ij}$ by other transfer costs in its row.

Such a multiple potential model however has some undesirable properties: It takes account of cross-price-effects within a row of the transport table but neglects the effects within a column. It assumes a priori a given relation between the own price effect in the numerator and the cross price effects in the denominator. Last but not least, the estimation of the parameters in $f(c^{ij})$ is hampered by the unattractive functional form of the specification.

An alternative model, removing cross price effects from the disturbance term, is /4.3/

$$x^{ij} = O^i . D^j . \alpha (c^{ij})^{\beta} . (\overline{c^{i\cdot}})^{\gamma} (\overline{c^{\cdot j}})^{\theta} . e^{\varepsilon \delta_{ij}} . e^{u^{ij}} \qquad /4.3/$$

with $\overline{c^{i\cdot}} = \prod_{j=1}^{I} (c^{ij})^{d^j}$ (weighted geometric average of transfer costs in row i)

$\overline{c^{\cdot j}} = \prod_{i=1}^{I} (c^{ij})^{a^i}$ (weighted geometric average of transfer costs in column j).

The weights are defined as $d^j = D^j / \sum_{v=1}^{I} D^v$ and $a^i = O^i / \sum_{u=1}^{I} O^u$

The least-squares estimates are:

$\hat{\beta}$ = -2,4599 with standard error 0,3398

$\hat{\gamma}$ = -0,6041 with standard error 0,8665

$\hat{\theta}$ = 0,6731 with standard error 0,8665

$\hat{\varepsilon}$ = 1,3349 with standard error 0,4071

$R^2$ = 0,7212

$F(4;95)$ = 61,44

The potentials $O^i$ and $D^j$ in this model were geographic surfaces of the regions.

Clearly the contribution of the cross price effects $(\overline{c^{i\cdot}})^{\gamma}$ and $(\overline{c^{\cdot j}})^{\theta}$ is insignificant. The estimates certainly do not invalidate the estimate of $\hat{\beta}$=-2,4534 which was obtained in model /4.1/ and which we pretended was a valid estimate of the long run own price elasticity of transport between two regions.

(1) See e.g. G.R.WELLS, Highway Planning Techniques, The balance of Cost and Benefit, London 1971, p.122.

## §5. SUMMARY CONCLUSION

We have developed a general model of spatial price equilibrium Max $Z_K^I$, from which the various interregional programming problems can be derived by introducing suitable assumptions. This contribution is important because it provides the link between different models that have evolved independently, often without an explicitation of their assumptions and without referring one to another.

As it is possible to derive a potential model from this general model Max $Z_K^I$ ,it is untrue that potential models are not related with general equilibrium theory, while interregional programming models are. To the contrary, it seems that aggregation affects the potential model less than it affects interregional programming models.

It has been shown that, provided the potentials and the transfer costs are defined adequately, one may advocate the use of potential models for estimation of the long-run price elasticity of transport demand. This elasticity is to be interpreted as the effect of changing one single transfer cost between two regions, ceteris paribus the other transfer costs in the economy. This is quite different of course from the effect, resulting from a general increase of all transfer costs in the economy. For Belgium the elasticity estimate is -2,4534.